Preface

This *Complete Solutions Manual* contains detailed solutions to all exercises in the text *Multivariable Calculus, Fourth Edition* (Chapters 11–18 of *Calculus, Fourth Edition* and Chapters 10–17 of *Calculus: Early Transcendentals, Fourth Edition*) by James Stewart. A *Student Solutions Manual* is also available, which contains solutions to the odd-numbered exercises in each chapter section, review section, True-False Quiz, and Problems Plus section as well as all solutions to the Concept Check questions. (It does not, however, include solutions to any of the projects.)

The *Early Transcendentals* version of the text uses different chapter and page numbers; consequently, all section numbers and references are given in a dual format. Users of the *Early Transcendentals* text should use the references denoted by "ET."

While we have extended every effort to ensure the accuracy of the solutions presented, we would appreciate correspondence regarding any errors that may exist. Other suggestions or comments are also welcome, and can be sent to dan clegg at `dclegg@palomar.edu` or in care of the publisher: Brooks/Cole Publishing Company, 511 Forest Lodge Road, Pacific Grove CA 93950.

We would like to thank James Stewart for entrusting us with the writing of this manual and offering suggestions, Kathi Townes of TECH-arts for typesetting and producing this manual, and Brian Betsill of TECH-arts for creating the illustrations. We also thank Gary W. Ostedt and Carol Ann Benedict of Brooks/Cole Publishing Company for their trust, assistance, and patience.

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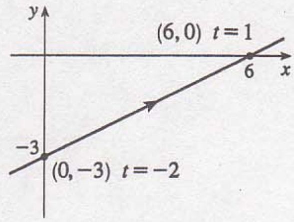
11 Parametric Equations and Polar Coordinates ET 10

11.1 Curves Defined by Parametric Equations ET 10.1

1. (a) $x = 2t + 4,\ y = t - 1$

t	-3	-2	-1	0	1	2
x	-2	0	2	4	6	8
y	-4	-3	-2	-1	0	1

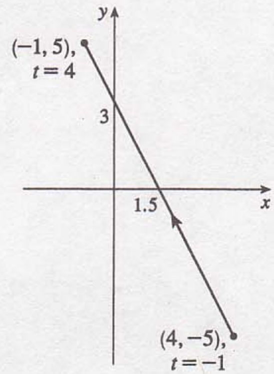


(b) $x = 2t + 4,\ y = t - 1 \quad \Rightarrow \quad x = 2(y + 1) + 4 = 2y + 6$ or

$y = \frac{1}{2}x - 3$

2. (a) $x = 3 - t,\ y = 2t - 3,\ -1 \le t \le 4$

t	-1	0	1	2	3	4
x	4	3	2	1	0	-1
y	-5	-3	-1	1	3	5

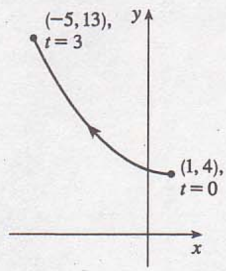


(b) $x = 3 - t \quad \Rightarrow \quad t = 3 - x \quad \Rightarrow \quad y = 2t - 3 = 2(3 - x) - 3 \quad \Rightarrow$

$y = 3 - 2x$

3. (a) $x = 1 - 2t,\ y = t^2 + 4,\ 0 \le t \le 3$

t	0	1	2	3
x	1	-1	-3	-5
y	4	5	8	13



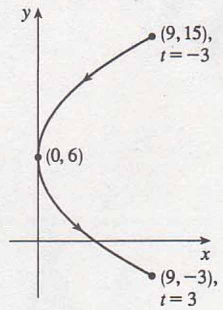
(b) $x = 1 - 2t \quad \Rightarrow \quad 2t = 1 - x \quad \Rightarrow \quad t = \dfrac{1 - x}{2} \quad \Rightarrow$

$y = t^2 + 4 = \left(\dfrac{1 - x}{2}\right)^2 + 4 = \frac{1}{4}(x - 1)^2 + 4$ or

$y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{17}{4}$

4. (a) $x = t^2, y = 6 - 3t$

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	15	12	9	6	3	0	-3

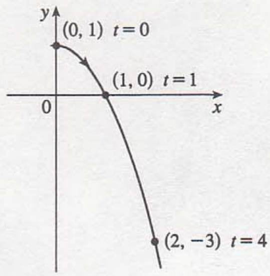


(b) $y = 6 - 3t \;\;\Rightarrow\;\; 3t = 6 - y \;\;\Rightarrow\;\; t = \dfrac{6-y}{3} \;\;\Rightarrow$

$$x = t^2 = \left(\frac{6-y}{3}\right)^2 = \tfrac{1}{9}(y-6)^2$$

5. (a) $x = \sqrt{t}, y = 1 - t$

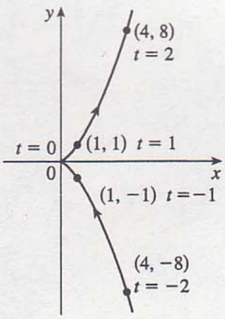
t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



(b) $x = \sqrt{t} \;\;\Rightarrow\;\; t = x^2$. $y = 1 - t = 1 - x^2$. Since $t \geq 0$, $x \geq 0$.

6. (a) $x = t^2, y = t^3$

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

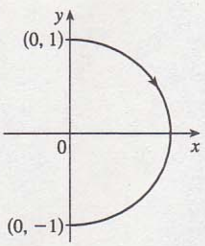


(b) $y = t^3 \;\;\Rightarrow\;\; t = \sqrt[3]{y}$. $x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}$. $t \in \mathbb{R}, y \in \mathbb{R}$, $x \geq 0$.

7. (a) $x = \sin\theta, y = \cos\theta, 0 \leq \theta \leq \pi$.

$x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1, 0 \leq x \leq 1$.

(b)

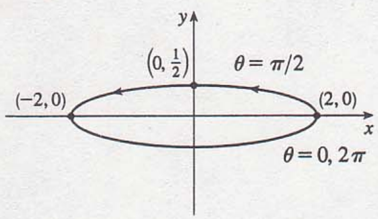


8. (a) $x = 2\cos\theta, y = \frac{1}{2}\sin\theta, 0 \leq \theta \leq 2\pi$.

$$1 = \cos^2\theta + \sin^2\theta = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{1/2}\right)^2, \text{ so}$$

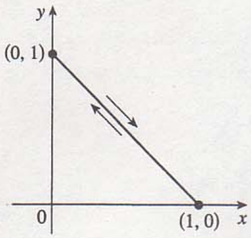
$$\frac{x^2}{2^2} + \frac{y^2}{(1/2)^2} = 1.$$

(b)



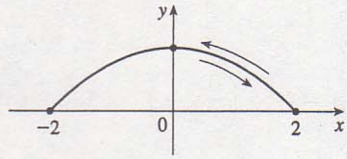
9. (a) $x = \sin^2\theta$, $y = \cos^2\theta$.
$x + y = \sin^2\theta + \cos^2\theta = 1$, $0 \le x \le 1$.
Note that the curve is at $(0, 1)$ whenever $\theta = \pi n$ and is at $(1, 0)$ whenever $\theta = \frac{\pi}{2}n$ for every integer n.

(b)



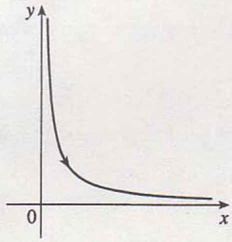
10. (a) $x = 2\cos\theta$, $y = \sin^2\theta$.
$1 = \cos^2\theta + \sin^2\theta = \left(\frac{x}{2}\right)^2 + y$, so
$y = 1 - \frac{x^2}{4}$, $-2 \le x \le 2$. The curve is at $(2, 0)$ whenever $\theta = 2\pi n$.

(b)



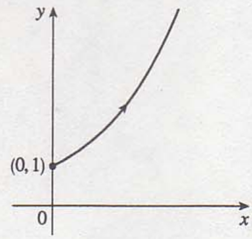
11. (a) $x = e^t$, $y = e^{-t}$, $y = 1/x$, $x > 0$

(b)



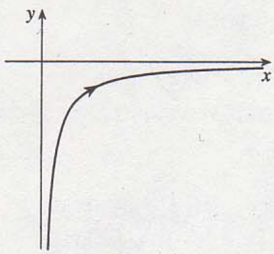
12. (a) $x = \ln t$, $y = \sqrt{t}$, $t \ge 1$. $x = \ln t \Rightarrow t = e^x \Rightarrow y = \sqrt{t} = e^{x/2}$, $x \ge 0$.

(b)



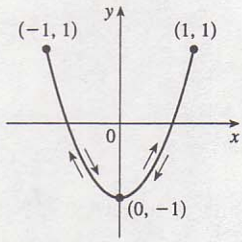
13. (a) $x = \tan\theta + \sec\theta$, $y = \tan\theta - \sec\theta$,
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. $xy = \tan^2\theta - \sec^2\theta = -1$
$\Rightarrow y = -1/x$, $x > 0$.

(b)



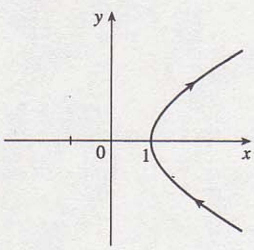
14. (a) $x = \cos t$, $y = \cos 2t$.
$y = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$, so
$y + 1 = 2x^2$, $-1 \le x \le 1$.

(b)



15. (a) $x = \cosh t$, $y = \sinh t$, $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$, $x \ge 1$

(b)



16. $x = 4 - 4t$, $y = 2t + 5$, $0 \le t \le 2$. $x = 4 - 2\,(2t) = 4 - 2\,(y - 5) = -2y + 14$, so the particle moves along the line $y = -\frac{1}{2}x + 7$ from $(4, 5)$ to $(-4, 9)$.

17. $x^2 + y^2 = \cos^2 \pi t + \sin^2 \pi t = 1$, $1 \le t \le 2$, so the particle moves counterclockwise along the circle $x^2 + y^2 = 1$ from $(-1, 0)$ to $(1, 0)$, along the lower half of the circle.

18. $(x - 2)^2 + (y - 3)^2 = \cos^2 t + \sin^2 t = 1$, so the motion takes place on a unit circle centered at $(2, 3)$. As t goes from 0 to 2π, the particle makes one complete counterclockwise rotation around the circle, starting and ending at $(3, 3)$.

19. $\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{3}y\right)^2 = \sin^2 t + \cos^2 t = 1$, so the particle moves once clockwise along the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$, starting and ending at $(0, 3)$.

20. $x = \cos^2 t = y^2$, so the particle moves along the parabola $x = y^2$. As t goes from 0 to 4π, the particle moves from $(1, 1)$ down to $(1, -1)$ (at $t = \pi$), back up to $(1, 1)$ again (at $t = 2\pi$), and then repeats this entire cycle between $t = 2\pi$ and $t = 4\pi$.

21. $x = \tan t$, $y = \cot t$, $\frac{\pi}{6} \le t \le \frac{\pi}{3}$. $y = 1/x$ for $\frac{1}{\sqrt{3}} \le x \le \sqrt{3}$. The particle moves along the first quadrant branch of the hyperbola $y = 1/x$ from $\left(\frac{1}{\sqrt{3}}, \sqrt{3}\right)$ to $\left(\sqrt{3}, \frac{1}{\sqrt{3}}\right)$.

22. (a) Note that as $t \to -\infty$, we have $x \to -\infty$ and $y \to \infty$, whereas when $t \to \infty$, both x and $y \to \infty$. This description fits only IV. [But also note that $x\,(t)$ increases, then decreases, then increases again.]

(b) Note that as $t \to \pm\infty$, $y \to -\infty$. This is only the case with VI.

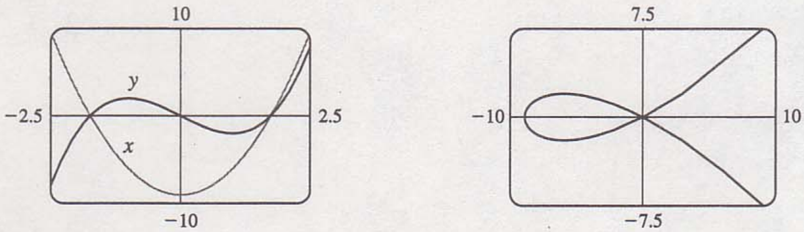
(c) If $t = 0$, then $(x, y) = (\sin 0, \sin 0) = (0, 0)$. Also, $|x| = |\sin 3t| \le 1$ for all t, and $|y| = |\sin 4t| \le 1$ for all t. The only graph which includes the point $(0, 0)$ and which has $|x| \le 1$ and $|y| \le 1$, is V.

(d) Note that as $t \to -\infty$, both x and $y \to -\infty$, and as $t \to \infty$, both x and $y \to \infty$. This description fits only III. (Also note that, since $\sin 2t$ and $\sin 3t$ lie between -1 and 1, the curve never strays very far from the line $y = x$.)

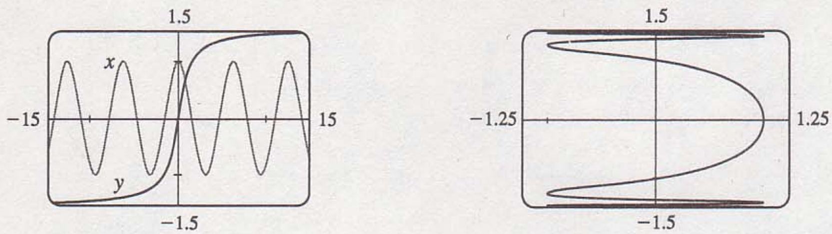
(e) Note that both $x\,(t)$ and $y\,(t)$ are periodic with period 2π and satisfy $|x| \le 1$ and $|y| \le 1$. Now the only y-intercepts occur when $x = \sin\,(t + \sin t) = 0 \quad \Leftrightarrow \quad t = 0$ or π. So there should be two y-intercepts: $y\,(0) = \cos 1 \approx 0.54$ and $y\,(\pi) = \cos\,(\pi - 1) \approx -0.54$. Similarly, there should be two x-intercepts: $x\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 1\right) \approx 0.54$ and $x\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2} - 1\right) \approx -0.54$. The only curve with these x- and y-intercepts is I.

(f) Note that $x\,(t)$ is periodic with period 2π, so the only y-intercepts occur when $x = \cos t = 0 \quad \Leftrightarrow \quad t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Also, the graph is symmetric about the x-axis, since $y\,(-t) = \sin\,(-t + \sin 5\,(-t)) = \sin\,(-t - \sin 5t) = -\sin\,(t + \sin 5t) = -y\,(t)$, and $x\,(-t) = \cos\,(-t) = \cos t = x\,(t)$. The only graph which has only two y-intercepts, and is symmetric about the x-axis, is II.

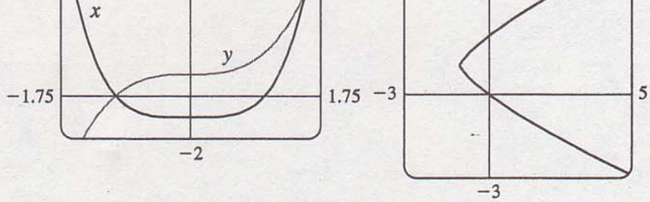
23. From the graphs, it seems that as $t \to -\infty$, $x \to \infty$ and $y \to -\infty$. So the point $(x(t), y(t))$ will move from far out in the fourth quadrant as t increases. At $t = -\sqrt{3}$, both x and y are 0, so the graph passes through the origin. After that the graph passes through the second quadrant (x is negative, y is positive), then intersects the x-axis at $x = -9$ when $t = 0$. After this, the graph passes through the third quadrant, going through the origin again at $t = \sqrt{3}$, and then as $t \to \infty$, $x \to \infty$ and $y \to \infty$. Note that for every point $(x(t), y(t)) = \left(3\left(t^2 - 3\right), t^3 - 3t\right)$, we can substitute $-t$ to get the corresponding point $(x(-t), y(-t)) = \left(3\left[(-t)^2 - 3\right], (-t)^3 - 3(-t)\right) = (x(t), -y(t))$, and so the graph is symmetric about the x-axis. The first figure was obtained using $x_1 = t$, $y_1 = 3\left(t^2 - 3\right)$; $x_2 = t$, $y_2 = t^3 - 3t$; and $-2\pi \le t \le 2\pi$.



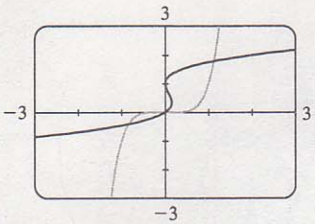
24. As $t \to -\infty$, $y \to -\frac{\pi}{2}$ and x oscillates between 1 and -1. Then, as t increases through 0, y increases while x continues to oscillate, and the graph passes through the origin. Then, as $t \to \infty$, $y \to \frac{\pi}{2}$ as x oscillates.



25. As $t \to -\infty$, $x \to \infty$ and $y \to -\infty$. The graph passes through the origin at $t = -1$, and then goes through the second quadrant (x negative, y positive), passing through the point $(-1, 1)$ at $t = 0$. As t increases, the graph passes through the point $(0, 2)$ at $t = 1$, and then as $t \to \infty$, both x and y approach ∞. The first figure was obtained using $x_1 = t$, $y_1 = t^4 - 1$; $x_2 = t$, $y_2 = t^3 + 1$; and $-2\pi \le t \le 2\pi$.

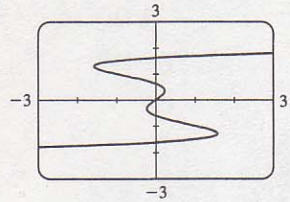


26. We use $x_1 = t$, $y_1 = t^5$ and $x_2 = t\,(t-1)^2$, $y_2 = t$ with $-2\pi \le t \le 2\pi$.



There are 3 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant III is approximately $(-0.8, -0.4)$ and the point in quadrant I is approximately $(1.1, 1.8)$.

27. As in Example 4, we let $y = t$ and $x = t - 3t^3 + t^5$ and use a t-interval of $[-2\pi, 2\pi]$.



28. (a) Clearly the curve passes through (x_1, y_1) when $t = 0$ and through (x_2, y_2) when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}\,(x - x_1)$, which is the equation of the line through (x_1, y_1) and (x_2, y_2).

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between (x_1, y_1) and (x_2, y_2) yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from (x_1, y_1) to (x_2, y_2).

(b) $x = -2 + [3 - (-2)]\,t = -2 + 5t$ and $y = 7 + (-1 - 7)\,t = 7 - 8t$ for $0 \le t \le 1$.

29. The circle $x^2 + y^2 = 4$ can be represented parametrically by $x = 2\cos t$, $y = 2\sin t$; $0 \le t \le 2\pi$. The circle $x^2 + (y-1)^2 = 4$ can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$; $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

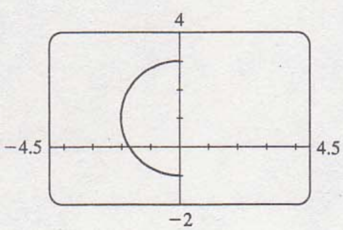
(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$; $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

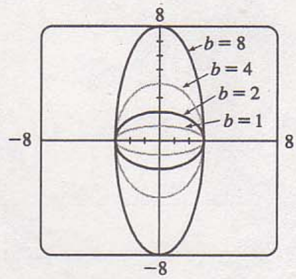
Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

30.



31. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.



(c) As b increases, the ellipse is stretched vertically.

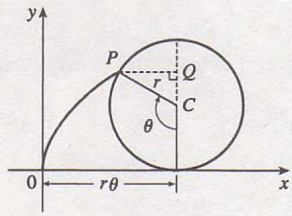
32. The possible parametrizations of the curve $y = x^3$ include

(1) $x = t, y = t^3, t \in \mathbb{R}$

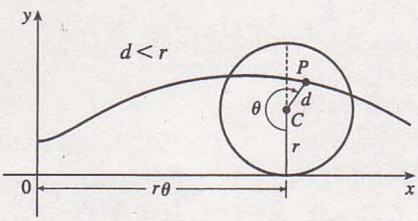
(2) $x = -t, y = -t^3, t \in \mathbb{R}$

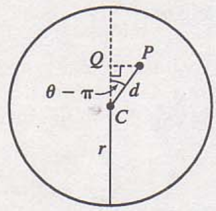
(3) $x = t + 1, y = (t+1)^3, t \in \mathbb{R}$

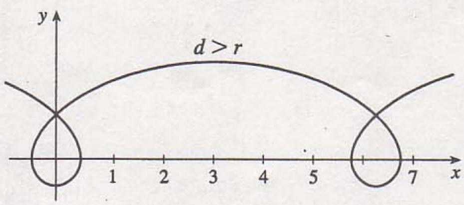
33. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as before, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



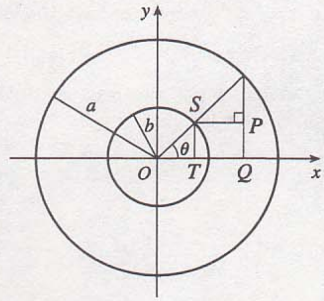
34. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Exercise 33, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.





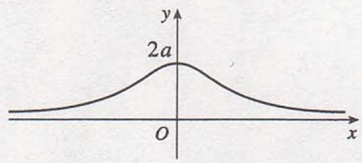


35. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \;\Rightarrow\; \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a$ $\Rightarrow\; \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



36. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\,\theta)$. Thus, the parametric equations are $x = a\sec\theta$, $y = b\sin\theta$.

37. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = \left(2a\sin\theta\cos\theta, 2a\sin^2\theta\right)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.

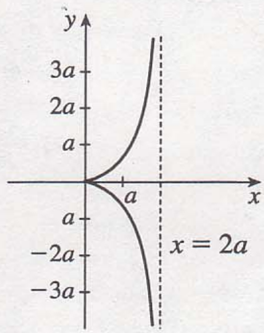


38. Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$. Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$. Now

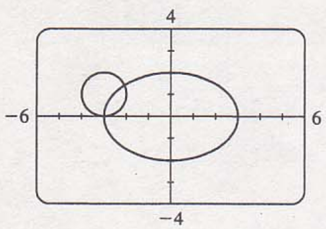
$$|OP| = |AB| = |OB| - |OA| = 2a\left(\frac{1}{\cos\theta} - \cos\theta\right)$$

$$= 2a\frac{1-\cos^2\theta}{\cos\theta} = 2a\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\tan\theta$$

So P has coordinates $x = 2a\sin\theta\tan\theta\cdot\cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\tan\theta\cdot\sin\theta = 2a\sin^2\theta\tan\theta$.



39. (a)



There are 2 points of intersection: $(-3, 0)$ and approximately $(-2.1, 1.4)$.

(b) As an aid in finding collision points, set your graphing utility to graph both curves simultaneously and closely observe the drawing of the graphs. In this case, we have one collision point: both particles are at $(-3, 0)$ when $t = \frac{3\pi}{2}$. [Notice that the first curve passes through $(-2.1, 1.4)$ when $t \approx 5.5$, but the second curve passes through $(-2.1, 1.4)$ when $t \approx 0.4$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points.

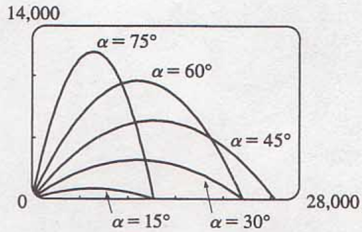
40. (a) If $\alpha = 30°$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30°)\,t = 250\sqrt{3}t$ and $y = (500\sin 30°)\,t - \frac{1}{2}(9.8)\,t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \frac{250}{4.9}t\right) = -4.9\left[t^2 - \frac{250}{4.9}t + \left(\frac{125}{4.9}\right)^2\right] + \frac{125^2}{4.9} = -4.9\left(t - \frac{125}{4.9}\right)^2 + \frac{125^2}{4.9} \le \frac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)

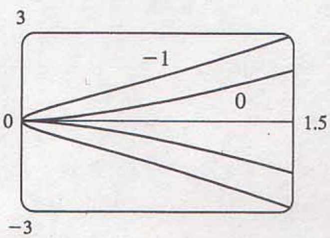


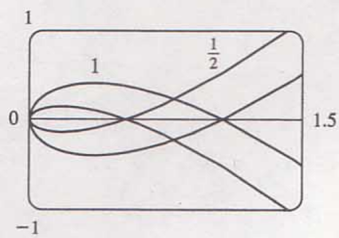
As α $(0° < \alpha < 90°)$ increases up to 45°, the projectile attains a greater height and a greater range. As α increases past 45°, the projectile attains a greater height, but its range decreases.

(c) $x = (v_0\cos\alpha)\,t \;\Rightarrow\; t = \dfrac{x}{v_0\cos\alpha}$.

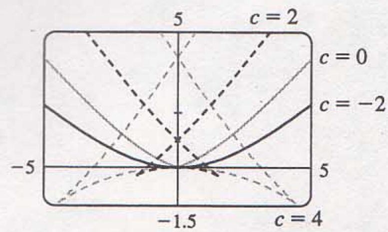
$$y = (v_0\sin\alpha)\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)\,x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2,$$ which is the equation of a parabola (quadratic in x).

41. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.

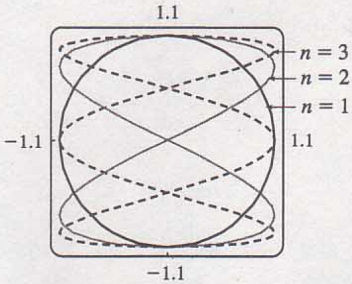




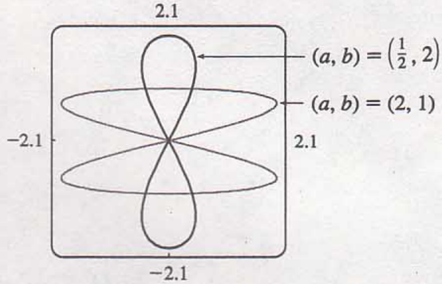
42. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.



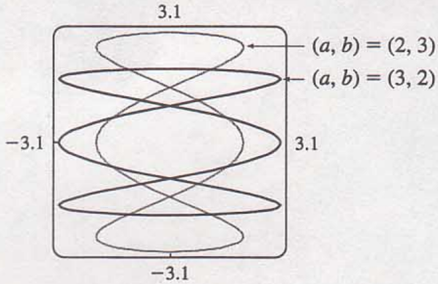
43. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x-axis. In general, the figures have $n - 1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.



$a = b = 1$

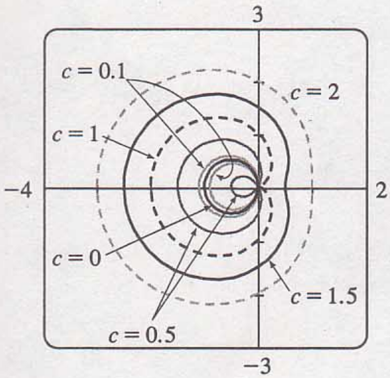


$n = 2$



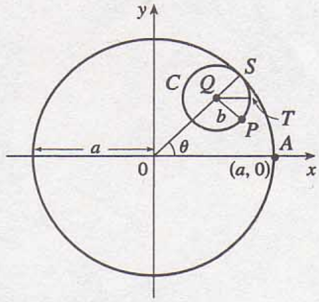
$n = 3$

44. We use $-\pi \le t \le \pi$ in the viewing rectangle $[-4, 2] \times [-3, 3]$. We first observe that for $c = 0$, we obtain a circle with center $\left(-\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. As the value of c increases, there is a larger outer loop and a smaller inner loop until $c = 1$, when we obtain a curve with a dent (called a **cardioid**). As c increases, we get curve with a dimple (called a **limaçon**) until $c = 2$. For $c > 2$, we have convex limaçons. For negative values of c, we obtain the same graphs as for positive c, but with different values of t corresponding to the points on the curve.



Laboratory Project □ Families of Hypocycloids

1. The center Q of the smaller circle has coordinates $((a-b)\cos\theta, (a-b)\sin\theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.) Thus, $\angle PQS = \frac{a}{b}\theta$ and $\angle PQT = \frac{a}{b}\theta - \theta$, so P has coordinates

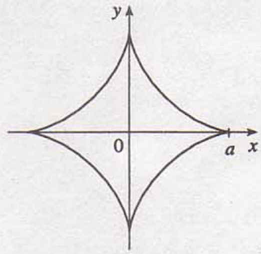


$$x = (a-b)\cos\theta + b\cos(\angle PQT) = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

and

$$y = (a-b)\sin\theta - b\sin(\angle PQT) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

2. With $b=1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a=4$. Let $a=4$ and $b=1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain

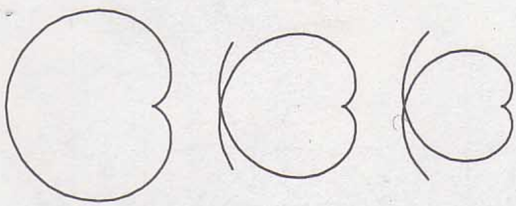


$$x = 3\cos\theta + \cos 3\theta = 3\cos\theta + \left(4\cos^3\theta - 3\cos\theta\right) = 4\cos^3\theta$$

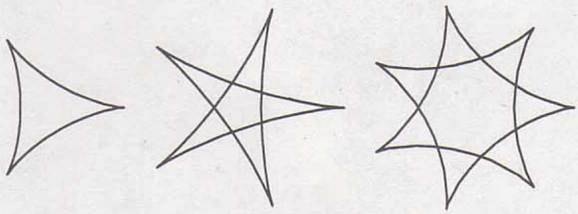
and

$$y = 3\sin\theta - \sin 3\theta = 3\sin\theta - \left(3\sin\theta - 4\sin^3\theta\right) = 4\sin^3\theta$$

3. The following graphs are obtained with $b=1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{10}$ with $-2\pi \le \theta \le 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.

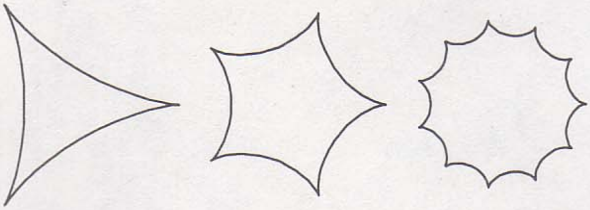


Letting $d=2$ and $n = 3, 5$, and 7 with $-2\pi \le \theta \le 2\pi$ gives us the following:



[continued]

So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form). When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}$, $\frac{5}{4}$, and $\frac{11}{10}$.

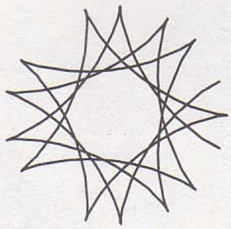


If $b = 1$, the equations for the hypocycloid are

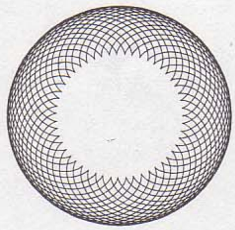
$$x = (a-1)\cos\theta + \cos((a-1)\theta),\ y = (a-1)\sin\theta - \sin((a-1)\theta)$$

which is a hypocycloid of a cusps (from Problem 2). If $n = d + 1$, then $a = (d+1)/d$, and the equations become $x = \dfrac{1}{d}\cos\theta + \cos\dfrac{\theta}{d}$, $y = \dfrac{1}{d}\sin\theta - \sin\dfrac{\theta}{d}$. Now letting $\varphi = -\theta/d$ and multiplying by d (from the hint) gives us $X = \cos(-d\varphi) + d\cos(-\varphi)$, $Y = \sin(-d\varphi) - d\sin(-\varphi)$ or, equivalently, $X = d\cos\varphi + \cos d\varphi$, $Y = d\sin\varphi - \sin d\varphi$. We recognize these equations as those of a hypocycloid with $(d+1)$ cusps.

4. In general, if $a > 1$, we get a figure with cusps on the "outside ring" and if $a < 1$, the cusps are on the "inside ring". In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ, every point on the washer would eventually be arbitrarily close to a point on the curve.

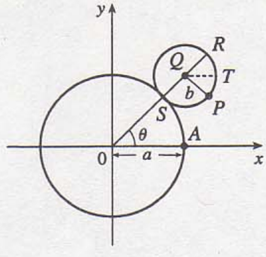


$a = \sqrt{2}$, $-10\pi \le \theta \le 10\pi$



$a = e - 2$, $0 \le \theta \le 446$

5. The center Q of the smaller circle has coordinates $((a+b)\cos\theta, (a+b)\sin\theta)$. Arc PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \dfrac{a\theta}{b}$, $\angle PQR = \pi - \dfrac{a\theta}{b}$, and $\angle PQT = \pi - \dfrac{a\theta}{b} - \theta = \pi - \left(\dfrac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.



Thus, the coordinates of P are

$$x = (a+b)\cos\theta + b\cos\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

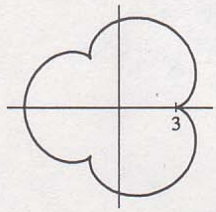
and

$$y = (a+b)\sin\theta - b\sin\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

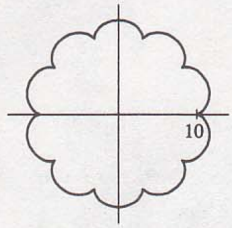
6. Let $b = 1$ and the equations become

$$x = (a+1)\cos\theta - \cos((a+1)\theta) \qquad y = (a+1)\sin\theta - \sin((a+1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an "a-leafed clover", with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

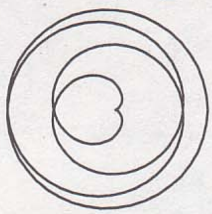


$a = 3,\ -2\pi \le \theta \le 2\pi$

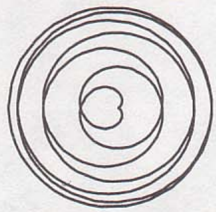


$a = 10,\ -2\pi \le \theta \le 2\pi$

If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \le \theta \le d\pi$ to be a closed curve traced exactly once.



$a = \frac{1}{4},\ -4\pi \le \theta \le 4\pi$



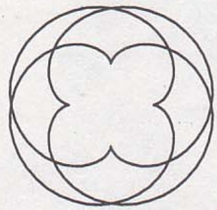
$a = \frac{1}{7},\ -7\pi \le \theta \le 7\pi$

Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n-pointed star in the middle.

$a = \frac{2}{5},\ -5\pi \le \theta \le 5\pi$

$a = \frac{7}{5},\ -5\pi \le \theta \le 5\pi$

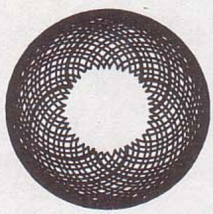
Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.



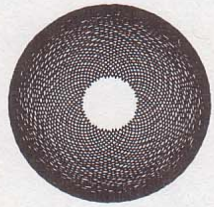
$a = \frac{4}{3},\ -3\pi \le \theta \le 3\pi$

$a = \frac{7}{6},\ -6\pi \le \theta \le 6\pi$

If a is irrational, we get washers that increase in size as a increases.



$a = \sqrt{2},\ 0 \le \theta \le 200$



$a = e - 2,\ 0 \le \theta \le 446$

7. The equations for the epicycloid are

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right) \qquad y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$$

For the first part of the problem, we set $a = n$ and $b = 1$, so these equations become

$$x = (n+1)\cos\theta - \cos\left((n+1)\,\theta\right) \qquad y = (n+1)\sin\theta - \sin\left((n+1)\,\theta\right)$$

If we substitute $\varphi = (n+1)\,\theta$ and rearrange the terms, these become

$$x = -\cos\varphi + (n+1)\cos\frac{\varphi}{n+1} \qquad y = -\sin\varphi + (n+1)\sin\frac{\varphi}{n+1}$$

so

$$X = \frac{1}{n+1}x = -\frac{\cos\varphi}{n+1} + \cos\frac{\varphi}{n+1} \qquad Y = \frac{1}{n+1}y = -\frac{\sin\varphi}{n+1} + \sin\frac{\varphi}{n+1}$$

whereas the equations of a hypocycloid with $a = \dfrac{n}{n+1}$ and $b = 1$ are

$$x = \left(\frac{n}{n+1} - 1\right)\cos\theta + \cos\left(\left(\frac{n}{n+1} - 1\right)\theta\right) = -\frac{\cos\theta}{n+1} + \cos\frac{\theta}{n+1}$$

$$y = \left(\frac{n}{n+1} - 1\right)\sin\theta - \sin\left(\left(\frac{n}{n+1} - 1\right)\theta\right) = -\frac{\sin\theta}{n+1} + \sin\frac{\theta}{n+1}$$

For the second part of the problem, we set $a = \dfrac{1}{n}$ in the equations for the epicycloid:

$$x = \frac{n+1}{n}\cos\theta - \cos\left(\frac{n+1}{n}\theta\right) \qquad y = \frac{n+1}{n}\sin\theta - \sin\left(\frac{n+1}{n}\theta\right)$$

Multiplying by $\dfrac{n}{n+1}$, substituting $\varphi = \dfrac{n+1}{n}\theta$ and rearranging the terms, we get

$$X = -\frac{n}{n+1}\cos\varphi + \cos\left(\frac{n}{n+1}\varphi\right) \qquad Y = -\frac{n}{n+1}\sin\varphi + \sin\left(\frac{n}{n+1}\varphi\right)$$

But these are exactly the equations of a hypocycloid $\{(X,Y)\}$ with $a = \dfrac{1}{n+1}$.

11.2 Tangents and Areas ET 10.2

1. $x = t - t^3,\ y = 2 - 5t \;\Rightarrow\; \dfrac{dy}{dt} = -5,\ \dfrac{dx}{dt} = 1 - 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-5}{1-3t^2}$ or $\dfrac{5}{3t^2-1}$.

2. $x = \sqrt{t} - t,\ y = t^3 - t \;\Rightarrow\; \dfrac{dy}{dt} = 3t^2 - 1,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}} - 1$, and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-1}{1/\left(2\sqrt{t}\right) - 1} = \frac{\left(3t^2-1\right)\left(2\sqrt{t}\right)}{1 - 2\sqrt{t}}$$

3. $x = t\ln t,\ y = \sin^2 t \;\Rightarrow\; \dfrac{dy}{dt} = 2\sin t\cos t,\ \dfrac{dx}{dt} = t\left(\dfrac{1}{t}\right) + (\ln t)\cdot 1 = 1 + \ln t$, and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin t\cos t}{1 + \ln t}$$

4. $x = te^t,\ y = t + e^t \quad \Rightarrow \quad \dfrac{dy}{dt} = 1 + e^t,\ \dfrac{dx}{dt} = te^t + e^t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + e^t}{te^t + e^t}$.

5. $x = t^2 + t,\ y = t^2 - t;\ t = 0.$ $\dfrac{dy}{dt} = 2t - 1,\ \dfrac{dx}{dt} = 2t + 1$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t - 1}{2t + 1}$. When $t = 0$, $x = y = 0$ and $\dfrac{dy}{dx} = -1$. An equation of the tangent is $y - 0 = (-1)(x - 0)$ or $y = -x$.

6. $x = 2t^2 + 1,\ y = \frac{1}{3}t^3 - t;\ t = 3.$ $\dfrac{dy}{dt} = t^2 - 1,\ \dfrac{dx}{dt} = 4t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t^2 - 1}{4t}$. When $t = 3$, $(x, y) = (19, 6)$ and $dy/dx = \frac{8}{12} = \frac{2}{3}$, so an equation of the tangent line is $y - 6 = \frac{2}{3}(x - 19)$ or $y = \frac{2}{3}x - \frac{20}{3}$.

7. $x = e^{\sqrt{t}},\ y = t - \ln t^2;\ t = 1.$ $\dfrac{dy}{dt} = 1 - \dfrac{2t}{t^2} = 1 - \dfrac{2}{t},\ \dfrac{dx}{dt} = \dfrac{e^{\sqrt{t}}}{2\sqrt{t}}$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 2/t}{e^{\sqrt{t}}/\left(2\sqrt{t}\right)} \cdot \dfrac{2t}{2t} = \dfrac{2t - 4}{\sqrt{t}e^{\sqrt{t}}}$. When $t = 1$, $(x, y) = (e, 1)$ and $\dfrac{dy}{dx} = -\dfrac{2}{e}$, so an equation of the tangent line is $y - 1 = -\frac{2}{e}(x - e)$ or $y = -\frac{2}{e}x + 3$.

8. $x = t\sin t,\ y = t\cos t;\ t = \pi.$ $\dfrac{dy}{dt} = \cos t - t\sin t,\ \dfrac{dx}{dt} = \sin t + t\cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t - t\sin t}{\sin t + t\cos t}$.

When $t = \pi$, $(x, y) = (0, -\pi)$ and $\dfrac{dy}{dx} = \dfrac{-1}{-\pi} = \dfrac{1}{\pi}$, so an equation of the tangent is $y + \pi = \frac{1}{\pi}(x - 0)$ or $y = \frac{1}{\pi}x - \pi$.

9. (a) $x = e^t,\ y = (t - 1)^2;\ (1, 1).$ $\dfrac{dy}{dt} = 2(t - 1),\ \dfrac{dx}{dt} = e^t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2(t - 1)}{e^t}$.

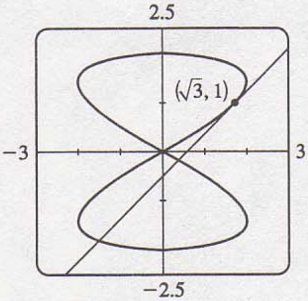
At $(1, 1)$, $t = 0$ and $\dfrac{dy}{dx} = -2$, so an equation of the tangent is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.

(b) $x = e^t \quad \Rightarrow \quad t = \ln x$, so $y = (t - 1)^2 = (\ln x - 1)^2$ and $\dfrac{dy}{dx} = 2(\ln x - 1)\left(\dfrac{1}{x}\right)$. When $x = 1$, $\dfrac{dy}{dx} = 2(-1)(1) = -2$, so an equation of the tangent is $y = -2x + 3$, as in part (a).

10. (a) $x = 5\cos t,\ y = 5\sin t;\ (3, 4).$ $\dfrac{dy}{dt} = 5\cos t,\ \dfrac{dx}{dt} = -5\sin t,\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = -\cot t$. At $(3, 4)$, $t = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\frac{4}{3}$, so $\dfrac{dy}{dx} = -\dfrac{3}{4}$, and an equation of the tangent is $y - 4 = -\frac{3}{4}(x - 3)$, or $y = -\frac{3}{4}x + \frac{25}{4}$.

(b) $x^2 + y^2 = 25$, so $2x + 2y\dfrac{dy}{dx} = 0$, or $\dfrac{dy}{dx} = -\dfrac{x}{y}$. At $(3, 4)$, $\dfrac{dy}{dx} = -\dfrac{3}{4}$, and as in part (a), an equation of the tangent is $y = -\frac{3}{4}x + \frac{25}{4}$.

11. $x = 2\sin 2t,\ y = 2\sin t;\ \left(\sqrt{3}, 1\right)$.

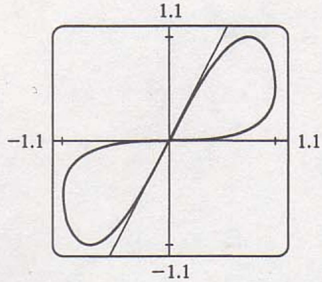


$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos t}{2 \cdot 2\cos 2t} = \dfrac{\cos t}{2\cos 2t}$. The point $\left(\sqrt{3}, 1\right)$ corresponds to $t = \frac{\pi}{6}$, so the slope of the tangent at that point is

$\dfrac{\cos\frac{\pi}{6}}{2\cos\frac{\pi}{3}} = \dfrac{\frac{\sqrt{3}}{2}}{2 \cdot \frac{1}{2}} = \dfrac{\sqrt{3}}{2}$. An equation of the tangent is therefore

$(y - 1) = \frac{\sqrt{3}}{2}\left(x - \sqrt{3}\right)$ or $y = \frac{\sqrt{3}}{2}x - \frac{1}{2}$.

12. $x = \sin t$, $y = \sin(t + \sin t)$; $(0, 0)$.



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t + \sin t)(1 + \cos t)}{\cos t} = (\sec t + 1)\cos(t + \sin t)$$

Note that there are two tangents at the point $(0, 0)$, since both $t = 0$ and $t = \pi$ correspond to the origin. The tangent corresponding to $t = 0$ has slope $(\sec 0 + 1)\cos(0 + \sin 0) = 2\cos 0 = 2$, and its equation is $y = 2x$. The tangent corresponding to $t = \pi$ has slope $(\sec\pi + 1)\cos(\pi + \sin\pi) = 0$, so it is the x-axis.

13. $x = t^4 - 1$, $y = t - t^2 \Rightarrow \dfrac{dy}{dt} = 1 - 2t$, $\dfrac{dx}{dt} = 4t^3$, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 2t}{4t^3} = \frac{1}{4}t^{-3} - \frac{1}{2}t^{-2}$;

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\tfrac{3}{4}t^{-4} + t^{-3},\ \frac{d^2y}{dx^2} = \frac{d\,(dy/dx)\,/dt}{dx/dt} = \frac{-\frac{3}{4}t^{-4} + t^{-3}}{4t^3} \cdot \frac{4t^4}{4t^4} = \frac{-3 + 4t}{16t^7}.$$

14. $x = t^3 + t^2 + 1$, $y = 1 - t^2$. $\dfrac{dy}{dt} = -2t$, $\dfrac{dx}{dt} = 3t^2 + 2t$; $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2t}{3t^2 + 2t} = -\dfrac{2}{3t + 2}$;

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{6}{(3t + 2)^2};\ \frac{d^2y}{dx^2} = \frac{d\,(dy/dx)\,/dt}{dx/dt} = \frac{6}{t\,(3t + 2)^3}.$$

15. $x = \sin\pi t$, $y = \cos\pi t$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-\pi\sin\pi t}{\pi\cos\pi t} = -\tan\pi t$;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,(dy/dx)\,/dt}{dx/dt} = \frac{-\pi\sec^2\pi t}{\pi\cos\pi t} = -\sec^3\pi t.$$

16. $x = 1 + \tan t$, $y = \cos 2t \Rightarrow \dfrac{dy}{dt} = -2\sin 2t$, $\dfrac{dx}{dt} = \sec^2 t$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\sec^2 t} = -4\sin t\cos t \cdot \cos^2 t = -4\sin t\cos^3 t;$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -4\sin t\,(3\cos^2 t)\,(-\sin t) - 4\cos^4 t = 12\sin^2 t\cos^2 t - 4\cos^4 t,$$

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)\,/dt}{dx/dt} = \frac{4\cos^2 t\,(3\sin^2 t - \cos^2 t)}{\sec^2 t} = 4\cos^4 t\,(3\sin^2 t - \cos^2 t).$$

17. $x = e^{-t}$, $y = te^{2t}$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{(2t+1)\,e^{2t}}{-e^{-t}} = -\,(2t+1)\,e^{3t}$;

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = -3\,(2t+1)\,e^{3t} - 2e^{3t} = -\,(6t+5)\,e^{3t}$;

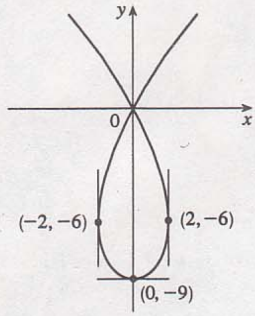
$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d\,(dy/dx)\,/dt}{dx/dt} = \dfrac{-\,(6t+5)\,e^{3t}}{-e^{-t}} = (6t+5)\,e^{4t}$.

18. $x = 1 + t^2$, $y = t\ln t$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\ln t}{2t}$; $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{2t\,(1/t) - (1+\ln t)\,2}{(2t)^2} = -\dfrac{\ln t}{2t^2}$;

$\dfrac{d^2y}{dx^2} = \dfrac{d\,(dy/dx)\,/dt}{dx/dt} = -\dfrac{\ln t}{4t^3}$.

19. $x = t\,(t^2-3) = t^3 - 3t$, $y = 3\,(t^2-3)$. $\dfrac{dx}{dt} = 3t^2 - 3 = 3\,(t-1)\,(t+1)$; $\dfrac{dy}{dt} = 6t$. $\dfrac{dy}{dt} = 0 \;\Leftrightarrow\; t = 0$ $\Leftrightarrow\; (x,y) = (0,-9)$. $\dfrac{dx}{dt} = 0 \;\Leftrightarrow\; t = \pm 1 \;\Leftrightarrow\; (x,y) = (-2,-6)$ or $(2,-6)$. So there is a horizontal tangent at $(0,-9)$ and there are vertical tangents at $(-2,-6)$ and $(2,-6)$.

	$t<-1$	$-1<t<0$	$0<t<1$	$t>1$
dx/dt	+	−	−	+
dy/dt	−	−	+	+
x	→	←	←	→
y	↓	↓	↑	↑
curve	↘	↙	↖	↗

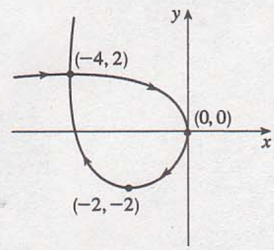


20. $x = t^3 - 3t^2$, $y = t^3 - 3t$. $\dfrac{dx}{dt} = 3t^2 - 6t = 3t\,(t-2)$,

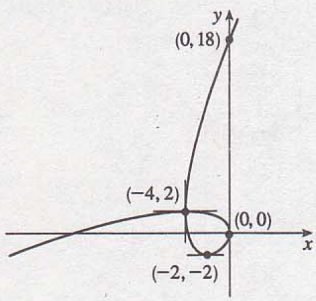
$\dfrac{dy}{dt} = 3t^2 - 3 = 3\,(t-1)\,(t+1)$. $\dfrac{dy}{dt} = 0 \;\Leftrightarrow\; t = +1$ or -1 $\Leftrightarrow$

$(x,y) = (-2,-2)$ or $(-4,2)$. $\dfrac{dx}{dt} = 0 \;\Leftrightarrow\; t = 0$ or 2 $\Leftrightarrow$

$(x,y) = (0,0)$ or $(-4,2)$. So the tangent is horizontal at $(-2,-2)$ and vertical at $(0,0)$. At $(-4,2)$ the curve crosses itself and there are two tangents, one horizontal and one vertical.



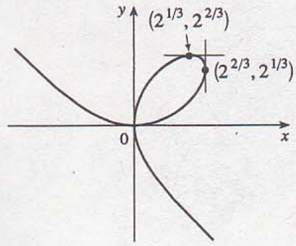
	$t<-1$	$-1<t<0$	$0<t<1$	$1<t<2$	$t>2$
dx/dt	+	+	−	−	+
dy/dt	+	−	−	+	+
x	→	→	←	←	→
y	↑	↓	↓	↑	↑
curve	↗	↘	↙	↖	↗



21. $x = \dfrac{3t}{1+t^3}$, $y = \dfrac{3t^2}{1+t^3}$. $\dfrac{dx}{dt} = \dfrac{(1+t^3)\,3 - 3t\,(3t^2)}{(1+t^3)^2} = \dfrac{3-6t^3}{(1+t^3)^2}$,

$\dfrac{dy}{dt} = \dfrac{(1+t^3)\,(6t) - 3t^2\,(3t^2)}{(1+t^3)^2} = \dfrac{6t-3t^4}{(1+t^3)^2} = \dfrac{3t\,(2-t^3)}{(1+t^3)^2}$. $\dfrac{dy}{dt} = 0 \;\Leftrightarrow\; t = 0$ or $\sqrt[3]{2} \;\Leftrightarrow$

$(x,y) = (0,0)$ or $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. $\dfrac{dx}{dt} = 0 \;\Leftrightarrow\; t^3 = \frac{1}{2} \;\Leftrightarrow\; t = 2^{-1/3}$



$\Leftrightarrow\; (x,y) = \left(\sqrt[3]{4}, \sqrt[3]{2}\right)$. There are horizontal tangents at $(0,0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$, and there are vertical tangents at $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$ and $(0,0)$. [The vertical tangent at $(0,0)$ is undetectable by the methods of this section because that tangent corresponds to the limiting position of the point (x,y) as $t \to \pm\infty$.] In the following table, $\alpha = \sqrt[3]{2}$.

	$t < -1$	$-1 < t < 0$	$0 < t < 1/\alpha$	$1/\alpha < t < \alpha$	$t > \alpha$
dx/dt	$+$	$+$	$+$	$-$	$-$
dy/dt	$-$	$-$	$+$	$+$	$-$
x	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\leftarrow$	$\leftarrow$
y	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$
curve	$\searrow$	$\searrow$	$\nearrow$	$\nwarrow$	$\swarrow$

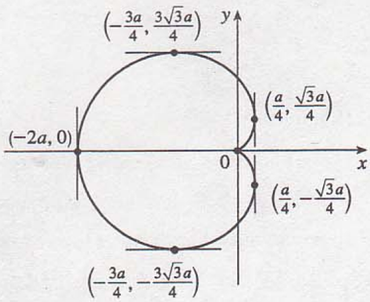
22. $x = a\left(\cos\theta - \cos^2\theta\right)$, $y = a\,(\sin\theta - \sin\theta\cos\theta)$. $\dfrac{dx}{d\theta} = a\,(-\sin\theta + 2\cos\theta\sin\theta)$,

$\dfrac{dy}{d\theta} = a\left(\cos\theta + \sin^2\theta - \cos^2\theta\right) = a\left(\cos\theta + 1 - 2\cos^2\theta\right)$, $\dfrac{dy}{d\theta} = 0 \;\Leftrightarrow$

$0 = 2\cos^2\theta - \cos\theta - 1 = (2\cos\theta + 1)(\cos\theta - 1) \;\Leftrightarrow\; \cos\theta = -\frac{1}{2}$ or $1 \;\Leftrightarrow\; (x,y) = \left(-\frac{3}{4}a, \pm\frac{3\sqrt{3}}{4}a\right)$ or

$(0,0)$. $\dfrac{dx}{d\theta} = 0 \;\Leftrightarrow\; (2\cos\theta - 1)\sin\theta = 0 \;\Leftrightarrow\; \cos\theta = \frac{1}{2}$ or $\sin\theta = 0 \;\Leftrightarrow\; (x,y) = (0,0)$ or

$\left(\frac{1}{4}a, \pm\frac{\sqrt{3}}{4}a\right)$ or $(-2a, 0)$. The curve has horizontal tangents at $\left(-\frac{3}{4}a, \pm\frac{3\sqrt{3}}{4}a\right)$ and vertical tangents at $(-2a,0)$ and $\left(\frac{1}{4}a, \pm\frac{\sqrt{3}}{4}a\right)$.

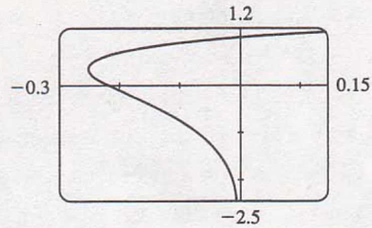


Since $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(2\cos\theta+1)(1-\cos\theta)}{(2\cos\theta-1)\sin\theta}$, we see that

$\displaystyle\lim_{\theta\to 0}\frac{dy}{dx} = \lim_{\theta\to 0}\frac{2\cos\theta+1}{2\cos\theta-1}\cdot\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = 3\cdot 0 = 0$ (using l'Hospital's Rule). Thus, the curve has a horizontal tangent at $(0,0)$, where both $dx/d\theta$ and $dy/d\theta$ are 0.

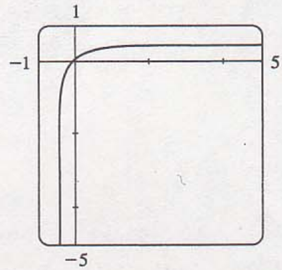
	$0 < t < \frac{\pi}{3}$	$\frac{\pi}{3} < t < \frac{2\pi}{3}$	$\frac{2\pi}{3} < t < \pi$	$\pi < t < \frac{4\pi}{3}$	$\frac{4\pi}{3} < t < \frac{5\pi}{3}$	$\frac{5\pi}{3} < t < 2\pi$
dx/dt	$+$	$-$	$-$	$+$	$+$	$-$
dy/dt	$+$	$+$	$-$	$-$	$+$	$+$
x	$\rightarrow$	$\leftarrow$	$\leftarrow$	$\rightarrow$	$\rightarrow$	$\leftarrow$
y	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$
curve	$\nearrow$	$\nwarrow$	$\swarrow$	$\searrow$	$\nearrow$	$\nwarrow$

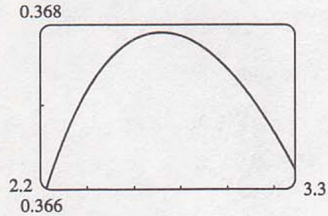
23. From the graph, it appears that the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$ is about $(-0.25, 0.36)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 4t^3 - 2t \quad \Leftrightarrow \quad 2t\left(2t^2 - 1\right) = 0 \quad \Leftrightarrow$ $2t\left(\sqrt{2}t + 1\right)\left(\sqrt{2}t - 1\right) = 0 \quad \Leftrightarrow \quad t = 0$ or $\pm\frac{1}{\sqrt{2}}$. The negative and 0 roots are inadmissible since $y(t)$ is only defined for $t > 0$, so the leftmost point must be



$$\left(x\left(\tfrac{1}{\sqrt{2}}\right), y\left(\tfrac{1}{\sqrt{2}}\right)\right) = \left(\left(\tfrac{1}{\sqrt{2}}\right)^4 - \left(\tfrac{1}{\sqrt{2}}\right)^2, \tfrac{1}{\sqrt{2}} + \ln\tfrac{1}{\sqrt{2}}\right) = \left(-\tfrac{1}{4}, \tfrac{1}{\sqrt{2}} - \tfrac{1}{2}\ln 2\right)$$

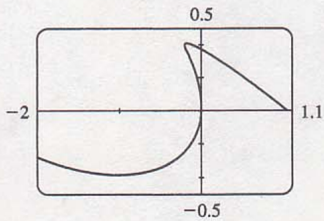
24. The curve is symmetric about the line $y = -x$, so if we can find the highest point (x_h, y_h), then the leftmost point is $(x_l, y_l) = (-y_h, -x_h)$. After carefully zooming in, we estimate that the highest point on the curve $x = te^t$, $y = te^{-t}$ is about $(2.7, 0.37)$.

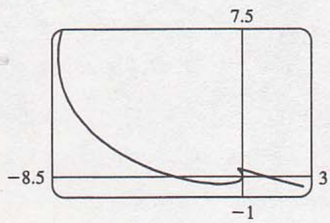




To find the exact coordinates of the highest point, we find the value of t for which the curve has a horizontal tangent, that is, $dy/dt = 0 \quad \Leftrightarrow \quad t\left(-e^{-t}\right) + e^{-t} = 0 \quad \Leftrightarrow \quad (1 - t)\, e^{-t} = 0 \quad \Leftrightarrow \quad t = 1$. This corresponds to the point $(x(1), y(1)) = (e, 1/e)$. To find the leftmost point, we find the value of t for which $0 = dx/dt = te^t + e^t \quad \Leftrightarrow$ $(1 + t)\, e^t = 0 \quad \Leftrightarrow \quad t = -1$. This corresponds to the point $(x(-1), y(-1)) = (-1/e, -e)$. As $t \to -\infty$, $x(t) = te^t \to 0^-$ by l'Hospital's Rule and $y(t) = te^{-t} \to -\infty$, so the y-axis is an asymptote. As $t \to \infty$, $x(t) \to \infty$ and $y(t) \to 0^+$, so the x-axis is the other asymptote. The asymptotes can also be determined from the graph, if we use a larger t-interval.

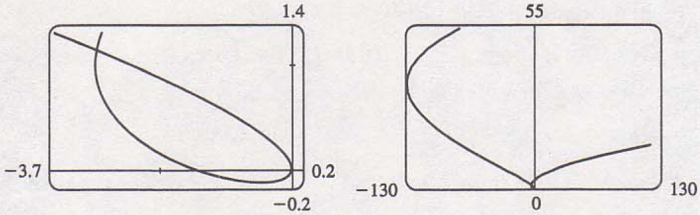
25. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$. We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate





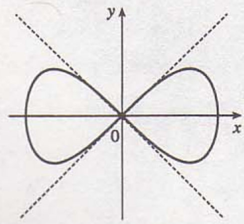
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \quad \Leftrightarrow \quad t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt = 2t\left(2t^2 - 3t - 2\right) = 0$ $\Leftrightarrow \quad 2t(2t + 1)(t - 2) = 0 \quad \Leftrightarrow \quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

26. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.

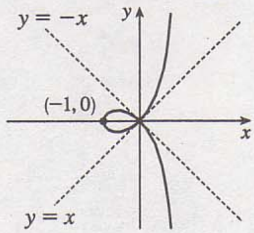


We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t-1}{4t^3+12t^2-16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \Leftrightarrow t = \frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \Leftrightarrow 4t\left(t^2 + 3t - 4\right) = 0 \Leftrightarrow 4t\,(t+4)\,(t-1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.

27. $x = \cos t$, $y = \sin t \cos t$. $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \Leftrightarrow \cos t = 0 \Leftrightarrow$ t is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = -1$, so $\frac{dy}{dx} = 1$. When $t = \frac{3\pi}{2}$, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -1$. So $\frac{dy}{dx} = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



28. $x = 1 - 2\cos^2 t = -\cos 2t$, $y = (\tan t)\left(1 - 2\cos^2 t\right) = -(\tan t)\cos 2t$. To find a point where the curve crosses itself, we look for two values of t that give the same point (x, y). Call these values t_1 and t_2. Then $\cos^2 t_1 = \cos^2 t_2$ (from the equation for x) and either $\tan t_1 = \tan t_2$ or $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$ (from the equation for y). We can satisfy $\cos^2 t_1 = \cos^2 t_2$ and $\tan t_1 = \tan t_2$ by choosing t_1 arbitrarily and taking $t_2 = t_1 + \pi$, so evidently the whole curve is retraced every time t traverses an interval of length π. Thus, we can restrict our attention to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $t_2 = -t_1$, then $\cos^2 t_2 = \cos^2 t_1$, but $\tan t_2 = -\tan t_1$. This suggests that we try to satisfy the condition $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$. Taking $t_1 = \frac{\pi}{4}$ and $t_2 = -\frac{\pi}{4}$ gives $(x, y) = (0, 0)$ for both values of t. $\frac{dx}{dt} = 2\sin 2t$, and $\frac{dy}{dt} = 2\sin 2t \tan t - \cos 2t \sec^2 t$. When $t = \frac{\pi}{4}$, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = 1$. When $t = -\frac{\pi}{4}$, $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = -1$. Thus, the equations of the two tangents at $(0, 0)$ are $y = x$ and $y = -x$.



29. (a) $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$; $\dfrac{dx}{d\theta} = r - d\cos\theta$, $\dfrac{dy}{d\theta} = d\sin\theta$. So $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

30. $x = a\cos^3\theta$, $y = a\sin^3\theta$.

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow dy/dx = 0 \Leftrightarrow \tan\theta = 0 \Leftrightarrow \theta = n\pi \Leftrightarrow (x, y) = (\pm a, 0)$. The tangent is vertical $\Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{2} \Leftrightarrow (x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1 \Leftrightarrow \tan\theta = \pm 1 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{4} \Leftrightarrow (x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$ (All sign choices are valid.)

31. The line with parametric equations $x = -7t$, $y = 12t - 5$ is $y = 12\left(-\frac{1}{7}x\right) - 5$, which has slope $-\frac{12}{7}$. The curve $x = t^3 + 4t$, $y = 6t^2$ has slope $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12t}{3t^2+4}$. This equals $-\frac{12}{7} \Leftrightarrow 3t^2 + 4 = -7t \Leftrightarrow$ $(3t+4)(t+1) = 0 \Leftrightarrow t = -1$ or $t = -\frac{4}{3} \Leftrightarrow (x,y) = (-5,6)$ or $\left(-\frac{208}{27}, \frac{32}{3}\right)$.

32. $x = 3t^2 + 1$, $y = 2t^3 + 1$, $\dfrac{dx}{dt} = 6t$, $\dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t$ (even where $t = 0$).
So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3+1) = t\left[x - (3t^2+1)\right]$. If this line is to pass through $(4,3)$, we must have $3 - (2t^3+1) = t\left[4 - (3t^2+1)\right] \Leftrightarrow 2t^3 - 2 = 3t^3 - 3t \Leftrightarrow t^3 - 3t + 2 = 0 \Leftrightarrow (t-1)^2(t+2) = 0$ $\Leftrightarrow t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4,3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13,-15)$.

33. $A = \int_0^1 (y-1)\,dx = \int_{\pi/2}^0 (e^t - 1)(-\sin t)\,dt = \int_0^{\pi/2} (e^t \sin t - \sin t)\,dt \overset{98}{=} \left[\frac{1}{2}e^t(\sin t - \cos t) + \cos t\right]_0^{\pi/2}$
$= \frac{1}{2}\left(e^{\pi/2} - 1\right)$

34. $t + 1/t = 2.5 \Leftrightarrow t = \frac{1}{2}$ or 2, and for $\frac{1}{2} < t < 2$, we have $t + 1/t < 2.5$. $x = -\frac{3}{2}$ when $t = \frac{1}{2}$ and $x = \frac{3}{2}$ when $t = 2$.

$$A = \int_{-3/2}^{3/2} (2.5 - y)\,dx = \int_{1/2}^{2} \left(\tfrac{5}{2} - t - 1/t\right)\left(1 + 1/t^2\right)dt \quad \left[x = t - 1/t,\ dx = \left(1 + 1/t^2\right)dt\right]$$
$$= \int_{1/2}^{2} \left(-t + \tfrac{5}{2} - 2t^{-1} + \tfrac{5}{2}t^{-2} - t^{-3}\right)dt = \left[\frac{-t^2}{2} + \frac{5t}{2} - 2\ln|t| - \frac{5}{2t} + \frac{1}{2t^2}\right]_{1/2}^{2}$$
$$= \left(-2 + 5 - 2\ln 2 - \tfrac{5}{4} + \tfrac{1}{8}\right) - \left(-\tfrac{1}{8} + \tfrac{5}{4} + 2\ln 2 - 5 + 2\right) = \tfrac{15}{4} - 4\ln 2$$

35. By symmetry of the ellipse about the x- and y-axes,

$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^{0} b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$
$$= 2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

36. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^{0} a\sin^3\theta\left(-3a\cos^2\theta\sin\theta\right)d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$. Now

$$\int \sin^4\theta\cos^2\theta\,d\theta = \int \sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int (1 - \cos 2\theta)\sin^2 2\theta\,d\theta$$
$$= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1 - \cos 4\theta) - \sin^2 2\theta\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C$$

so $\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

37. $A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi}\left(r^2 - 2dr\cos\theta + d^2\cos^2\theta\right)d\theta$
$= \left[r^2\theta - 2dr\sin\theta + \frac{1}{2}d^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2$

38. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x-axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt{3} \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\text{area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}} (t^3 - 3t)\,2t\,dt = 2\int_0^{-\sqrt{3}} (2t^4 - 6t^2)\,dt = 2\left[\tfrac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}}$$
$$= 2\left[\tfrac{2}{5}\left(-3^{1/2}\right)^5 - 2\left(-3^{1/2}\right)^3\right] = 2\left[\tfrac{2}{5}\left(-9\sqrt{3}\right) - 2\left(-3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

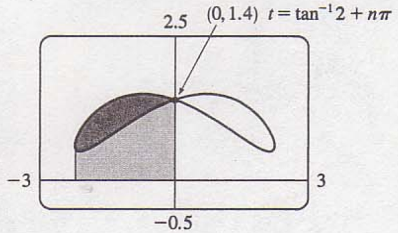
$$\text{volume} = \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt{3}} (t^3 - 3t)^2\,2t\,dt = 2\pi\int_0^{-\sqrt{3}} (t^6 - 6t^4 + 9t^2)\,t\,dt$$
$$= 2\pi\left[\tfrac{1}{8}t^8 - t^6 + \tfrac{9}{4}t^4\right]_0^{-\sqrt{3}} = 2\pi\left[\tfrac{1}{8}\left(-3^{1/2}\right)^8 - \left(-3^{1/2}\right)^6 + \tfrac{9}{4}\left(-3^{1/2}\right)^4\right]$$
$$= 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coordinate of the centroid of the top half of $\mathcal{R}$, the area of which is $\frac{1}{2} \cdot \frac{24}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$. So, using Formula 9.3.8 [ET 8.3.8] with $A = \frac{12}{5}\sqrt{3}$, we get

$$\overline{x} = \tfrac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \tfrac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2\,(t^3 - 3t)\,2t\,dt = \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}t^7 - \tfrac{3}{5}t^5\right]_0^{-\sqrt{3}}$$
$$= \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}\left(-3^{1/2}\right)^7 - \tfrac{3}{5}\left(-3^{1/2}\right)^5\right] = \tfrac{5}{6\sqrt{3}}\left[-\tfrac{27}{7}\sqrt{3} + \tfrac{27}{5}\sqrt{3}\right] = \tfrac{9}{7}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\frac{9}{7}, 0\right)$.

39. The graph of $x = \sin t - 2\cos t$, $y = 1 + \sin t\cos t$ is symmetric about the y-axis. The graph intersects the y-axis when $x = 0 \;\Rightarrow\; \sin t - 2\cos t = 0 \;\Rightarrow\; \sin t = 2\cos t \;\Rightarrow\; \tan t = 2 \;\Rightarrow\; t = \tan^{-1} 2 + n\pi$. The left loop is traced in a clockwise direction from $t = \tan^{-1} 2 - \pi$ to $t = \tan^{-1} 2$, so the area of the loop is given (as in Example 4) by



$$A = \int_{\tan^{-1} 2 - \pi}^{\tan^{-1} 2} y\,dx \approx \int_{-2.0344}^{1.1071} (1 + \sin t\cos t)(\cos t + 2\sin t)\,dt \approx 0.8944$$

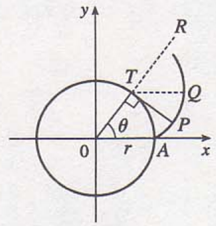
This integral can be evaluated exactly; its value is $\frac{2}{5}\sqrt{5}$.

40. If f' is continuous and $f'(t) \ne 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g\left(f^{-1}(x)\right)$. Then $x = f(t) \;\Rightarrow\; f^{-1}(x) = t$, so $y = g(t) = g\left(f^{-1}(x)\right) = F(x)$.

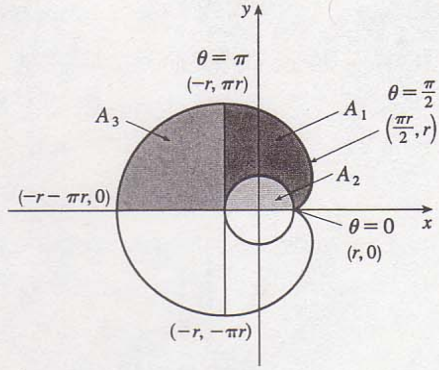
41. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates

$x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$,

$y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



42. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 41 corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. (This corresponds to the range $-\pi \le \theta \le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.] The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$. Now $y\,dx = r(\sin\theta - \theta\cos\theta)\,r\theta\cos\theta\,d\theta = r^2\left(\theta\sin\theta\cos\theta - \theta^2\cos^2\theta\right)d\theta$. Integrate: $(1/r^2)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$\begin{aligned} A_1 + A_2 &= r^2\left[-\theta\cos^2\theta - \tfrac{1}{2}\left(\theta^2 - 1\right)\sin\theta\cos\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{2}\theta\right]_\pi^0 = r^2\left[0 - \left(-\pi - \tfrac{\pi^3}{6} + \tfrac{\pi}{2}\right)\right] \\ &= r^2\left(\tfrac{\pi}{2} + \tfrac{\pi^3}{6}\right) \end{aligned}$$

Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$.

Laboratory Project □ Bézier Curves

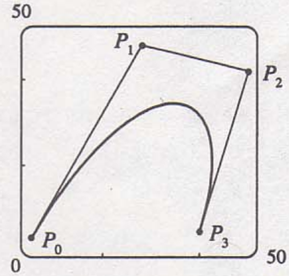
1. We are given the points $P_0\,(x_0, y_0) = (4, 1)$, $P_1\,(x_1, y_1) = (28, 48)$, $P_2\,(x_2, y_2) = (50, 42)$, and $P_3\,(x_3, y_3) = (40, 5)$. The curve is then given by

$$x\,(t) = 4\,(1-t)^3 + 3 \cdot 28t\,(1-t)^2 + 3 \cdot 50t^2\,(1-t) + 40t^3$$

$$y\,(t) = 1\,(1-t)^3 + 3 \cdot 48t\,(1-t)^2 + 3 \cdot 42t^2\,(1-t) + 5t^3$$

where $0 \le t \le 1$. The line segments are of the form $x = x_0 + (x_1 - x_0)\,t$, $y = y_0 + (y_1 - y_0)\,t$:

P_0P_1	$x = 4 + 24t,$	$y = 1 + 47t$
P_1P_2	$x = 28 + 22t,$	$y = 48 - 6t$
P_2P_3	$x = 50 - 10t,$	$y = 42 - 37t$



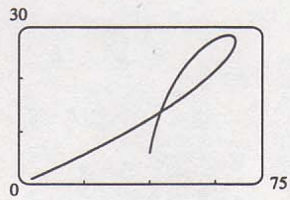
2. It suffices to show that the slope of the tangent at P_0 is the same as that of line segment P_0P_1, namely $\dfrac{y_1 - y_0}{x_1 - x_0}$.

We calculate the slope of the tangent to the Bézier curve:

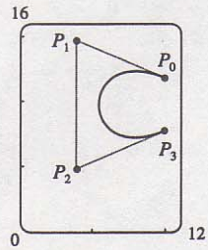
$$\frac{dy/dt}{dx/dt} = \frac{-3y_0\,(1-t)^2 + 3y_1\left[-2t\,(1-t) + (1-t)^2\right] + 3y_2\left[-t^2 + (2t)\,(1-t)\right] + 3y_3t^2}{-3x_0\,(1-t)^2 + 3x_1\left[-2t\,(1-t) + (1-t)^2\right] + 3x_2\left[-t^2 + (2t)\,(1-t)\right] + 3x_3t^2}$$

At point P_0, $t = 0$, so the slope of the tangent is $\dfrac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \dfrac{y_1 - y_0}{x_1 - x_0}$. So the tangent to the curve at P_0 passes through P_1. Similarly, the slope of the tangent at point P_3 (where $t = 1$) is $\dfrac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \dfrac{y_3 - y_2}{x_3 - x_2}$, which is also the slope of line P_2P_3.

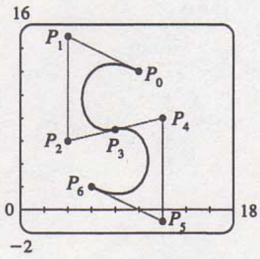
3. It seems that if P_1 were to the right of P_2, a loop would appear. We try setting $P_1 = (110, 30)$, and the resulting curve does indeed have a loop.



4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try $P_0\,(10, 12)$, $P_1\,(4, 15)$, $P_2\,(4, 5)$, and $P_3\,(10, 8)$, and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the x- and y-scales the same so as not to distort the figure (this is called a "constrained projection" in Maple.)



5. We use the same P_0 and P_1 as in part (a), and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4, 6)$ and P_3 down and to the left, to $(8, 7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4, P_5, and P_6 by rotating points P_2, P_1, and P_0 about the center of the letter (point P_3). The points are therefore $P_4\,(12, 8)$, $P_5\,(12, -1)$, and $P_6\,(6, 2)$.



11.3 Arc Length and Surface Area **ET 10.3**

1. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$. $dx/dt = 1 - 2t$ and $dy/dt = 2t^{1/2}$, so

$(dx/dt)^2 + (dy/dt)^2 = (1-2t)^2 + \left(2t^{1/2}\right)^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2$ and

$L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_1^2 \sqrt{1 + 4t^2}\, dt.$

2. $x = 1 + e^t$, $y = t^2$, $-3 \le t \le 3$. $dx/dt = e^t$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = e^{2t} + 4t^2$ and $L = \int_{-3}^3 \sqrt{e^{2t} + 4t^2}\, dt.$

3. $x = t\sin t$, $y = t\cos t$, $0 \le t \le \frac{\pi}{2}$. $dx/dt = t\cos t + \sin t$ and $dy/dt = t\,(-\sin t) + \cos t$, so

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (t\cos t + \sin t)^2 + (\cos t - t\sin t)^2 \\ &= t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t \\ &= t^2\left(\cos^2 t + \sin^2 t\right) + \sin^2 t + \cos^2 t = t^2 + 1\end{aligned}$$

and $L = \int_0^{\pi/2} \sqrt{t^2 + 1}\, dt.$

4. $x = \ln t$, $y = \sqrt{t+1}$, $1 \le t \le 5$. $\dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{t+1}}$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \dfrac{1}{t^2} + \dfrac{1}{4\,(t+1)} = \dfrac{t^2+4t+4}{4t^2\,(t+1)}$ and $L = \displaystyle\int_1^5 \sqrt{\frac{t^2+4t+4}{4t^2\,(t+1)}}\, dt = \int_1^5 \frac{t+2}{2t\sqrt{t+1}}\, dt.$

5. $x = t^3$, $y = t^2$, $0 \le t \le 4$. $(dx/dt)^2 + (dy/dt)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2$.

$L = \int_0^4 \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^4 \sqrt{9t^4 + 4t^2}\, dt = \int_0^4 t\sqrt{9t^2+4}\, dt = \frac{1}{18}\int_4^{148} \sqrt{u}\, du$

(where $u = 9t^2 + 4$). So $L = \frac{1}{18}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_4^{148} = \frac{1}{27}\left(148^{3/2} - 4^{3/2}\right) = \frac{8}{27}\left(37^{3/2} - 1\right)$.

6. $x = a\,(\cos\theta + \theta\sin\theta)$, $y = a\,(\sin\theta - \theta\cos\theta)$, $0 \le \theta \le \pi$.

$$\begin{aligned}(dx/d\theta)^2 + (dy/d\theta)^2 &= a^2\left[(-\sin\theta + \sin\theta + \theta\cos\theta)^2 + (\cos\theta - \cos\theta + \theta\sin\theta)^2\right] \\ &= a^2\theta^2\left(\cos^2\theta + \sin^2\theta\right) = (a\theta)^2\end{aligned}$$

$L = \int_0^\pi a\theta\, d\theta = \frac{1}{2}\pi^2 a$

7. $x = \dfrac{t}{1+t}$, $y = \ln(1+t)$, $0 \le t \le 2$. $\dfrac{dx}{dt} = \dfrac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \dfrac{1}{(1+t)^2}$ and $\dfrac{dy}{dt} = \dfrac{1}{1+t}$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \dfrac{1}{(1+t)^4} + \dfrac{1}{(1+t)^2} = \dfrac{1}{(1+t)^4}\left[1 + (1+t)^2\right] = \dfrac{t^2+2t+2}{(1+t)^4}$ and

$$L = \int_0^2 \frac{\sqrt{t^2+2t+2}}{(1+t)^2}\, dt = \int_1^3 \frac{\sqrt{u^2+1}}{u^2}\, du \;\; [u = t+1,\; du = dt] \;\stackrel{24}{=}\; \left[-\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right)\right]_1^3$$

$= -\frac{\sqrt{10}}{3} + \ln\left(3 + \sqrt{10}\right) + \sqrt{2} - \ln\left(1 + \sqrt{2}\right)$

8. $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 3$. $dx/dt = e^t - e^{-t}$ and $dy/dt = -2$, so

$(dx/dt)^2 + (dy/dt)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = \left(e^t + e^{-t}\right)^2$ and

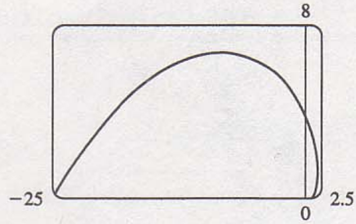
$L = \int_0^3 \left(e^t + e^{-t}\right) dt = \left[e^t - e^{-t}\right]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}.$

9. $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left[e^t(\cos t - \sin t)\right]^2 + \left[e^t(\sin t + \cos t)\right]^2$$

$$= e^{2t}\left(2\cos^2 t + 2\sin^2 t\right) = 2e^{2t}$$

$$\Rightarrow \quad L = \int_0^\pi \sqrt{2}e^t\,dt = \sqrt{2}\left(e^\pi - 1\right)$$

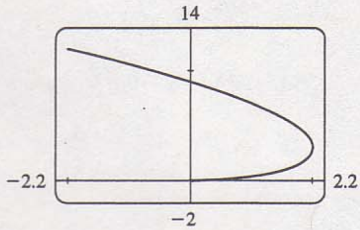


10. $x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 2$.

$$(dx/dt)^2 + (dy/dt)^2 = \left(3 - 3t^2\right)^2 + (6t)^2$$

$$= 9\left(1 + 2t^2 + t^4\right) = \left[3\left(1 + t^2\right)\right]^2$$

$$L = \int_0^2 3\left(1 + t^2\right)dt = \left[3t + t^3\right]_0^2 = 14$$



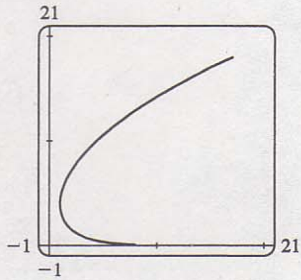
11. $x = e^t - t$, $y = 4e^{t/2}$, $-8 \le t \le 3$.

$$(dx/dt)^2 + (dy/dt)^2 = \left(e^t - 1\right)^2 + \left(2e^{t/2}\right)^2 = e^{2t} - 2e^t + 1 + 4e^t$$

$$= e^{2t} + 2e^t + 1 = \left(e^t + 1\right)^2$$

$$L = \int_{-8}^3 \sqrt{\left(e^t + 1\right)^2}\,dt = \int_{-8}^3 \left(e^t + 1\right)dt = \left[e^t + t\right]_{-8}^3$$

$$= \left(e^3 + 3\right) - \left(e^{-8} - 8\right) = e^3 - e^{-8} + 11$$



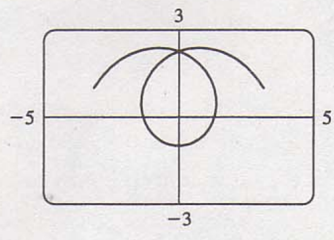
12. $x = t\cos t + \sin t$, $y = t\sin t - \cos t$, $-\pi \le t \le \pi$. $dx/dt = -t\sin t + 2\cos t$ and $dy/dt = t\cos t + 2\sin t$, so $(dx/dt)^2 + (dy/dt)^2 = t^2\sin^2 t - 4t\sin t\cos t + 4\cos^2 t + t^2\cos^2 t + 4t\sin t\cos t + 4\sin^2 t = t^2 + 4$ and

$$L = \int_{-\pi}^{\pi} \sqrt{t^2 + 4}\,dt = 2\int_0^\pi \sqrt{t^2 + 4}\,dt$$

$$\overset{21}{=} 2\left[\tfrac{1}{2}t\sqrt{t^2 + 4} + 2\ln\left(t + \sqrt{t^2 + 4}\right)\right]_0^\pi$$

$$= 2\left[\tfrac{\pi}{2}\sqrt{\pi^2 + 4} + 2\ln\left(\pi + \sqrt{\pi^2 + 4}\right) - 2\ln 2\right]$$

$$= \pi\sqrt{\pi^2 + 4} + 4\ln\left(\pi + \sqrt{\pi^2 + 4}\right) - 4\ln 2 \approx 16.633506$$



13. $x = \ln t$ and $y = e^{-t}$ $\Rightarrow$ $\dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = -e^{-t}$ $\Rightarrow$ $L = \int_1^2 \sqrt{t^{-2} + e^{-2t}}\,dt$. Using Simpson's Rule with $n = 10$, $\Delta x = (2 - 1)/10 = 0.1$ and $f(t) = \sqrt{t^{-2} + e^{-2t}}$ we get $L \approx \frac{0.1}{3}\left[f(1.0) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2.0)\right] \approx 0.7314$.

14. $x = 2a\cot\theta$ $\Rightarrow$ $dx/dt = -2a\csc^2\theta$ and $y = 2a\sin^2\theta$ $\Rightarrow$ $dy/dt = 4a\sin\theta\cos\theta = 2a\sin 2\theta$. So $L = \int_{\pi/4}^{\pi/2} \sqrt{4a^2\csc^4\theta + 4a^2\sin^2 2\theta}\,d\theta = 2a\int_{\pi/4}^{\pi/2} \sqrt{\csc^4\theta + \sin^2 2\theta}\,d\theta$. Using Simpson's Rule with $n = 4$, $\Delta x = \frac{\pi/4}{4} = \frac{\pi}{16}$ and $f(\theta) = \sqrt{\csc^4\theta + \sin^2 2\theta}$, we get $L \approx 2a \cdot S_4 = (2a)\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 2.2605a$.

15. $x = \sin^2\theta$, $y = \cos^2\theta$, $0 \le \theta \le 3\pi$.
$(dx/d\theta)^2 + (dy/d\theta)^2 = (2\sin\theta\cos\theta)^2 + (-2\cos\theta\sin\theta)^2 = 8\sin^2\theta\cos^2\theta = 2\sin^2 2\theta \quad\Rightarrow$

$$\begin{aligned}\text{Distance} &= \textstyle\int_0^{3\pi}\sqrt{2}\,|\sin 2\theta|\,d\theta = 6\sqrt{2}\int_0^{\pi/2}\sin 2\theta\,d\theta \text{ (by symmetry)} = \left[-3\sqrt{2}\cos 2\theta\right]_0^{\pi/2}\\ &= -3\sqrt{2}\,(-1-1) = 6\sqrt{2}\end{aligned}$$

The full curve is traversed as θ goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \ge 0$), and this segment is completely traversed as θ goes from 0 to $\frac{\pi}{2}$.
Thus $L = \int_0^{\pi/2}\sin 2\theta\,d\theta = \sqrt{2}$, as above.

16. $x = \cos^2 t$, $y = \cos t$, $0 \le t \le 4\pi$. $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (-2\cos t\sin t)^2 + (-\sin t)^2 = \sin^2 t\,(4\cos^2 t + 1)$

$$\begin{aligned}\text{Distance} &= \textstyle\int_0^{4\pi}|\sin t|\sqrt{4\cos^2 t+1}\,dt = 4\int_0^{\pi}\sin t\sqrt{4\cos^2 t+1}\,dt\\ &= \textstyle -4\int_1^{-1}\sqrt{4u^2+1}\,du \;\; [u=\cos t,\, du = -\sin t\,dt] = 4\int_{-1}^{1}\sqrt{4u^2+1}\,du = 8\int_0^1\sqrt{4u^2+1}\,du\\ &= \textstyle 8\int_0^{\tan^{-1}2}\sec\theta\,\frac{1}{2}\sec^2\theta\,d\theta = 4\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta \overset{71}{=} \left[2\sec\theta\tan\theta + 2\ln|\sec\theta+\tan\theta|\right]_0^{\tan^{-1}2}\\ &= 4\sqrt{5} + 2\ln\left(\sqrt{5}+2\right)\end{aligned}$$

$L = \int_0^{\pi}|\sin t|\sqrt{4\cos^2 t+1}\,dt = \sqrt{5} + \frac{1}{2}\ln\left(\sqrt{5}+2\right)$

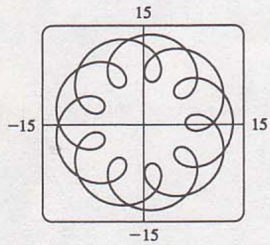
17. $x = a\sin\theta$, $y = b\cos\theta$, $0 \le \theta \le 2\pi$.

$$\begin{aligned}\left(\tfrac{dx}{d\theta}\right)^2 + \left(\tfrac{dy}{d\theta}\right)^2 &= (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2\left(1-\sin^2\theta\right) + b^2\sin^2\theta\\ &= a^2 - \left(a^2-b^2\right)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2\left(1 - e^2\sin^2\theta\right)\end{aligned}$$

So $L = 4\int_0^{\pi/2}\sqrt{a^2\left(1-e^2\sin^2\theta\right)}\,d\theta$ (by symmetry) $= 4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta$

18. $x = a\cos^3\theta$, $y = a\sin^3\theta$.
$(dx/d\theta)^2 + (dy/d\theta)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2 = 9a^2\sin^2\theta\cos^2\theta$.
$L = 4\int_0^{\pi/2}3a\sin\theta\cos\theta\,d\theta = \left[12a\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 6a$.

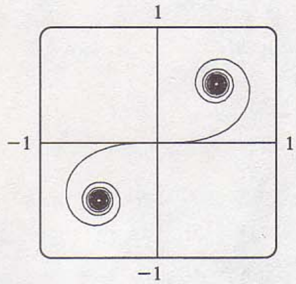
19. (a) Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \ne x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.



(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Theorem 4 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi}\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ as $88E\left(2\sqrt{2}i\right)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1\frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\,dt$ and i is the imaginary number $\sqrt{-1}$. Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03.

20. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.

(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Theorem 4, the length of the curve from the origin to the point with parameter value t is



$$L = \int_0^t \sqrt{(dx/du)^2 + (dy/du)^2}\, du = \int_0^t \sqrt{\cos^2\left(\tfrac{\pi}{2}u^2\right) + \sin^2\left(\tfrac{\pi}{2}u^2\right)}\, du = \int_0^t 1\, du = t \ \ (\text{or } -t \text{ if } t < 0)$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

21. $x = t^3$ and $y = t^4 \Rightarrow dx/dt = 3t^2$ and $dy/dt = 4t^3$. So
$S = \int_0^1 2\pi t^4 \sqrt{9t^4 + 16t^6}\, dt = \int_0^1 2\pi t^6 \sqrt{9 + 16t^2}\, dt.$

22. $x = \sin^2 t$, $y = \sin 3t$, $0 \le t \le \frac{\pi}{3}$. $dx/dt = 2\sin t\cos t = \sin 2t$ and $dy/dt = 3\cos 3t$, so
$(dx/dt)^2 + (dy/dt)^2 = \sin^2 2t + 9\cos^2 3t$ and $S = \int 2\pi y\, ds = \int_0^{\pi/3} 2\pi \sin 3t \sqrt{\sin^2 2t + 9\cos^2 3t}\, dt.$

23. $x = t^3, y = t^2, 0 \le t \le 1.$ $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (3t^2)^2 + (2t)^2 = 9t^4 + 4t^2.$

$$S = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt$$
$$= 2\pi \int_4^{13} \frac{u-4}{9} \sqrt{u}\left(\tfrac{1}{18}du\right) \ \ (\text{where } u = 9t^2 + 4) \ = \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)$$

24. $x = 3t - t^3, y = 3t^2, 0 \le t \le 1.$ $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = 9\left(1 + 2t^2 + t^4\right) = \left[3\left(1 + t^2\right)\right]^2.$
$S = \int_0^1 2\pi 3t^2 3\left(1 + t^2\right) dt = 18\pi \int_0^1 \left(t^2 + t^4\right) dt = 18\pi \left[\frac{1}{3}t^3 + \frac{1}{5}t^5\right]_0^1 = \frac{48}{5}\pi$

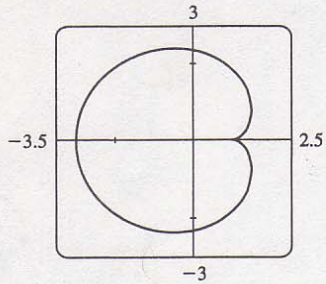
25. $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \le \theta \le \frac{\pi}{2}$.
$\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2 = 9a^2\sin^2\theta\cos^2\theta.$
$S = \int_0^{\pi/2} 2\pi a \sin^3\theta\, 3a\sin\theta\cos\theta\, d\theta = 6\pi a^2 \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta = \frac{6}{5}\pi a^2 \left[\sin^5\theta\right]_0^{\pi/2} = \frac{6}{5}\pi a^2$

26. $\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2 = (-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2$

$$= 4\left[\left(\sin^2\theta - 2\sin\theta\sin 2\theta + \sin^2 2\theta\right) + \left(\cos^2\theta - 2\cos\theta\cos 2\theta + \cos^2 2\theta\right)\right]$$

$$= 4\left[1 + 1 - 2(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)\right] = 8\left[1 - \cos(2\theta - \theta)\right] = 8(1 - \cos\theta)$$

We plot the graph with parameter interval $[0, 2\pi]$, and see that we should only integrate between 0 and π. (If the interval $[0, 2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y = 2\sin\theta - \sin 2\theta = 2\sin\theta\,(1 - \cos\theta)$. So



$S = \int_0^\pi 2\pi 2\sin\theta\,(1 - \cos\theta)\,2\sqrt{2}\sqrt{1 - \cos\theta}\,d\theta$

$= 8\sqrt{2}\pi \displaystyle\int_0^\pi (1 - \cos\theta)^{3/2}\sin\theta\,d\theta = 8\sqrt{2}\pi \int_0^2 \sqrt{u^3}\,du$

[where $u = 1 - \cos\theta$, $du = \sin\theta\,d\theta$] $= \left[8\sqrt{2}\pi\left(\frac{2}{5}\right)u^{5/2}\right]_0^2 = \frac{128}{5}\pi$

27. $x = t + t^3$, $y = t - \dfrac{1}{t^2}$, $1 \le t \le 2$. $\dfrac{dx}{dt} = 1 + 3t^2$ and $\dfrac{dy}{dt} = 1 + \dfrac{2}{t^3}$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (1 + 3t^2)^2 + \left(1 + \dfrac{2}{t^3}\right)^2$ and

$S = \displaystyle\int 2\pi y\,ds = \int_1^2 2\pi\left(t - \frac{1}{t^2}\right)\sqrt{(1 + 3t^2)^2 + \left(1 + \frac{2}{t^3}\right)^2}\,dt \approx 59.101.$

28. $S = \int_{\pi/4}^{\pi/2} 2\pi \cdot 2a\sin^2\theta\sqrt{\csc^4\theta + \sin^2 2\theta}\,dt = 4\pi a\int_{\pi/4}^{\pi/2}\sin^2\theta\sqrt{\csc^4\theta + \sin^2 2\theta}\,d\theta$. Using Simpson's Rule with $n = 4$, $\Delta x = \frac{\pi}{16}$ and $f(\theta) = \sin^2\theta\sqrt{\csc^4\theta + \sin^2 2\theta}$, we get

$S \approx (4\pi a)\,\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 11.0893a.$

29. $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \quad\Rightarrow$

$S = \int_0^5 2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^5 2\pi\,(3t^2)\,6t\sqrt{1 + t^2}\,dt = 18\pi\int_0^5 t^2\sqrt{1 + t^2}\,2t\,dt$

$= 18\pi\int_1^{26}(u - 1)\sqrt{u}\,du$ (where $u = 1 + t^2$) $= 18\pi\int_1^{26}\left(u^{3/2} - u^{1/2}\right)du = 18\pi\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^{26}$

$= 18\pi\left[\left(\frac{2}{5}\cdot 676\sqrt{26} - \frac{2}{3}\cdot 26\sqrt{26}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right] = \frac{24}{5}\pi\left(949\sqrt{26} + 1\right)$

30. $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 1$. $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (e^t - 1)^2 + \left(2e^{t/2}\right)^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2$.

$$S = \int_0^1 2\pi\,(e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt = \int_0^1 2\pi\,(e^t - t)\,(e^t + 1)\,dt$$

$$= 2\pi\left[\tfrac{1}{2}e^{2t} + e^t - (t - 1)\,e^t - \tfrac{1}{2}t^2\right]_0^1 = \pi\left(e^2 + 2e - 6\right)$$

31. $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$.

$$\begin{aligned}(dx/d\theta)^2 + (dy/d\theta)^2 &= (-a\sin\theta)^2 + (b\cos\theta)^2 = a^2\sin^2\theta + b^2\cos^2\theta = a^2\left(1-\cos^2\theta\right) + b^2\cos^2\theta \\ &= a^2 - \left(a^2-b^2\right)\cos^2\theta = a^2 - c^2\cos^2\theta = a^2\left(1 - \frac{c^2}{a^2}\cos^2\theta\right) = a^2\left(1 - e^2\cos^2\theta\right)\end{aligned}$$

(a) $S = \int_0^\pi 2\pi b\sin\theta a\sqrt{1-e^2\cos^2\theta}\,d\theta = 2\pi ab\int_{-e}^{e}\sqrt{1-u^2}\left(\frac{1}{e}\right)du$ (where $u = -e\cos\theta$, $du = e\sin\theta\,d\theta$)

$= \frac{4\pi ab}{e}\int_0^e\left(1-u^2\right)^{1/2}du = \frac{4\pi ab}{e}\int_0^{\sin^{-1}e}\cos^2 v\,dv$ (where $u = \sin v$) $= \frac{2\pi ab}{e}\int_0^{\sin^{-1}e}(1+\cos 2v)\,dv$

$= \frac{2\pi ab}{e}\left[v + \frac{1}{2}\sin 2v\right]_0^{\sin^{-1}e} = \frac{2\pi ab}{e}\left[v + \sin v\cos v\right]_0^{\sin^{-1}e} = \frac{2\pi ab}{e}\left(\sin^{-1}e + e\sqrt{1-e^2}\right)$

But $\sqrt{1-e^2} = \sqrt{1-\dfrac{c^2}{a^2}} = \sqrt{\dfrac{a^2-c^2}{a^2}} = \sqrt{\dfrac{b^2}{a^2}} = \dfrac{b}{a}$, so $S = \dfrac{2\pi ab}{e}\sin^{-1}e + 2\pi b^2$.

(b) $S = \int_{-\pi/2}^{\pi/2} 2\pi a\cos\theta a\sqrt{1-e^2\cos^2\theta}\,d\theta = 4\pi a^2\int_0^{\pi/2}\cos\theta\sqrt{(1-e^2)+e^2\sin^2\theta}\,d\theta$

$$\begin{aligned}&= \frac{4\pi a^2\left(1-e^2\right)}{e}\int_0^{\pi/2}\frac{e}{\sqrt{1-e^2}}\cos\theta\sqrt{1+\left(\frac{e\sin\theta}{\sqrt{1-e^2}}\right)^2}\,d\theta \\ &= \frac{4\pi a^2\left(1-e^2\right)}{e}\int_0^{e/\sqrt{1-e^2}}\sqrt{1+u^2}\,du\ \left(\text{where } u = \frac{e\sin\theta}{\sqrt{1-e^2}}\right) \\ &= \frac{4\pi a^2\left(1-e^2\right)}{e}\int_0^{\sin^{-1}e}\sec^3 v\,dv \quad (\text{where } u = \tan v,\ du = \sec^2 v\,dv) \\ &= \frac{2\pi a^2\left(1-e^2\right)}{e}\left[\sec v\tan v + \ln\left|\sec v + \tan v\right|\right]_0^{\sin^{-1}e} \\ &= \frac{2\pi a^2\left(1-e^2\right)}{e}\left[\frac{1}{\sqrt{1-e^2}}\frac{e}{\sqrt{1-e^2}} + \ln\left|\frac{1}{\sqrt{1-e^2}} + \frac{e}{\sqrt{1-e^2}}\right|\right] \\ &= 2\pi a^2 + \frac{2\pi a^2\left(1-e^2\right)}{e}\ln\sqrt{\frac{1+e}{1-e}} = 2\pi a^2 + \frac{2\pi b^2}{e}\frac{1}{2}\ln\left(\frac{1+e}{1-e}\right) \quad \left(\text{since } 1-e^2 = \frac{b^2}{a^2}\right) \\ &= 2\pi\left[a^2 + \frac{b^2}{2e}\ln\frac{1+e}{1-e}\right]\end{aligned}$$

32. By Formula 11.3.5 [ET 10.3.5], $S = \int_a^b 2\pi F(x)\sqrt{1+F'(x)^2}\,dx$. Now

$1 + F'(x)^2 = 1 + \left(\dfrac{dy/dt}{dx/dt}\right)^2 = \dfrac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with $x = x(t)$ $\Rightarrow$

$dx = \dfrac{dx}{dt}\,dt$, we have $S = \displaystyle\int_\alpha^\beta 2\pi y\sqrt{\frac{(dx/dt)^2+(dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

33. (a) $\phi = \tan^{-1}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d\phi}{dt} = \frac{d}{dt}\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{1}{1+(dy/dx)^2}\left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]$. But $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}$

$\Rightarrow \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \Rightarrow \frac{d\phi}{dt} = \frac{1}{1+(\dot{y}/\dot{x})^2}\left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$.

Using the Chain Rule, and the fact that $s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \Rightarrow$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$, we have that

$\frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \left(\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)\frac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$. So

$\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right| = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

(b) $x = x$ and $y = f(x) \Rightarrow \dot{x} = 1, \ddot{x} = 0$ and $\dot{y} = \frac{dy}{dx}, \ddot{y} = \frac{d^2y}{dx^2}$.

So $\kappa = \frac{|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)|}{\left[1 + (dy/dx)^2\right]^{3/2}} = \frac{|d^2y/dx^2|}{\left[1 + (dy/dx)^2\right]^{3/2}}$.

34. (a) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \frac{d^2y}{dx^2} = 2$. So $\kappa = \frac{|d^2y/dx^2|}{\left[1 + (dy/dx)^2\right]^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$, and at $(1, 1)$,

$\kappa = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}}$.

(b) $\kappa' = \frac{d\kappa}{dx} = -3\left(1 + 4x^2\right)^{-5/2}(8x) = 0 \Leftrightarrow x = 0 \Rightarrow y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and $\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

35. $x = \theta - \sin\theta \Rightarrow \dot{x} = 1 - \cos\theta \Rightarrow \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \Rightarrow \dot{y} = \sin\theta \Rightarrow \ddot{y} = \cos\theta$.

Therefore, $\kappa = \frac{|\cos\theta - \cos^2\theta - \sin^2\theta|}{\left[(1-\cos\theta)^2 + \sin^2\theta\right]^{3/2}} = \frac{|\cos\theta - (\cos^2\theta + \sin^2\theta)|}{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)^{3/2}} = \frac{|\cos\theta - 1|}{(2 - 2\cos\theta)^{3/2}}$. The top of the arch is characterized by a horizontal tangent, and from Example 1 in Section 11.2 [ET 10.2], the tangent is horizontal when $\theta = (2n-1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ:

$\kappa = \frac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \frac{|-1-1|}{[2 - 2(-1)]^{3/2}} = \frac{1}{4}$.

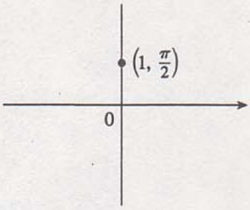
36. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and v, $w \neq 0$. For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve $x = a + (c-a)\,t$, $y = b + (d-b)\,t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$, and $\kappa = \frac{|v \cdot 0 - w \cdot 0|}{(v^2 + w^2)^{3/2}} = 0$.

(b) Parametric equations for a circle of radius r are $x = r\cos\theta$ and $y = r\sin\theta$. We can take the center to be the origin. So $\dot{x} = -r\sin\theta \Rightarrow \ddot{x} = -r\cos\theta$ and $\dot{y} = r\cos\theta \Rightarrow \ddot{y} = -r\sin\theta$. Therefore,

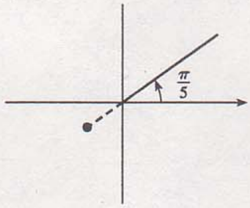
$\kappa = \frac{|r^2\sin^2\theta + r^2\cos^2\theta|}{\left(r^2\sin^2\theta + r^2\cos^2\theta\right)^{3/2}} = \frac{r^2}{r^3} = \frac{1}{r}$. And so for any θ (and thus any point), $\kappa = \frac{1}{r}$.

11.4 Polar Coordinates ET 10.4

1. (a) By adding 2π to $\frac{\pi}{2}$, we obtain the point $\left(1, \frac{5\pi}{2}\right)$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $\left(-1, \frac{3\pi}{2}\right)$ is a point that satisfies the $r < 0$ requirement.

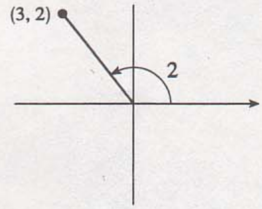


(b) $\left(-2, \frac{\pi}{4}\right)$



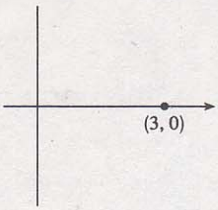
$\left(2, \frac{5\pi}{4}\right), \left(-2, \frac{9\pi}{4}\right)$

(c) $(3, 2)$



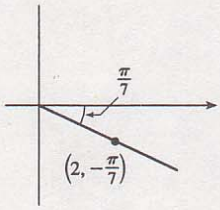
$(3, 2 + 2\pi), (-3, 2 + \pi)$

2. (a) $(3, 0)$



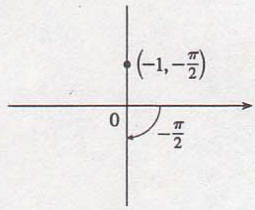
$(3, 2\pi), (-3, \pi)$

(b) $\left(2, -\frac{\pi}{7}\right)$



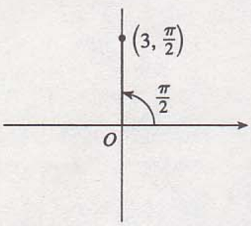
$\left(2, \frac{13\pi}{7}\right), \left(-2, \frac{6\pi}{7}\right)$

(c) $\left(-1, -\frac{\pi}{2}\right)$



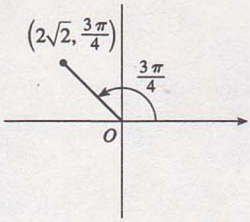
$\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$

3. (a)



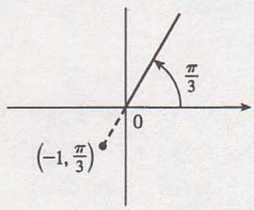
$x = 3\cos\frac{\pi}{2} = 3\,(0) = 0$ and $y = 3\sin\frac{\pi}{2} = 3\,(1) = 3$ give us $(0, 3)$.

(b)



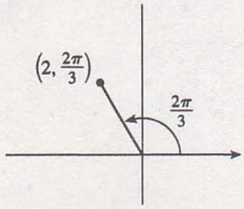
$x = 2\sqrt{2}\cos\frac{3\pi}{4}$
$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -2$ and
$y = 2\sqrt{2}\sin\frac{3\pi}{4} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$ give us $(-2, 2)$.

(c)



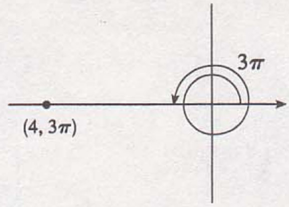
$x = -1\cos\frac{\pi}{3} = -\frac{1}{2}$ and $y = -1\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ give us $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

4. (a)



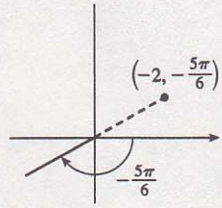
$x = 2\cos\frac{2\pi}{3} = -1,$
$y = 2\sin\frac{2\pi}{3} = \sqrt{3}$

(b)



$x = 4\cos 3\pi = -4,$
$y = 4\sin 3\pi = 0$

(c)



$x = -2\cos\left(-\frac{5\pi}{6}\right) = \sqrt{3},$
$y = -2\sin\left(-\frac{5\pi}{6}\right) = 1$

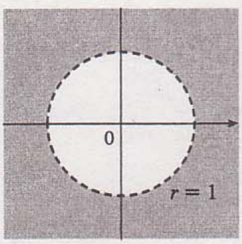
5. (a) $x = 1$ and $y = 1 \;\Rightarrow\; r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$. Since $(1, 1)$ is in the first quadrant, the polar coordinates are (i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ and (ii) $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$.

(b) $x = 2\sqrt{3}$ and $y = -2 \;\Rightarrow\; r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = 4$ and $\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6}$. Since $\left(2\sqrt{3}, -2\right)$ is in the fourth quadrant and $0 \le \theta \le 2\pi$, the polar coordinates are (i) $\left(4, \frac{11\pi}{6}\right)$ and (ii) $\left(-4, \frac{5\pi}{6}\right)$.

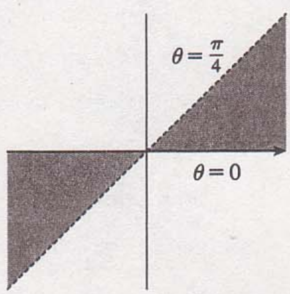
6. (a) $(x, y) = \left(-1, -\sqrt{3}\right)$, $r = \sqrt{1 + 3} = 2$, $\tan\theta = y/x = \sqrt{3}$ and (x, y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$. The polar coordinates are (i) $\left(2, \frac{4\pi}{3}\right)$ and (ii) $\left(-2, \frac{\pi}{3}\right)$.

(b) $(x, y) = (-2, 3)$, $r = \sqrt{4 + 9} = \sqrt{13}$, $\tan\theta = y/x = -\frac{3}{2}$ and (x, y) is in the second quadrant, so $\theta = \tan^{-1}\left(-\frac{3}{2}\right) + \pi$. The polar coordinates are (i) $\left(\sqrt{13}, \theta\right)$ and (ii) $\left(-\sqrt{13}, \theta + \pi\right)$.

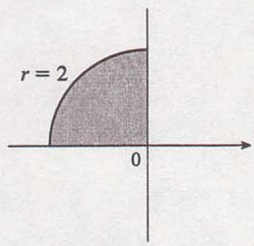
7. $r > 1$



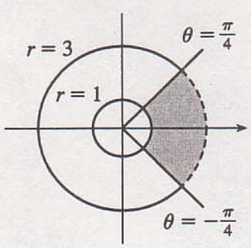
8. $0 \le \theta < \frac{\pi}{4}$



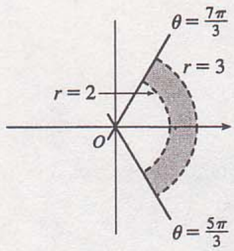
9. $0 \le r \le 2$, $\frac{\pi}{2} \le \theta \le \pi$



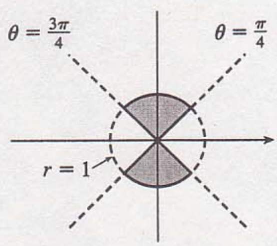
10. $1 \le r < 3$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



11. $2 < r < 3$, $\frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



12. $-1 \le r \le 1$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



13. $\left(1, \frac{\pi}{6}\right)$ is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ Cartesian and $\left(3, \frac{3\pi}{4}\right)$ is $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ Cartesian. The square of the distance between them is $\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2} - \frac{3}{\sqrt{2}}\right)^2 = \frac{1}{4}\left(40 + 6\sqrt{6} - 6\sqrt{2}\right)$, so the distance is $\frac{1}{2}\sqrt{40 + 6\sqrt{6} - 6\sqrt{2}}$.

14. The points in Cartesian coordinates are $(r_1 \cos\theta_1, r_1 \sin\theta_1)$ and $(r_2 \cos\theta_2, r_2 \sin\theta_2)$ respectively. So the square of the distance between them is $(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 = r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2$, and the distance is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r = 2 \quad\Leftrightarrow\quad \sqrt{x^2 + y^2} = 2 \quad\Leftrightarrow\quad x^2 + y^2 = 4$, a circle of radius 2 centered at the origin.

16. $r\cos\theta = 1 \quad\Leftrightarrow\quad x = 1$, a vertical line.

17. $r = 3\sin\theta \quad\Rightarrow\quad r^2 = 3r\sin\theta \quad\Leftrightarrow\quad x^2 + y^2 = 3y \quad\Leftrightarrow\quad x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0, \frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2 = 3r\sin\theta \quad\Rightarrow\quad r(r - 3\sin\theta) = 0 \quad\Rightarrow\quad r = 0$ or $r = 3\sin\theta$. But $r = 3\sin\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the single equation $r = 3\sin\theta$ is equivalent to the compound condition ($r = 0$ or $r = 3\sin\theta$).

18. $r = \dfrac{1}{1 + 2\sin\theta} \quad\Rightarrow\quad r + 2r\sin\theta = 1 \quad\Leftrightarrow\quad r = 1 - 2r\sin\theta \quad\Leftrightarrow\quad \sqrt{x^2 + y^2} = 1 - 2y \quad\Rightarrow$ $x^2 + y^2 = 1 - 4y + 4y^2 \quad\Leftrightarrow\quad 3y^2 - 4y - x^2 = -1 \quad\Leftrightarrow\quad 3\left(y^2 - \frac{4}{3}y + \frac{4}{9}\right) - x^2 = \frac{4}{3} - 1 \quad\Leftrightarrow$ $3\left(y - \frac{2}{3}\right)^2 - x^2 = \frac{1}{3} \quad\Leftrightarrow\quad 9\left(y - \frac{2}{3}\right)^2 - 3x^2 = 1 \quad\Leftrightarrow\quad \dfrac{\left(y - \frac{2}{3}\right)^2}{\left(\frac{1}{3}\right)^2} - \dfrac{x^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1$. This is a hyperbola opening up and down and centered at $\left(0, \frac{2}{3}\right)$.

19. $r^2 = \sin 2\theta = 2\sin\theta\cos\theta \quad\Leftrightarrow\quad r^4 = 2r\sin\theta\, r\cos\theta \quad\Leftrightarrow\quad \left(x^2 + y^2\right)^2 = 2yx$

20. $r^2 = \theta \quad\Rightarrow\quad \tan\left(r^2\right) = \tan\theta \quad\Rightarrow\quad \tan\left(x^2 + y^2\right) = y/x$

21. $y = 5 \quad\Leftrightarrow\quad r\sin\theta = 5$

22. $y = 2x - 1 \quad\Leftrightarrow\quad r\sin\theta = 2r\cos\theta - 1 \quad\Leftrightarrow\quad r(2\cos\theta - \sin\theta) = 1 \quad\Leftrightarrow\quad r = \dfrac{1}{2\cos\theta - \sin\theta}$. (We can divide by $2\cos\theta - \sin\theta$ because it must be nonzero in order that its product with r equal 1.)

23. $x^2 + y^2 = 25 \quad\Leftrightarrow\quad r^2 = 25 \quad\Rightarrow\quad r = 5$

24. $x^2 = 4y \quad\Leftrightarrow\quad r^2\cos^2\theta = 4r\sin\theta \quad\Leftrightarrow\quad r\cos^2\theta = 4\sin\theta \quad\Leftrightarrow\quad r = 4\tan\theta\sec\theta$

25. $2xy = 1 \quad\Leftrightarrow\quad 2r\cos\theta\, r\sin\theta = 1 \quad\Leftrightarrow\quad r^2\sin 2\theta = 1 \quad\Leftrightarrow\quad r^2 = \csc 2\theta$

26. $x^2 - y^2 = 1 \quad\Leftrightarrow\quad r^2\left(\cos^2\theta - \sin^2\theta\right) = 1 \quad\Leftrightarrow\quad r^2\cos 2\theta = 1 \quad\Rightarrow\quad r^2 = \sec 2\theta$

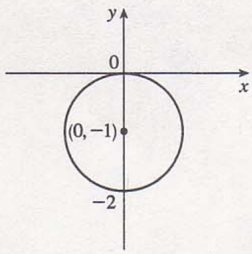
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $y = \tan\left(\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}x$ is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation $x = 3$.

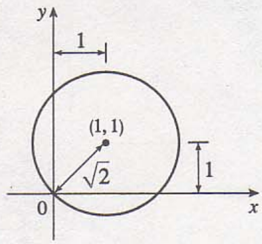
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x - 2)^2 + (y - 3)^2 = 5^2$.

(b) This circle is more easily given in polar coordinates: $r = 4$. The Cartesian equation is also simple: $x^2 + y^2 = 16$.

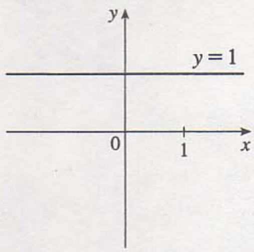
29. $r = -2\sin\theta \iff r^2 = -2r\sin\theta$ (since the possibility $r = 0$ is covered by the equation $r = -2\sin\theta$) $\iff x^2 + y^2 = -2y \iff$ $x^2 + y^2 + 2y + 1 = 1 \iff x^2 + (y+1)^2 = 1$.



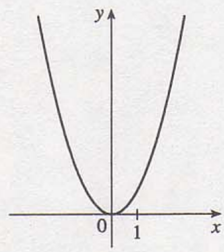
30. $r = 2\sin\theta + 2\cos\theta \iff$ $r^2 = 2r\sin\theta + 2r\cos\theta$, $x^2 + y^2 = 2y + 2x$ $\iff (x-1)^2 + (y-1)^2 = 2$



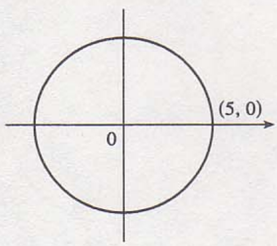
31. $r = \csc\theta = \dfrac{1}{\sin\theta} \iff r\sin\theta = 1$. (The right-hand equation implies that $\sin\theta \neq 0$, so we can divide by $\sin\theta$ to get the left-hand equation) $\iff y = 1$.



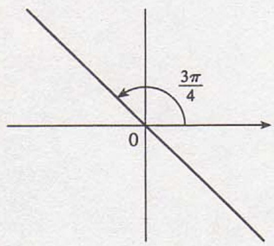
32. $r = \tan\theta\sec\theta \Rightarrow r = \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\cos\theta} \Rightarrow$ $r\cos^2\theta = \sin\theta \Rightarrow r^2\cos^2\theta = r\sin\theta \Rightarrow$ $x^2 = y$



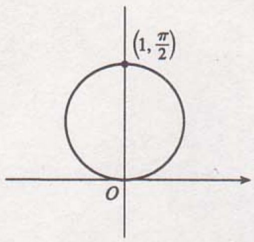
33. As in Example 4, $r = 5$ represents the circle with center O and radius 5.



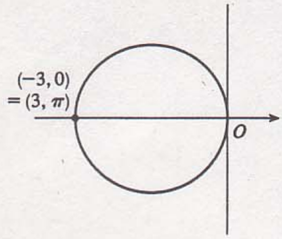
34. $\theta = \frac{3\pi}{4}$ is a line through the origin.



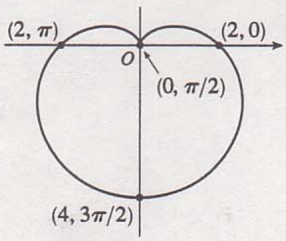
35. $r = \sin\theta \;\Leftrightarrow\; r^2 = r\sin\theta \;\Leftrightarrow\; x^2 + y^2 = y$ $\Leftrightarrow\; x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$. The reasoning here is the same as in Exercise 29. This is a circle of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}\right)$.



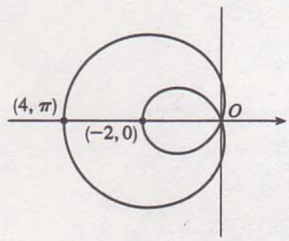
36. $r = -3\cos\theta \;\Leftrightarrow\; r^2 = -3r\cos\theta \;\Leftrightarrow$ $x^2 + y^2 = -3x \;\Leftrightarrow\; \left(x + \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$. This curve is a circle of radius $\frac{3}{2}$.



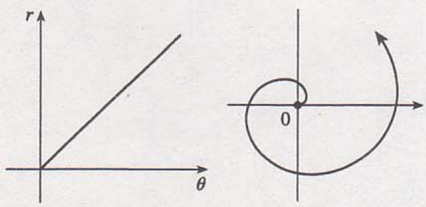
37. $r = 2\,(1 - \sin\theta)$. This curve is a cardioid.



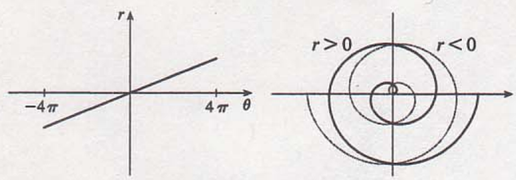
38. $r = 1 - 3\cos\theta$. This is a limaçon.



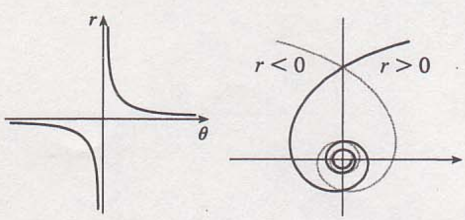
39. $r = \theta, \theta \geq 0$



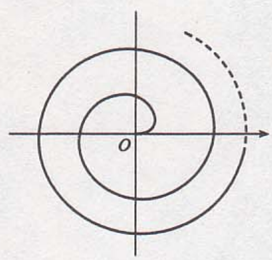
40. $r = \theta/2, -4\pi \leq \theta \leq 4\pi$



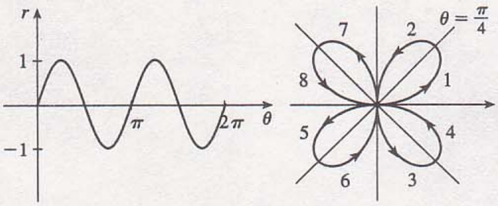
41. $r = 1/\theta$



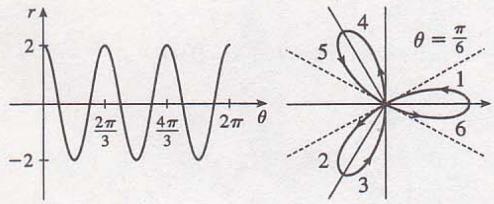
42. $r = \sqrt{\theta}$. This curve is a spiral.



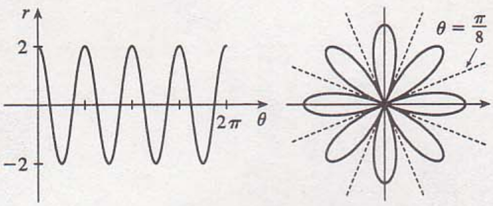
43. $r = \sin 2\theta$



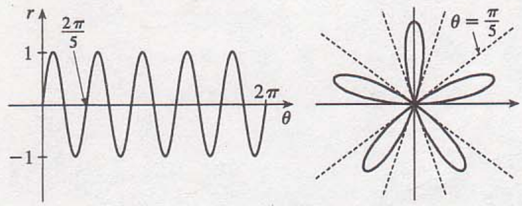
44. $r = 2\cos 3\theta$



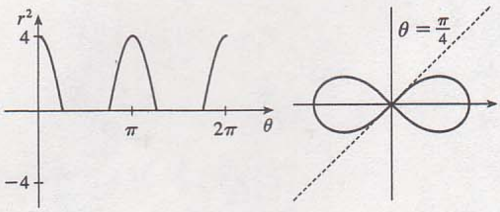
45. $r = 2\cos 4\theta$



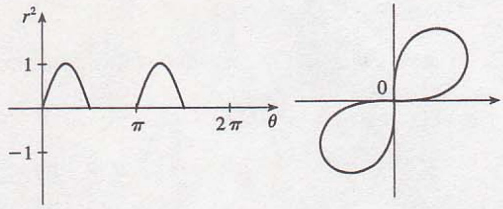
46. $r = \sin 5\theta$



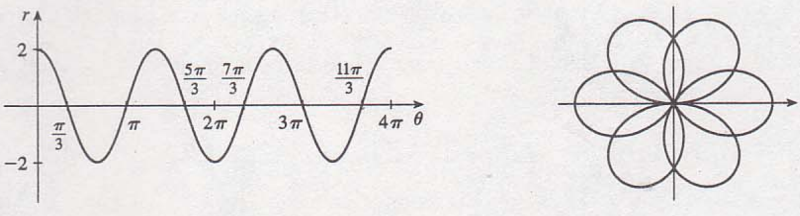
47. $r^2 = 4\cos 2\theta$



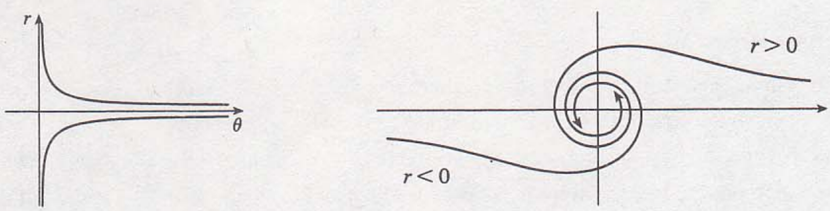
48. $r^2 = \sin 2\theta$



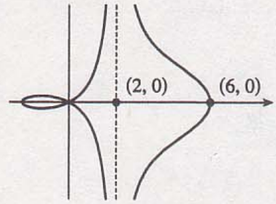
49. $r = 2\cos\left(\frac{3}{2}\theta\right)$



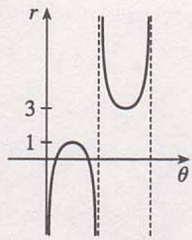
50. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$

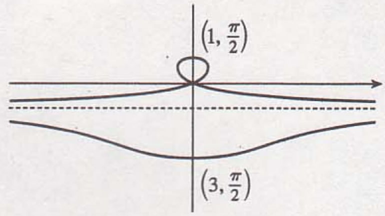


51. $x = (r)\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \;\Rightarrow$ $(4 + 2\sec\theta) \to \infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^-$ or $\theta \to \left(\frac{3\pi}{2}\right)^+$ (since we need only consider $0 \le \theta < 2\pi$), so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$. Also, $r \to -\infty \;\Rightarrow\; (4 + 2\sec\theta) \to -\infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^+$ or $\theta \to \left(\frac{3\pi}{2}\right)^-$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \;\Rightarrow\; x = 2$ is a vertical asymptote.



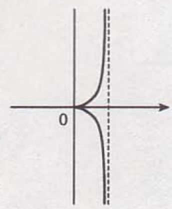
52. $y = r\sin\theta = 2\sin\theta - \csc\theta\sin\theta = 2\sin\theta - 1$. $r \to \infty \;\Rightarrow\; (2 - \csc\theta) \to \infty \;\Rightarrow$ $\csc\theta \to -\infty \;\Rightarrow\; \theta \to \pi^+$ (since we need only consider $0 \le \theta < 2\pi$) and so $\lim\limits_{r\to\infty} y = \lim\limits_{\theta\to\pi^+} 2\sin\theta - 1 = -1$. Also $r \to -\infty \;\Rightarrow\; (2 - \csc\theta) \to -\infty \;\Rightarrow$ $\csc\theta \to \infty \;\Rightarrow$ $\theta \to \pi^-$ and so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi^-} 2\sin\theta - 1 = -1$. Therefore $\lim\limits_{r\to\pm\infty} y = -1 \;\Rightarrow\; y = -1$ is a horizontal asymptote.





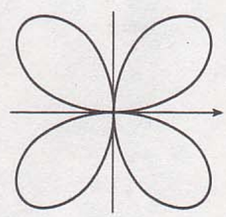
53. To show that $x = 1$ is an asymptote we must prove $\lim\limits_{r\to\pm\infty} x = 1$.

$x = (r)\cos\theta = (\sin\theta\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \;\Rightarrow$ $\sin\theta\tan\theta \to \infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \;\Rightarrow\; \sin\theta\tan\theta \to -\infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$.



Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \;\Rightarrow\; x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \ge 0$ for all θ, and $x = \sin^2\theta \le 1$ for all θ. And $x \ne 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \le x < 1$.

54. The equation is $\left(x^2 + y^2\right)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \;\Rightarrow\; r^2 = 4\cos^2\theta\sin^2\theta \;\Rightarrow\; r = \pm 2\cos\theta\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.



55. (a) We see that the curve crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values,) so we solve the equation of the limaçon for $r = 0 \;\Leftrightarrow\; c\sin\theta = -1 \;\Leftrightarrow\; \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum: $y = r\sin\theta = \sin\theta + c\sin^2\theta \;\Rightarrow\; \dfrac{dy}{d\theta} = \cos\theta + 2c\sin\theta\cos\theta = \cos\theta + c\sin 2\theta \;\Rightarrow$ $\dfrac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta$. At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

56. (a) $r = \sin(\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π.

(b) $r = \sin(\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π.

(c) $r = \sec(3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}$, and so on.

(d) $r = \theta\sin\theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π. This graph is unbounded, and each time θ moves through an interval of 2π, the same basic shape is repeated (because of the periodic $\sin\theta$ factor) but it gets larger each time (since θ increases each time we go around.)

(e) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.

(f) $r = 1\big/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.

57. Using Equation 3 with $r = 3\cos\theta$, we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)(\sin\theta) + r\cos\theta}{(dr/d\theta)(\cos\theta) - r\sin\theta} = \frac{-3\sin\theta\sin\theta + 3\cos\theta\cos\theta}{-3\sin\theta\cos\theta - 3\cos\theta\sin\theta} = \frac{3\left(\cos^2\theta - \sin^2\theta\right)}{-3\left(2\sin\theta\cos\theta\right)}$$

$$= -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta = \frac{1}{\sqrt{3}} \text{ when } \theta = \frac{\pi}{3}$$

Another Solution: $r = 3\cos\theta \;\Rightarrow\; x = r\cos\theta = 3\cos^2\theta,\ y = r\sin\theta = 3\sin\theta\cos\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3\sin^2\theta + 3\cos^2\theta}{-6\cos\theta\sin\theta} = \frac{\cos 2\theta}{-\sin 2\theta} = -\cot 2\theta = \frac{1}{\sqrt{3}} \text{ when } \theta = \frac{\pi}{3}$$

58. Using Equation 3 with $r = \cos\theta + \sin\theta$, we have

$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{(-\sin\theta + \cos\theta)\sin\theta + (\cos\theta + \sin\theta)\cos\theta}{(-\sin\theta + \cos\theta)\cos\theta - (\cos\theta + \sin\theta)\sin\theta} = -1 \text{ when } \theta = \frac{\pi}{4}$$

Another Solution: $r = \cos\theta + \sin\theta \;\Rightarrow\; x = r\cos\theta = (\cos\theta + \sin\theta)\cos\theta$, $y = r\sin\theta = (\cos\theta + \sin\theta)\sin\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-\sin\theta + \cos\theta) + (\cos\theta + \sin\theta)\cos\theta}{\cos\theta(-\sin\theta + \cos\theta) - (\cos\theta + \sin\theta)\sin\theta} = -1 \text{ when } \theta = \frac{\pi}{4}$$

59. $r = 1/\theta \;\Rightarrow\; x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\left(-1/\theta^2\right) + (1/\theta)\cos\theta}{\cos\theta\left(-1/\theta^2\right) - (1/\theta)\sin\theta} \cdot \frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta} = -\pi \text{ when } \theta = \pi$$

60. $r = \ln\theta \quad\Rightarrow\quad x = r\cos\theta = \ln\theta\cos\theta,\ y = r\sin\theta = \ln\theta\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\,(1/\theta) + \ln\theta\cos\theta}{\cos\theta\,(1/\theta) - \ln\theta\sin\theta} = \frac{\sin e + e\cos e}{\cos e - e\sin e} \text{ when } \theta = e$$

61. $r = 1 + \cos\theta \quad\Rightarrow\quad x = r\cos\theta = \cos\theta + \cos^2\theta,\ y = r\sin\theta = \sin\theta + \sin\theta\cos\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta} = \frac{\cos\theta + \cos 2\theta}{-\sin\theta - \sin 2\theta} = -1 \text{ when } \theta = \frac{\pi}{6}$$

62. $r = \sin 3\theta \quad\Rightarrow\quad x = r\cos\theta = \sin 3\theta\cos\theta,\ y = r\sin\theta = \sin 3\theta\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta} = -\sqrt{3} \text{ when } \theta = \frac{\pi}{6}$$

63. $r = 3\cos\theta \quad\Rightarrow\quad x = r\cos\theta = 3\cos\theta\cos\theta,\ y = r\sin\theta = 3\cos\theta\sin\theta \quad\Rightarrow$
$dy/d\theta = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \quad\Rightarrow\quad 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad\Leftrightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$. $dx/d\theta = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \quad\Rightarrow$
$2\theta = 0$ or $\pi \quad\Leftrightarrow\quad \theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3, 0)$ and $\left(0, \frac{\pi}{2}\right)$.

64. $y = r\sin\theta = \cos\theta\sin\theta + \sin^2\theta = \frac{1}{2}\sin 2\theta + \sin^2\theta \quad\Rightarrow\quad dy/d\theta = \cos 2\theta + \sin 2\theta = 0 \quad\Rightarrow\quad \tan 2\theta = -1$
$\Rightarrow\quad 2\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4} \quad\Leftrightarrow\quad \theta = \frac{3\pi}{8}$ or $\frac{7\pi}{8} \quad\Rightarrow\quad$ horizontal tangents at $\left(\cos\frac{3\pi}{8} + \sin\frac{3\pi}{8}, \frac{3\pi}{8}\right)$ and $\left(\cos\frac{7\pi}{8} + \sin\frac{7\pi}{8}, \frac{7\pi}{8}\right)$. $x = r\cos\theta = \cos^2\theta + \cos\theta\sin\theta \quad\Rightarrow\quad dx/d\theta = -\sin 2\theta + \cos 2\theta = 0 \quad\Rightarrow$
$\tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4} \quad\Leftrightarrow\quad \theta = \frac{\pi}{8}$ or $\frac{5\pi}{8} \quad\Rightarrow\quad$ vertical tangents at $\left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}, \frac{\pi}{8}\right)$ and $\left(\cos\frac{5\pi}{8} + \sin\frac{5\pi}{8}, \frac{5\pi}{8}\right)$.
Note: These expressions can be simplified using trigonometric identities. For example,
$\cos\frac{\pi}{8} + \sin\frac{\pi}{8} = \frac{1}{2}\sqrt{4 + 2\sqrt{2}}$.

65. $r = 1 + \cos\theta \quad\Rightarrow\quad x = r\cos\theta = \cos\theta\,(1 + \cos\theta),\ y = r\sin\theta = \sin\theta\,(1 + \cos\theta) \quad\Rightarrow$
$dy/d\theta = (1 + \cos\theta)\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \quad\Rightarrow\quad \cos\theta = \frac{1}{2}$ or $-1 \quad\Rightarrow\quad \theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \quad\Rightarrow\quad$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.
$dx/d\theta = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta\,(1 + 2\cos\theta) = 0 \quad\Rightarrow\quad \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \quad\Rightarrow$
$\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \quad\Rightarrow\quad$ vertical tangent at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$. Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\displaystyle\lim_{\theta\to\pi} \frac{dy/d\theta}{dx/d\theta} = 0$.

66. $\dfrac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \quad\Rightarrow\quad \sin\theta = -\cos\theta \quad\Rightarrow\quad \tan\theta = -1 \quad\Rightarrow$
$\theta = -\frac{1}{4}\pi + n\pi$ (n any integer) $\quad\Rightarrow\quad$ horizontal tangents at $\left(e^{\pi(n-1/4)}, \pi\left(n - \frac{1}{4}\right)\right)$.
$\dfrac{dx}{d\theta} = e^\theta\cos\theta - e^\theta\sin\theta = e^\theta(\cos\theta - \sin\theta) = 0 \quad\Rightarrow\quad \sin\theta = \cos\theta \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow\quad \theta = \frac{1}{4}\pi + n\pi$
(n any integer) $\quad\Rightarrow\quad$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n + \frac{1}{4}\right)\right)$.

67. $r = \cos 2\theta \quad\Rightarrow\quad x = r\cos\theta = \cos 2\theta\cos\theta,\ y = r\sin\theta = \cos 2\theta\sin\theta \quad\Rightarrow$

$$\begin{aligned} dy/d\theta &= -2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta = -4\sin^2\theta\cos\theta + \left(\cos^3\theta - \sin^2\theta\cos\theta\right) \\ &= \cos\theta\left(\cos^2\theta - 5\sin^2\theta\right) = \cos\theta\left(1 - 6\sin^2\theta\right) = 0 \quad\Rightarrow \end{aligned}$$

$\cos\theta = 0$ or $\sin\theta = \pm\frac{1}{\sqrt{6}} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \alpha, \pi - \alpha, \pi + \alpha$, or $2\pi - \alpha$ (where $\alpha = \sin^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is horizontal at $\left(-1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right), \left(\frac{2}{3}, \alpha\right), \left(\frac{2}{3}, \pi - \alpha\right), \left(\frac{2}{3}, \pi + \alpha\right)$, and $\left(\frac{2}{3}, 2\pi - \alpha\right)$.

$$\begin{aligned} dx/d\theta &= -2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta = -4\sin\theta\cos^2\theta - \left(2\cos^2\theta - 1\right)\sin\theta \\ &= \sin\theta\left(1 - 6\cos^2\theta\right) = 0 \quad \Rightarrow \end{aligned}$$

$\sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{6}} \Rightarrow \theta = 0, \pi, \frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha, \frac{3\pi}{2} - \alpha$, or $\frac{3\pi}{2} + \alpha$ (where $\alpha = \cos^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is vertical at $(1, 0), (1, \pi), \left(\frac{2}{3}, \frac{3\pi}{2} - \alpha\right), \left(\frac{2}{3}, \frac{3\pi}{2} + \alpha\right), \left(\frac{2}{3}, \frac{\pi}{2} - \alpha\right)$, and $\left(\frac{2}{3}, \frac{\pi}{2} + \alpha\right)$.

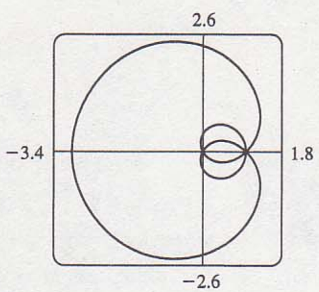
68. $dr/d\theta = (1/r)\cos 2\theta$ (by differentiating implicitly), so
$\dfrac{dy}{d\theta} = \dfrac{1}{r}\cos 2\theta\sin\theta + r\cos\theta = \dfrac{1}{r}\left(\cos 2\theta\sin\theta + r^2\cos\theta\right) = \dfrac{1}{r}\left(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta\right) = \dfrac{1}{r}\sin 3\theta$. This is 0 when $\sin 3\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{3}$ or $\frac{4\pi}{3}$ (restricting θ to the domain of the lemniscate), so there are horizontal tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3}\right)$, $\left(\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3}\right)$ and $(0, 0)$. Similarly, $dx/d\theta = (1/r)\cos 3\theta = 0$ when $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$, so there are vertical tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6}\right)$ and $\left(\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6}\right)$ [and $(0, 0)$]. See the sketch in Exercise 48.

69. $r = a\sin\theta + b\cos\theta \Rightarrow r^2 = ar\sin\theta + br\cos\theta \Rightarrow x^2 + y^2 = ay + bx \Rightarrow$
$\left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}\left(a^2 + b^2\right)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

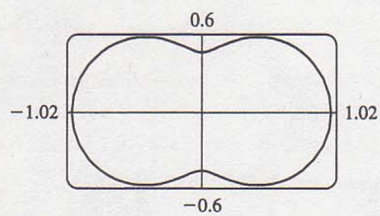
70. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle ($r = a\sin\theta$), $dy/d\theta = a\cos\theta\sin\theta + a\sin\theta\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle ($r = a\cos\theta$), $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

Note for Exercises 71–76: Maple is able to plot polar curves using the `polarplot` command, or using the `coords=polar` option in a regular `plot` command. In Mathematica, use `PolarPlot`. In Derive, change to `Polar` under `Options State`. If your graphing device cannot plot polar equations, you must convert to parametric equations. For example, in Exercise 71, $x = r\cos\theta = [1 + 2\sin(\theta/2)]\cos\theta$, $y = r\sin\theta = [1 + 2\sin(\theta/2)]\sin\theta$.

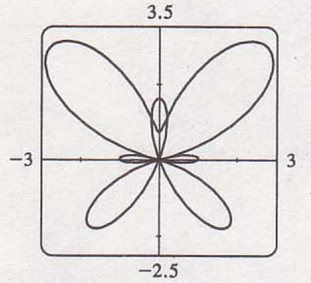
71. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.



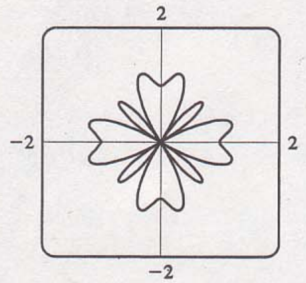
72. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



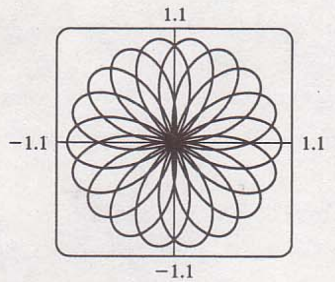
73. $r = e^{\sin\theta} - 2\cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



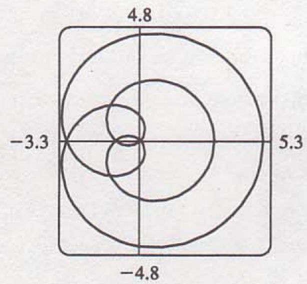
74. $r = \sin^2(4\theta) + \cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



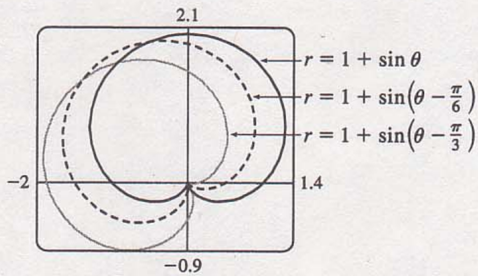
75. $r = \sin(9\theta/4)$. The parameter interval is $[0, 8\pi]$.



76. $r = 1 + 4\cos(\theta/3)$. The parameter interval is $[0, 6\pi]$.



77.



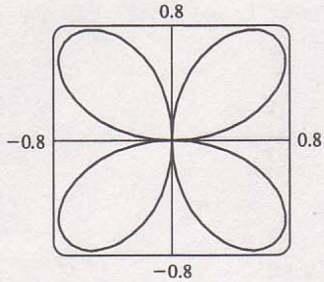
It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.

78. From the graph, the highest points seem to have $y \approx 0.77$.

To find the exact value, we solve $dy/d\theta = 0$.

$y = r\sin\theta = \sin\theta\sin 2\theta \quad \Rightarrow$

$$\begin{aligned} dy/d\theta &= 2\sin\theta\cos 2\theta + \cos\theta\sin 2\theta \\ &= 2\sin\theta\left(2\cos^2\theta - 1\right) + \cos\theta\left(2\sin\theta\cos\theta\right) \\ &= 2\sin\theta\left(3\cos^2\theta - 1\right) \end{aligned}$$



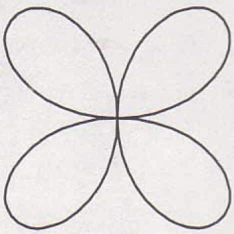
In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad \Leftrightarrow \quad \sin\theta = \sqrt{\frac{2}{3}} \quad \Leftrightarrow$

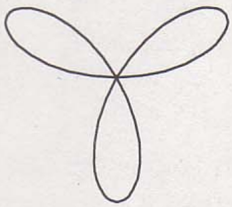
$y = 2\sin^2\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{9} \approx 0.77.$

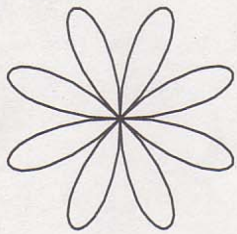
79. (a) $r = \sin n\theta$. From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n.

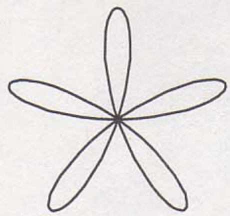
This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta\cos n\pi + \cos n\theta\sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

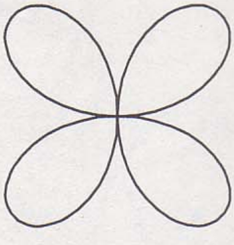

$n = 2$

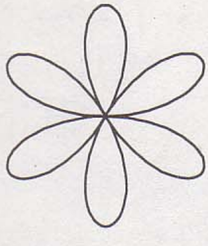

$n = 3$

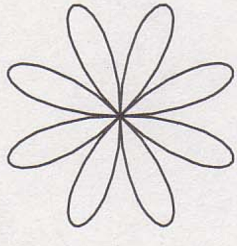

$n = 4$

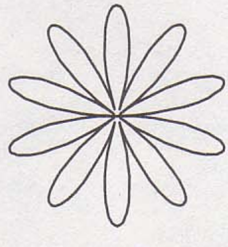

$n = 5$

(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.


$n = 2$

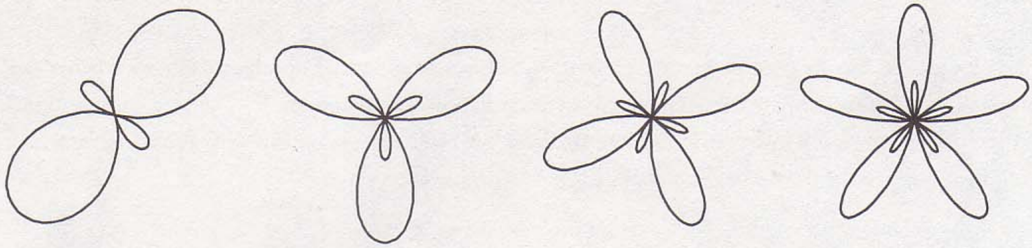

$n = 3$


$n = 4$


$n = 5$

80. $r = 1 + c\sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 79: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

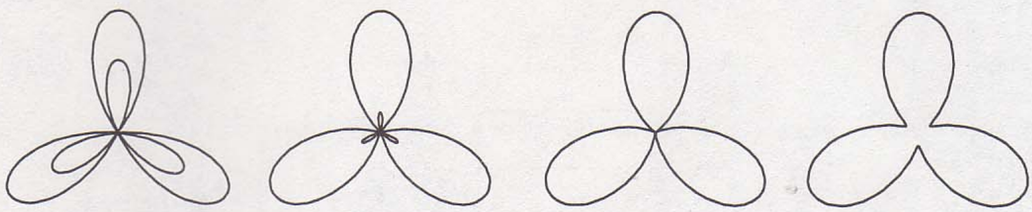
$c = 2$



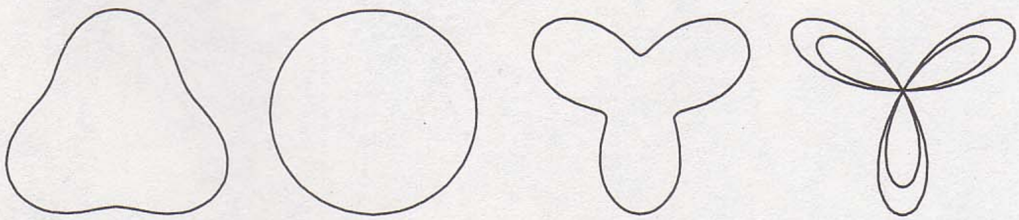
$n = 2$ $\quad$ $n = 3$ $\quad$ $n = 4$ $\quad$ $n = 5$

Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c)\sin n\theta = 1 + c\sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.

$n = 3$



$c = -4$ $\quad$ $c = -1.4$ $\quad$ $c = -1$ $\quad$ $c = -0.8$

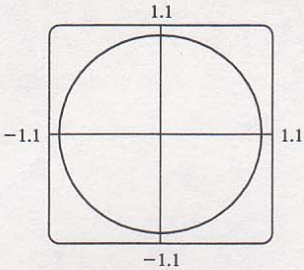


$c = -0.2$ $\quad$ $c = 0$ $\quad$ $c = 0.5$ $\quad$ $c = 8$

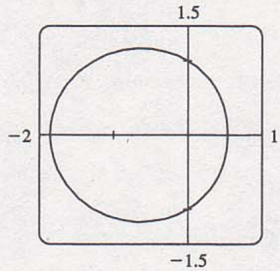
81. $r = \dfrac{1 - a\cos\theta}{1 + a\cos\theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$.

As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ-ranges) seems to appear at $a \approx 0.42$ (the actual value is $\sqrt{2} - 1$). As $a \to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \to \infty$ as $1 + a\cos\theta \to 0 \quad \Leftrightarrow$ $\theta \to \pm\cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ (actually, $\sqrt{2} + 1$). As a increases, the dimple grows more and more pronounced.

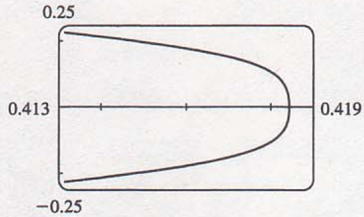
If $a < 0$, we get the same graph as we do for the corresponding positive a-value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 80.



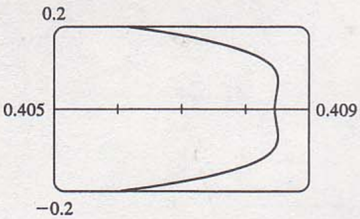
$a = 0$



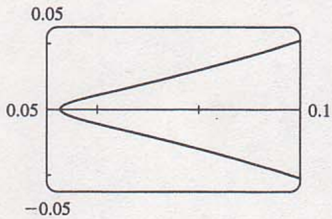
$a = 0.3$



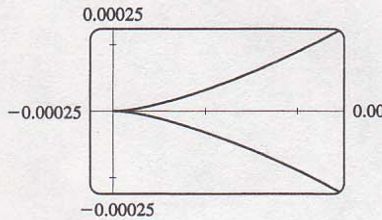
$a = 0.41, |\theta| \leq 0.5$



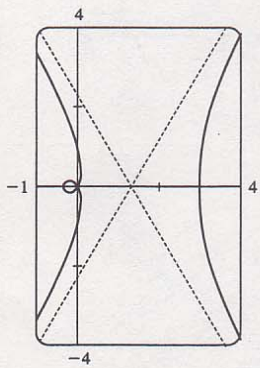
$a = 0.42, |\theta| \leq 0.5$



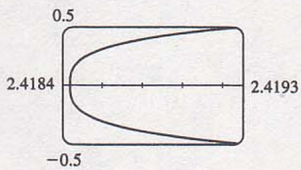
$a = 0.9, |\theta| \leq 0.5$



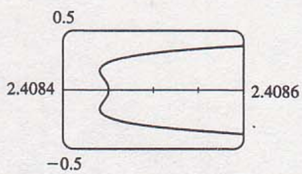
$a = 1, |\theta| \leq 0.1$



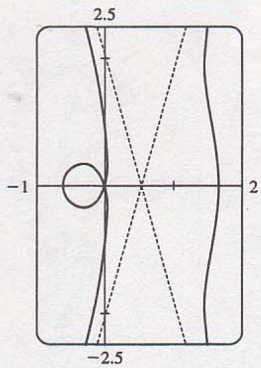
$a = 2$



$a = 2.41, |\theta - \pi| \leq 0.2$



$a = 2.42, |\theta - \pi| \leq 0.2$



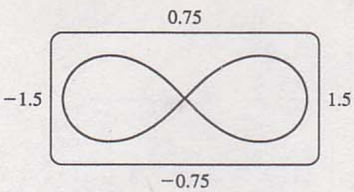
$a = 4$

82. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ, a, and c. We note that the given equation is a quadratic in r^2, so we use the quadratic formula and find that

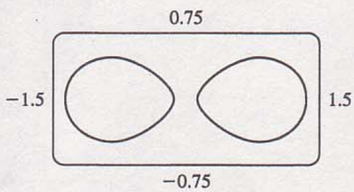
$$r^2 = \frac{2c^2 \cos 2\theta \pm \sqrt{4c^4 \cos^2 2\theta - 4\left(c^4 - a^4\right)}}{2} = c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta}$$

so $r = \pm\sqrt{c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π.

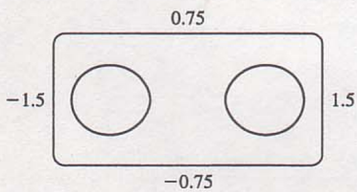
We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at $a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



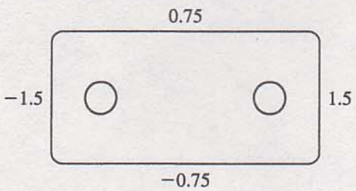
$(a, c) = (1, 1)$



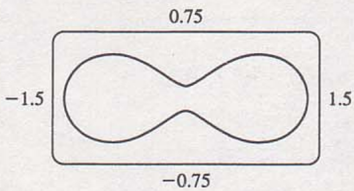
$(a, c) = (0.99, 1)$



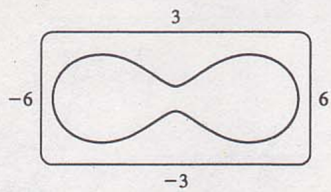
$(a, c) = (0.9, 1)$



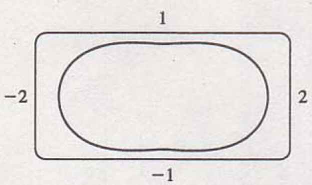
$(a, c) = (0.6, 1)$



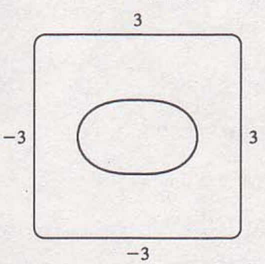
$(a, c) = (1.01, 1)$



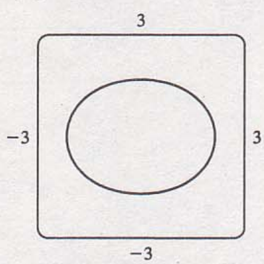
$(a, c) = (4.04, 4)$



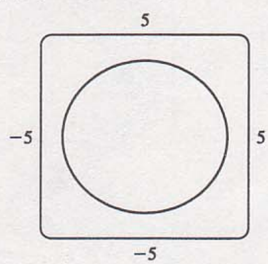
$(a, c) = (1.3, 1)$



$(a, c) = (1.5, 1)$



$(a, c) = (2, 1)$



$(a, c) = (4, 1)$

83. $\tan\psi = \tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta} = \dfrac{\dfrac{dy}{dx} - \tan\theta}{1 + \dfrac{dy}{dx}\tan\theta} = \dfrac{\dfrac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$

$$= \frac{\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta} + \dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right) - \tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right) + \tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)}$$

$$= \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta + \dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}} = \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta + \dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

84. (a) $r = e^\theta \quad\Rightarrow\quad dr/d\theta = e^\theta$, so by Exercise 83,

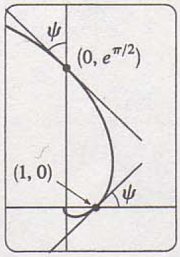
$\tan\psi = r/e^\theta = 1 \quad\Rightarrow\quad \psi = \arctan 1 = \frac{\pi}{4}$.

(b) The Cartesian equation of the tangent line at $(1, 0)$ is $y = x - 1$, and that of the tangent line at $\left(0, e^{\pi/2}\right)$ is $y = e^{\pi/2} - x$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$.

Then, by Exercise 83, $a = \dfrac{r}{dr/d\theta} \quad\Rightarrow$

$\dfrac{dr}{d\theta} = \dfrac{1}{a}r \quad\Rightarrow\quad r = Ce^{\theta/a}$ (by Theorem 10.4.2 [ET 9.4.2]).



11.5 Areas and Lengths in Polar Coordinates **ET 10.5**

1. $r = \sqrt{\theta}, 0 \le \theta \le \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = \int_0^{\pi/4} \frac{1}{2}\left(\sqrt{\theta}\right)^2 d\theta = \int_0^{\pi/4} \frac{1}{2}\theta\,d\theta = \left[\frac{1}{4}\theta^2\right]_0^{\pi/4} = \frac{1}{64}\pi^2$

2. $r = e^{\theta/2}, \pi \le \theta \le 2\pi$. $A = \int_\pi^{2\pi} \frac{1}{2}\left(e^{\theta/2}\right)^2 d\theta = \int_\pi^{2\pi} \frac{1}{2}e^\theta\,d\theta = \frac{1}{2}\left[e^\theta\right]_\pi^{2\pi} = \frac{1}{2}\left(e^{2\pi} - e^\pi\right)$

3. $r = \sin\theta, \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$.

$A = \int_{\pi/3}^{2\pi/3} \frac{1}{2}\sin^2\theta\,d\theta = \frac{1}{4}\int_{\pi/3}^{2\pi/3}(1 - \cos 2\theta)\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{2\pi/3}$

$= \frac{1}{4}\left[\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} - \frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right] = \frac{1}{4}\left[\frac{2\pi}{3} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{4}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$

4. $r = \sqrt{\sin\theta}, 0 \le \theta \le \pi$. $A = \int_0^\pi \frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 d\theta = \int_0^\pi \frac{1}{2}\sin\theta\,d\theta = \left[-\frac{1}{2}\cos\theta\right]_0^\pi = \frac{1}{2} + \frac{1}{2} = 1$

5. $r = \theta, 0 \le \theta \le \pi$. $A = \int_0^\pi \frac{1}{2}\theta^2\,d\theta = \left[\frac{1}{6}\theta^3\right]_0^\pi = \frac{1}{6}\pi^3$

6. $r = 1 + \sin\theta, \frac{\pi}{2} \le \theta \le \pi$.

$A = \int_{\pi/2}^\pi \frac{1}{2}(1 + \sin\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/2}^\pi (1 + 2\sin\theta + \sin^2\theta)\,d\theta = \frac{1}{2}\int_{\pi/2}^\pi \left[1 + 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right] d\theta$

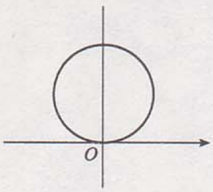
$= \frac{1}{2}\left[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/2}^\pi = \frac{1}{2}\left[\pi + 2 + \frac{\pi}{2} - 0 - \left(\frac{\pi}{2} - 0 + \frac{\pi}{4} - 0\right)\right] = \frac{1}{2}\left(\frac{3\pi}{4} + 2\right) = \frac{3\pi}{8} + 1$

7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$

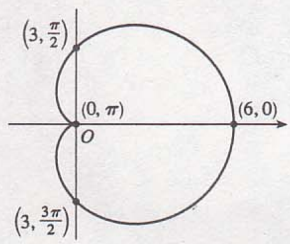
$$\begin{aligned}
A &= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}\,(4 + 3\sin\theta)^2\,d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\left(16 + 24\sin\theta + 9\sin^2\theta\right)d\theta \\
&= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\left(16 + 9\sin^2\theta\right)d\theta \quad \Big(\text{by Theorem 5.5.6(b) [ET 5.5.7(b)]}\Big) \\
&= \tfrac{1}{2}\cdot 2\int_0^{\pi/2}\left[16 + 9\cdot\tfrac{1}{2}\,(1 - \cos 2\theta)\right]d\theta \quad \Big(\text{by Theorem 5.5.6(a) [ET 5.5.7(a)]}\Big) \\
&= \int_0^{\pi/2}\left(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\right)d\theta = \left[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4} - 0\right) - (0 - 0) = \tfrac{41\pi}{4}
\end{aligned}$$

8. $r = \sin 4\theta,\ 0 \le \theta \le \frac{\pi}{4}.\ A = \int_0^{\pi/4} \frac{1}{2}\sin^2 4\theta\,d\theta = \int_0^{\pi/4}\frac{1}{4}\,(1 - \cos 8\theta)\,d\theta = \left[\frac{1}{4}\theta - \frac{1}{32}\sin 8\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

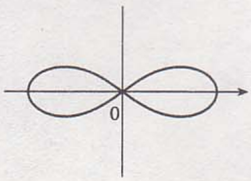
9.
$$\begin{aligned}
A &= \int_0^{\pi}\tfrac{1}{2}\,(5\sin\theta)^2\,d\theta \\
&= \tfrac{25}{4}\int_0^{\pi}(1 - \cos 2\theta)\,d\theta \\
&= \tfrac{25}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = \tfrac{25}{4}\pi
\end{aligned}$$



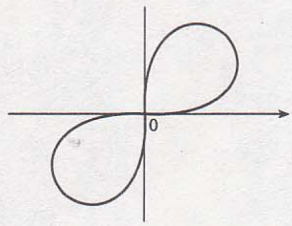
10.
$$\begin{aligned}
A &= \int_0^{2\pi}\tfrac{1}{2}r^2\,d\theta = \int_0^{2\pi}\tfrac{1}{2}\,[3\,(1 + \cos\theta)]^2\,d\theta \\
&= \tfrac{9}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \cos^2\theta\right)d\theta \\
&= \tfrac{9}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \tfrac{1}{2}\,(1 + \cos 2\theta)\right]d\theta \\
&= \tfrac{9}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{27}{2}\pi
\end{aligned}$$



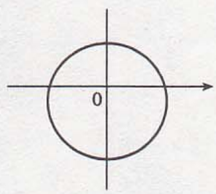
11.
$$\begin{aligned}
A &= 4\int_0^{\pi/4}\tfrac{1}{2}r^2 d\theta = 2\int_0^{\pi/4}(4\cos 2\theta)\,d\theta \\
&= 8\int_0^{\pi/4}\cos 2\theta d\theta = 4\,[\sin 2\theta]_0^{\pi/4} = 4
\end{aligned}$$



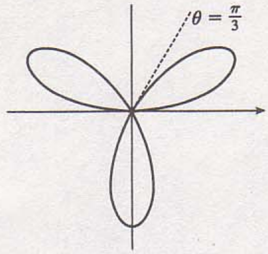
12.
$$\begin{aligned}
A &= 4\int_0^{\pi/4}\tfrac{1}{2}r^2\,d\theta = 2\int_0^{\pi/4}\sin 2\theta\,d\theta \\
&= [-\cos 2\theta]_0^{\pi/4} = 1
\end{aligned}$$



13.
$$\begin{aligned}
A &= 2\int_{-\pi/2}^{\pi/2}\tfrac{1}{2}\,(4 - \sin\theta)^2\,d\theta = \int_{-\pi/2}^{\pi/2}\left(16 - 8\sin\theta + \sin^2\theta\right)d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(16 + \sin^2\theta\right)d\theta \quad \Big(\text{by Theorem 5.5.6(b) [ET 5.5.7(b)]}\Big) \\
&= 2\int_0^{\pi/2}\left(16 + \sin^2\theta\right)d\theta \quad \Big(\text{by Theorem 5.5.6(a) [ET 5.5.7(a)]}\Big) \\
&= 2\int_0^{\pi/2}\left[16 + \tfrac{1}{2}\,(1 - \cos 2\theta)\right]d\theta = 2\left[\tfrac{33}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \\
&= \tfrac{33\pi}{2}
\end{aligned}$$

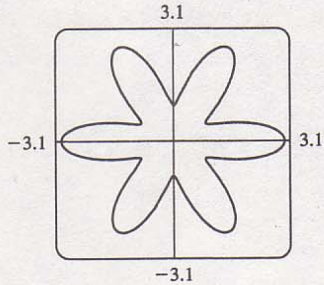


14. $A = 6\int_0^{\pi/6} \frac{1}{2}\sin^2 3\theta d\theta = 3\int_0^{\pi/6} \frac{1}{2}(1-\cos 6\theta)\, d\theta$
$= \frac{3}{2}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/6}$
$= \frac{\pi}{4}$



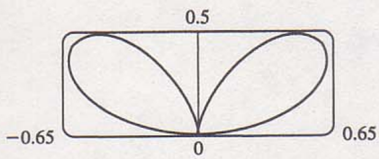
15. By symmetry, the total area is twice the area enclosed above the polar axis, so

$$A = 2\int_0^{\pi} \tfrac{1}{2}r^2 d\theta = \int_0^{\pi} [2+\cos 6\theta]^2\, d\theta = \int_0^{\pi} \left(4 + 4\cos 6\theta + \cos^2 6\theta\right) d\theta$$
$$= \left[4\theta + 4\left(\tfrac{1}{6}\sin 6\theta\right) + \left(\tfrac{1}{24}\sin 12\theta + \tfrac{1}{2}\theta\right)\right]_0^{\pi} = 4\pi + \tfrac{\pi}{2} = \tfrac{9\pi}{2}$$



16. Note that the entire curve $r = 2\sin\theta\cos^2\theta$ is generated by $\theta \in [0, \pi]$. The radius is positive on this interval, so the area enclosed is

$$A = \int_0^{\pi} \tfrac{1}{2}r^2 d\theta = \int_0^{\pi} \tfrac{1}{2}\left(2\sin\theta\cos^2\theta\right)^2 d\theta = 2\int_0^{\pi}\sin^2\theta\cos^4\theta d\theta = 2\int_0^{\pi}(\sin\theta\cos\theta)^2\cos^2\theta d\theta$$
$$= 2\int_0^{\pi}\left(\tfrac{1}{2}\sin 2\theta\right)^2\cos^2\theta d\theta = \tfrac{1}{4}\int_0^{\pi}\sin^2 2\theta\,(\cos 2\theta + 1)\, d\theta = \tfrac{1}{4}\left[\int_0^{\pi}\sin^2 2\theta\cos 2\theta d\theta + \int_0^{\pi}\sin^2 2\theta d\theta\right]$$
$$= \tfrac{1}{4}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi} \quad \text{(the first integral vanishes)} \quad = \tfrac{\pi}{8}$$



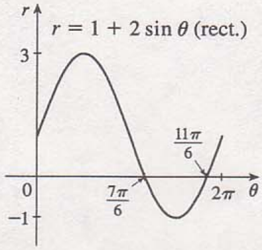
17. $A = \int_0^{\pi/2} \frac{1}{2}\sin^2 2\theta\, d\theta = \int_0^{\pi/2}\frac{1}{4}(1-\cos 4\theta)\, d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi}{8}$

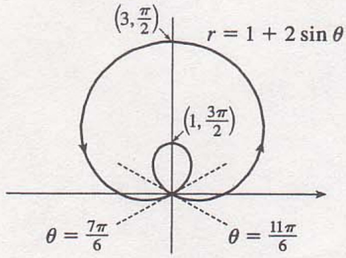
18. $A = \int_0^{\pi/3}\frac{1}{2}(4\sin 3\theta)^2\, d\theta = 8\int_0^{\pi/3}\sin^2 3\theta\, d\theta = 4\int_0^{\pi/3}(1-\cos 6\theta)\, d\theta = 4\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{4\pi}{3}$

19. $r = 0 \;\Rightarrow\; 3\cos 5\theta = 0 \;\Rightarrow\; 5\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{10}$.
$A = \int_{-\pi/10}^{\pi/10}\frac{1}{2}(3\cos 5\theta)^2\, d\theta = \int_0^{\pi/10} 9\cos^2 5\theta\, d\theta = \frac{9}{2}\int_0^{\pi/10}(1+\cos 10\theta)\, d\theta = \frac{9}{2}\left[\theta + \frac{1}{10}\sin 10\theta\right]_0^{\pi/10} = \frac{9\pi}{20}$

20. $A = 2\int_0^{\pi/8}\frac{1}{2}(2\cos 4\theta)^2\, d\theta = 2\int_0^{\pi/8}(1+\cos 8\theta)\, d\theta = 2\left[\theta + \frac{1}{8}\sin 8\theta\right]_0^{\pi/8} = \frac{\pi}{4}$

21.



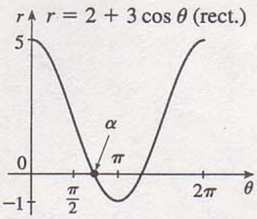


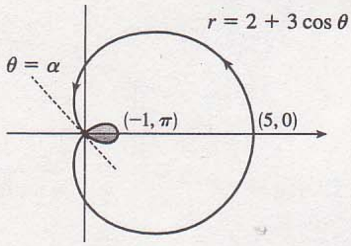
This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}\,(1 + 2\sin\theta)^2\,d\theta = \int_{7\pi/6}^{3\pi/2} \left(1 + 4\sin\theta + 4\sin^2\theta\right) d\theta = \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2}$$
$$= \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

22. $2 + 3\cos\theta = 0 \;\Rightarrow\; \cos\theta = -\frac{2}{3} \;\Rightarrow\; \theta = \cos^{-1}\left(-\frac{2}{3}\right)\ (= \alpha)$ or $2\pi - \cos^{-1}\left(-\frac{2}{3}\right) \;\Rightarrow$

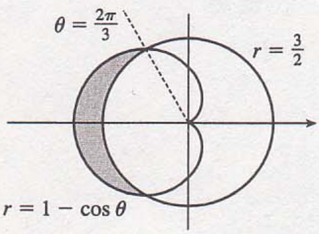
$$A = 2\int_{\alpha}^{\pi} \tfrac{1}{2}\,(2 + 3\cos\theta)^2\,d\theta = \int_{\alpha}^{\pi} \left(4 + 12\cos\theta + 9\cos^2\theta\right) d\theta = \int_{\alpha}^{\pi} \left(\tfrac{17}{2} + 12\cos\theta + \tfrac{9}{2}\cos 2\theta\right) d\theta$$
$$= \left[\tfrac{17}{2}\theta + 12\sin\theta + \tfrac{9}{4}\sin 2\theta\right]_{\alpha}^{\pi} = \tfrac{17}{2}(\pi - \alpha) - 12\sin\alpha - \tfrac{9}{2}\sin\alpha\cos\alpha$$
$$= \tfrac{17}{2}\left[\pi - \cos^{-1}\left(-\tfrac{2}{3}\right)\right] - 12\left(\tfrac{\sqrt{5}}{3}\right) - \tfrac{9}{2}\left(\tfrac{\sqrt{5}}{3}\right)\left(-\tfrac{2}{3}\right) = \tfrac{17}{2}\cos^{-1}\tfrac{2}{3} - 3\sqrt{5}$$





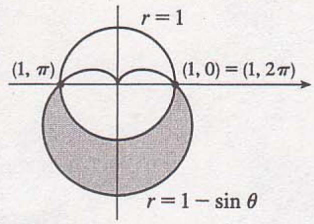
23. $1 - \cos\theta = \frac{3}{2} \;\Rightarrow\; \cos\theta = -\frac{1}{2} \;\Rightarrow\; \theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3} \;\Rightarrow$

$$A = 2\int_{2\pi/3}^{\pi} \tfrac{1}{2}\left[(1 - \cos\theta)^2 - \left(\tfrac{3}{2}\right)^2\right] d\theta = \int_{2\pi/3}^{\pi} \left(-\tfrac{5}{4} - 2\cos\theta + \cos^2\theta\right) d\theta$$
$$= \left[-\tfrac{5}{4}\theta - 2\sin\theta\right]_{2\pi/3}^{\pi} + \tfrac{1}{2}\int_{2\pi/3}^{\pi} (1 + \cos 2\theta)\,d\theta$$
$$= -\tfrac{5}{12}\pi + \sqrt{3} + \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{2\pi/3}^{\pi}$$
$$= -\tfrac{5}{12}\pi + \sqrt{3} + \tfrac{1}{6}\pi + \tfrac{\sqrt{3}}{8} = \tfrac{9\sqrt{3}}{8} - \tfrac{1}{4}\pi$$



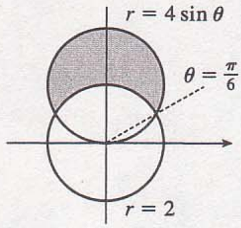
24. $1-\sin\theta=1 \;\Rightarrow\; \sin\theta=0 \;\Rightarrow\; \theta=0 \text{ or } \pi \;\Rightarrow$

$$A=\int_{\pi}^{2\pi}\tfrac{1}{2}\left[(1-\sin\theta)^2-1\right]d\theta=\tfrac{1}{2}\int_{\pi}^{2\pi}\left(\sin^2\theta-2\sin\theta\right)d\theta$$
$$=\tfrac{1}{4}\int_{\pi}^{2\pi}(1-\cos 2\theta-4\sin\theta)\,d\theta=\tfrac{1}{4}\left[\theta-\tfrac{1}{2}\sin 2\theta+4\cos\theta\right]_{\pi}^{2\pi}$$
$$=\tfrac{1}{4}\pi+2$$



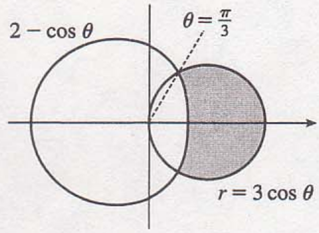
25. $4\sin\theta=2 \;\Leftrightarrow\; \sin\theta=\frac{1}{2} \;\Leftrightarrow\; \theta=\frac{\pi}{6} \text{ or } \frac{5\pi}{6} \;\Leftrightarrow$

$$A=2\int_{\pi/6}^{\pi/2}\tfrac{1}{2}\left[(4\sin\theta)^2-2^2\right]d\theta=\int_{\pi/6}^{\pi/2}\left(16\sin^2\theta-4\right)d\theta$$
$$=\int_{\pi/6}^{\pi/2}\left[8\left(1-\cos 2\theta\right)-4\right]d\theta=\left[4\theta-4\sin 2\theta\right]_{\pi/6}^{\pi/2}$$
$$=\tfrac{4}{3}\pi+2\sqrt{3}$$



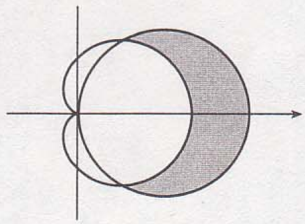
26. $3\cos\theta=2-\cos\theta \;\Rightarrow\; \cos\theta=\frac{1}{2} \;\Rightarrow\; \theta=\pm\frac{\pi}{3} \;\Rightarrow$

$$A=2\int_{0}^{\pi/3}\tfrac{1}{2}\left[(3\cos\theta)^2-(2-\cos\theta)^2\right]d\theta$$
$$=\int_{0}^{\pi/3}\left(8\cos^2\theta+4\cos\theta-4\right)d\theta$$
$$=\int_{0}^{\pi/3}(4\cos 2\theta+4\cos\theta)\,d\theta=\left[2\sin 2\theta+4\sin\theta\right]_{0}^{\pi/3}=3\sqrt{3}$$



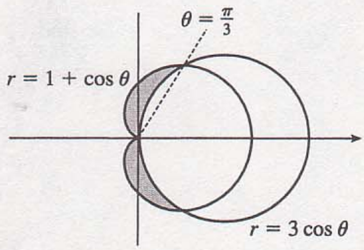
27. $3\cos\theta=1+\cos\theta \;\Leftrightarrow\; \cos\theta=\frac{1}{2} \;\Rightarrow\; \theta=\frac{\pi}{3} \text{ or } -\frac{\pi}{3}.$

$$A=2\int_{0}^{\pi/3}\tfrac{1}{2}\left[(3\cos\theta)^2-(1+\cos\theta)^2\right]d\theta$$
$$=\int_{0}^{\pi/3}\left(8\cos^2\theta-2\cos\theta-1\right)d\theta=\int_{0}^{\pi/3}\left[4\left(1+\cos 2\theta\right)-2\cos\theta-1\right]d\theta$$
$$=\left[3\theta+2\sin 2\theta-2\sin\theta\right]_{0}^{\pi/3}=\pi+\sqrt{3}-\sqrt{3}=\pi$$



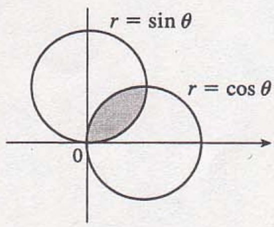
28. $A=2\int_{\pi/3}^{\pi}\frac{1}{2}\left(1+\cos\theta\right)^2d\theta-2\int_{\pi/3}^{\pi/2}\frac{1}{2}\left(3\cos\theta\right)^2d\theta$

$$=\left[\theta+2\sin\theta+\tfrac{1}{2}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)\right]_{\pi/3}^{\pi}-\tfrac{9}{2}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$
$$=\left(\pi-\tfrac{9}{8}\sqrt{3}\right)-\tfrac{9}{2}\left(\tfrac{\pi}{6}-\tfrac{1}{4}\sqrt{3}\right)=\tfrac{\pi}{4}$$

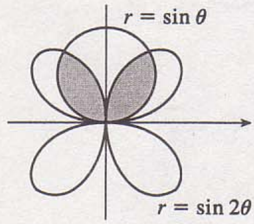


29. $A=2\int_{0}^{\pi/4}\frac{1}{2}\sin^2\theta d\theta=\int_{0}^{\pi/4}\frac{1}{2}\left(1-\cos 2\theta\right)d\theta$

$$=\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_{0}^{\pi/4}$$
$$=\tfrac{1}{8}\pi-\tfrac{1}{4}$$

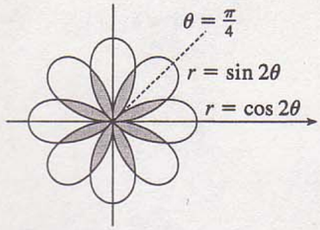


30. $\sin\theta = \pm\sin 2\theta = \pm 2\sin\theta\cos\theta \quad\Rightarrow\quad \sin\theta\,(1 \pm 2\cos\theta) = 0$. From the figure we can see that the intersections occur where $\cos\theta = \pm\frac{1}{2}$, or $\theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.



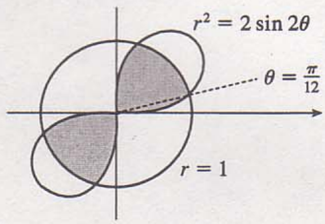
$$A = 2\left[\int_0^{\pi/3} \tfrac{1}{2}\sin^2\theta d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}\sin^2 2\theta d\theta\right]$$
$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \tfrac{1}{2}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{\pi/3}^{\pi/2} = \tfrac{4\pi - 3\sqrt{3}}{16}$$

31. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow\quad \theta = \frac{\pi}{8} \quad\Rightarrow$



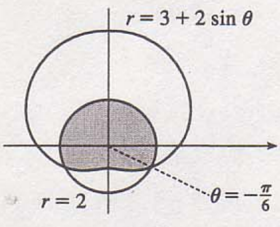
$$A = 16\int_0^{\pi/8} \tfrac{1}{2}\sin^2 2\theta d\theta = 4\int_0^{\pi/8} (1 - \cos 4\theta)\,d\theta$$
$$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = \tfrac{1}{2}\pi - 1$$

32. $2\sin 2\theta = 1^2 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \quad\Rightarrow\quad \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$.

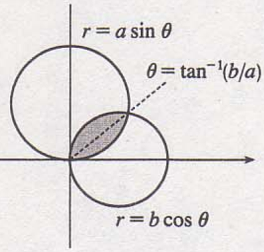


$$A = 4\left[\int_0^{\pi/12} \tfrac{1}{2}\cdot 2\sin 2\theta d\theta + \int_{\pi/12}^{\pi/4} \tfrac{1}{2}\left(1^2\right)d\theta\right]$$
$$= \left[-2\cos 2\theta\right]_0^{\pi/12} + 2\left(\tfrac{1}{4}\pi - \tfrac{1}{12}\pi\right) = 2 - \sqrt{3} + \tfrac{\pi}{3}$$

33. $A = 2\left[\int_{-\pi/2}^{-\pi/6} \frac{1}{2}(3 + 2\sin\theta)^2\,d\theta + \int_{-\pi/6}^{\pi/2} \frac{1}{2}2^2\,d\theta\right]$
$$= \int_{-\pi/2}^{-\pi/6} \left(9 + 12\sin\theta + 4\sin^2\theta\right)d\theta + \left[4\theta\right]_{-\pi/6}^{\pi/2}$$
$$= \left[9\theta - 12\cos\theta + 2\theta - \sin 2\theta\right]_{-\pi/2}^{-\pi/6} + \tfrac{8\pi}{3} = \tfrac{19\pi}{3} - \tfrac{11\sqrt{3}}{2}$$



34. Let $\alpha = \tan^{-1}(b/a)$. Then



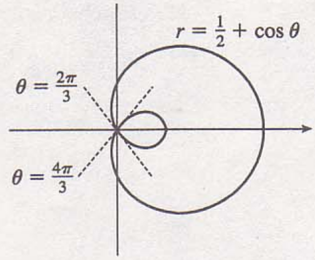
$$A = \int_0^{\alpha} \tfrac{1}{2}(a\sin\theta)^2\,d\theta + \int_{\alpha}^{\pi/2} \tfrac{1}{2}(b\cos\theta)^2\,d\theta$$
$$= \tfrac{1}{4}a^2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\alpha} + \tfrac{1}{4}b^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$$
$$= \tfrac{1}{4}\alpha\left(a^2 - b^2\right) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}\left(a^2 + b^2\right)(\sin\alpha\cos\alpha)$$
$$= \tfrac{1}{4}\left(a^2 - b^2\right)\tan^{-1}(b/a) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}ab$$

35. $A = 2\left[\int_0^{2\pi/3} \frac{1}{2}\left(\frac{1}{2} + \cos\theta\right)^2 d\theta - \int_{2\pi/3}^{\pi} \frac{1}{2}\left(\frac{1}{2} + \cos\theta\right)^2 d\theta\right]$

$$= \int_0^{2\pi/3} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta - \int_{2\pi/3}^{\pi} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta$$

$$= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$

$$= \left(\tfrac{\pi}{2} + \tfrac{3\sqrt{3}}{8}\right) - \left(\tfrac{3\pi}{4}\right) + \left(\tfrac{\pi}{2} + \tfrac{3\sqrt{3}}{8}\right) = \tfrac{1}{4}\left(\pi + 3\sqrt{3}\right)$$



36. The points of intersection occur where $\sqrt{1 - 0.8\sin^2\theta} = \sin\theta \quad \Leftrightarrow$

$1.8\sin^2\theta = 1 \quad \Leftrightarrow \quad \theta = \arcsin\sqrt{\frac{5}{9}}$ ($= \alpha$, so $\cos\alpha = \frac{2}{3}$). So the area is

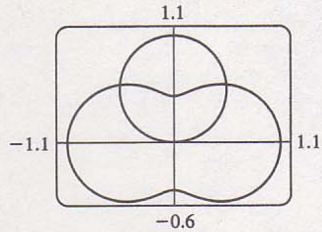
$$A = 2\int_0^{\alpha} \tfrac{1}{2}\sin^2\theta d\theta + 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}\left(\sqrt{1 - 0.8\sin^2\theta}\right)^2 d\theta$$

$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\alpha} + \left[\theta - 0.8\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2}$$

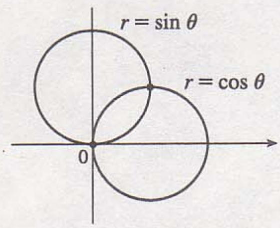
$$= \tfrac{1}{2}\alpha - \tfrac{1}{4}(2\sin\alpha\cos\alpha) + 0.6\cdot\tfrac{\pi}{2} - \left[0.6\alpha + 0.2(2\sin\alpha\cos\alpha)\right]$$

$$= \tfrac{1}{2}\arcsin\tfrac{\sqrt{5}}{3} - \tfrac{1}{2}\tfrac{\sqrt{5}}{3}\tfrac{2}{3} + 0.3\pi - 0.6\arcsin\tfrac{\sqrt{5}}{3} - 0.4\cdot\tfrac{\sqrt{5}}{3}\tfrac{2}{3}$$

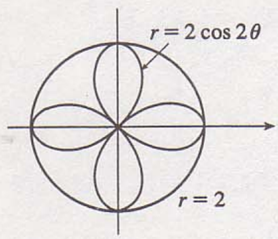
$$= \tfrac{3}{10}\pi - \tfrac{1}{10}\arcsin\tfrac{\sqrt{5}}{3} - \tfrac{1}{5}\sqrt{5} \approx 0.411$$



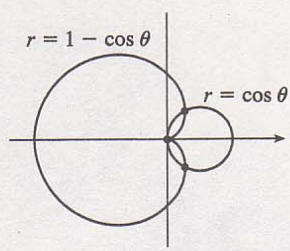
37. The two circles intersect at the pole since $(0, 0)$ satisfies the first equation and $\left(0, \frac{\pi}{2}\right)$ the second. The other intersection point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ occurs where $\sin\theta = \cos\theta$.



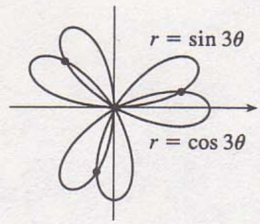
38. $2\cos 2\theta = \pm 2 \quad \Rightarrow \quad \cos 2\theta = \pm 1 \quad \Rightarrow \quad \theta = 0, \frac{\pi}{2}, \pi$, or $\frac{3\pi}{2}$, so the points are $(2, 0)$, $\left(2, \frac{\pi}{2}\right)$, $(2, \pi)$, and $\left(2, \frac{3\pi}{2}\right)$.



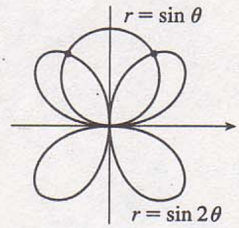
39. The curves intersect at the pole since $\left(0, \frac{\pi}{2}\right)$ satisfies $r = \cos\theta$ and $(0, 0)$ satisfies $r = 1 - \cos\theta$. $\cos\theta = 1 - \cos\theta \Longrightarrow \cos\theta = \frac{1}{2} \Longrightarrow$ $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3} \Longrightarrow$ the other intersection points are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$.



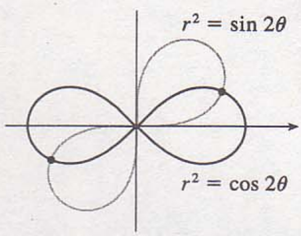
40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1 \Rightarrow 3\theta = \frac{1}{4}\pi + n\pi$ (n any integer) $\Rightarrow \theta = \frac{\pi}{12}$, $\frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.



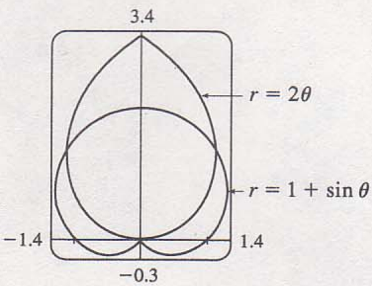
41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta \Leftrightarrow \sin\theta\,(1 - 2\cos\theta) = 0 \Leftrightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow \theta = 0$, $\pi, \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ (by symmetry) are the other intersection points.

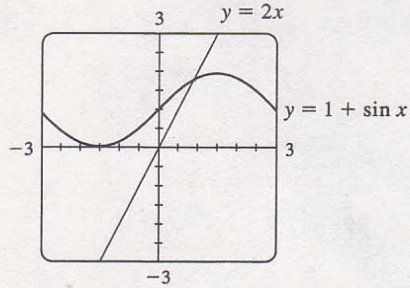


42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi$ (since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations) $\Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43.



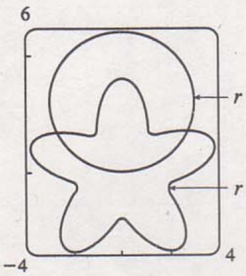


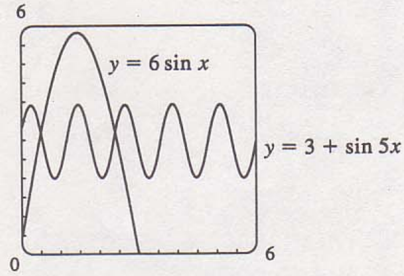
From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we estimate the θ-values of the intersection points to be about 0.89 and $\pi - 0.89 \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.)

By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$A \approx 2\int_0^{0.89} \tfrac{1}{2}(2\theta)^2\,d\theta + 2\int_{0.89}^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\,d\theta$$
$$= \left[\tfrac{4}{3}\theta^3\right]_0^{0.89} + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_{0.89}^{\pi/2} \approx 3.46$$

44.





From the first graph, it appears that the θ-values of the points of intersection are about 0.58 and 2.57. (These values may be more easily estimated by plotting $y = 3 + \sin 5x$ and $y = 6\sin x$ in rectangular coordinates; see the second graph.) By symmetry, the total area enclosed in both curves is

$$A \approx 2\int_0^{0.58} \tfrac{1}{2}(6\sin\theta)^2\,d\theta + 2\int_{0.58}^{\pi/2} \tfrac{1}{2}(3+\sin 5\theta)^2\,d\theta = \int_0^{0.58} 36\sin^2\theta d\theta + \int_{0.58}^{\pi/2}(9 + 6\sin 5\theta + \sin^2 5\theta)\,d\theta$$
$$= \left[36\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_0^{0.58} + \left[9\theta - \tfrac{6}{5}\cos 5\theta + \tfrac{1}{5}\left(\tfrac{5}{2}\theta - \tfrac{1}{4}\sin 10\theta\right)\right]_{0.58}^{\pi/2} \approx 10.41$$

45. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{3\pi/4}\sqrt{(5\cos\theta)^2 + (-5\sin\theta)^2}\,d\theta = 5\int_0^{3\pi/4}\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$
$= 5\int_0^{3\pi/4} d\theta = \frac{15}{4}\pi$

46. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi}\sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}\,d\theta = \int_0^{2\pi}\sqrt{e^{4\theta} + 4e^{4\theta}}\,d\theta = \int_0^{2\pi}\sqrt{5e^{4\theta}}\,d\theta$
$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\,d\theta = \frac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$

47. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi}\sqrt{(2^\theta)^2 + [(\ln 2)\,2^\theta]^2}\,d\theta = \int_0^{2\pi} 2^\theta\sqrt{1 + \ln^2 2}\,d\theta$
$$= \left[\sqrt{1+\ln^2 2}\left(\frac{2^\theta}{\ln 2}\right)\right]_0^{2\pi} = \frac{\sqrt{1+\ln^2 2}\,(2^{2\pi} - 1)}{\ln 2}$$

48. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi}\sqrt{\theta^2 + 1}\,d\theta \overset{21}{=} \left[\frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_0^{2\pi}$
$= \pi\sqrt{4\pi^2 + 1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2+1}\right)$

49. $L = \int_0^{2\pi}\sqrt{(\theta^2)^2 + (2\theta)^2}d\theta = \int_0^{2\pi}\theta\sqrt{\theta^2+4}d\theta = \frac{1}{2}\cdot\frac{2}{3}\left[(\theta^2+4)^{3/2}\right]_0^{2\pi} = \frac{8}{3}\left[(\pi^2+1)^{3/2} - 1\right]$

50. $L = 2\int_0^{\pi}\sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\,d\theta = 2\sqrt{2}\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta = 2\sqrt{2}\int_0^{\pi}\sqrt{2\cos^2(\theta/2)}\,d\theta$
$= [8\sin(\theta/2)]_0^{\pi} = 8$

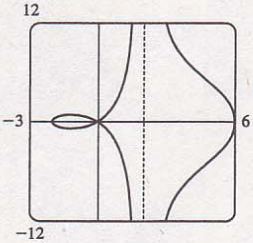
51. From Figure 4 in Example 1,

$$L = \int_{-\pi/4}^{\pi/4}\sqrt{r^2 + (r')^2}\,d\theta = 2\int_0^{\pi/4}\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\,d\theta \approx 2\,(1.211056) \approx 2.4221$$

52. $4 + 2\sec\theta = 0 \;\Rightarrow\; \sec\theta = -2 \;\Rightarrow\; \cos\theta = -\frac{1}{2} \;\Rightarrow$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

$L = \int_{2\pi/3}^{4\pi/3} \sqrt{(4 + 2\sec\theta)^2 + (2\sec\theta\tan\theta)^2}\,d\theta \approx 5.8128$



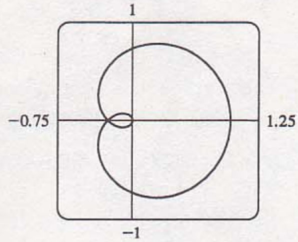
53. $L = 2\int_0^{2\pi} \sqrt{\cos^8(\theta/4) + \cos^6(\theta/4)\sin^2(\theta/4)}\,d\theta$

$= 2\int_0^{2\pi} \left|\cos^3(\theta/4)\right| \sqrt{\cos^2(\theta/4) + \sin^2(\theta/4)}\,d\theta$

$= 2\int_0^{2\pi} \left|\cos^3(\theta/4)\right| d\theta = 8\int_0^{\pi/2} \cos^3 u\,du \quad (\text{where } u = \frac{1}{4}\theta)$

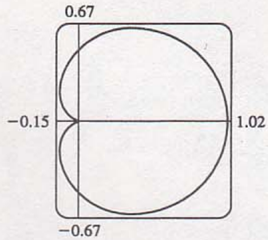
$= 8\left[\sin u - \frac{1}{3}\sin^3 u\right]_0^{\pi/2} = \frac{16}{3}$

Note that the curve is retraced after every interval of length 4π.



54. $L = 2\int_0^{\pi} \sqrt{\left[\cos^2\left(\frac{1}{2}\theta\right)\right]^2 + \left[-\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)\right]^2}\,d\theta$

$= 2\int_0^{\pi} \cos\left(\frac{1}{2}\theta\right)d\theta = 4\left[\sin\left(\frac{1}{2}\theta\right)\right]_0^{\pi} = 4$



55. (a) From (11.3.5 [ET 10.3.5]),

$S = \int_a^b 2\pi y\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta$

$= \int_a^b 2\pi y\sqrt{r^2 + (dr/d\theta)^2}\,d\theta$ (see the derivation of Equation 5) $= \int_a^b 2\pi r\sin\theta\sqrt{r^2 + (dr/d\theta)^2}\,d\theta$

(b) $r^2 = \cos 2\theta \;\Rightarrow\; 2r\dfrac{dr}{d\theta} = -2\sin 2\theta \;\Rightarrow\; \left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{\sin^2 2\theta}{r^2} = \dfrac{\sin^2 2\theta}{\cos 2\theta}$.

$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\,d\theta = 4\pi\int_0^{\pi/4}\sin\theta\,d\theta$$

$$= \left[-4\pi\cos\theta\right]_0^{\pi/4} = -4\pi\left(\tfrac{1}{\sqrt{2}} - 1\right) = 2\pi\left(2 - \sqrt{2}\right)$$

56. (a) Rotation around $\theta = \frac{\pi}{2}$ is the same as rotation around the y-axis, that is, $S = \int_a^b 2\pi x\,ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ for a parametric equation, and for the special case of a polar equation, $x = r\cos\theta$ and $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$ (see the derivation of Equation 5.) Therefore, for a polar equation, rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r\cos\theta\sqrt{r^2 + (dr/d\theta)^2}\,d\theta$.

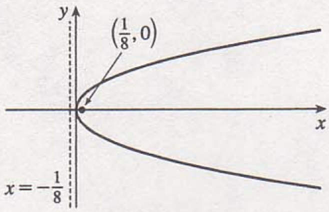
(b) In the case of the lemniscate we are concerned with $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ and $r^2 = \cos 2\theta \;\Rightarrow$ $2r\,dr/d\theta = -2\sin 2\theta \;\Rightarrow\; (dr/d\theta)^2 = (\sin^2 2\theta)/r^2 = (\sin^2 2\theta)/\cos 2\theta$. Therefore

$$S = \int_{-\pi/4}^{\pi/4} 2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\,d\theta$$

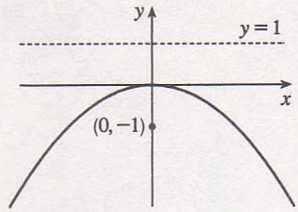
$$= 4\pi\int_0^{\pi/4}\cos\theta\sqrt{\cos 2\theta}\sqrt{1/\cos 2\theta}\,d\theta = 4\pi\int_0^{\pi/4}\cos\theta\,d\theta = 2\sqrt{2}\pi$$

11.6 Conic Sections

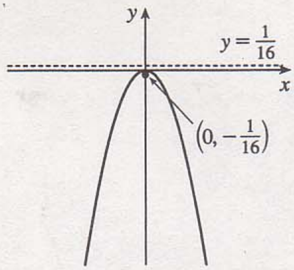
1. $x = 2y^2 \;\Rightarrow\; y^2 = \frac{1}{2}x$. $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(0,0)$, the focus is $\left(\frac{1}{8},0\right)$, and the directrix is $x = -\frac{1}{8}$.



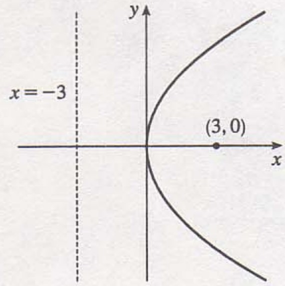
2. $4y + x^2 = 0 \;\Rightarrow\; x^2 = -4y$. $4p = -4$, so $p = -1$. The vertex is $(0,0)$, the focus is $(0,-1)$, and the directrix is $y = 1$.



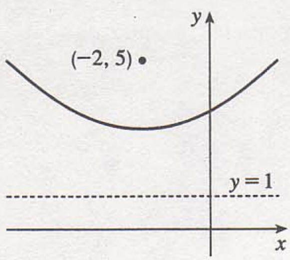
3. $4x^2 = -y \;\Rightarrow\; x^2 = -\frac{1}{4}y$. $4p = -\frac{1}{4}$, so $p = -\frac{1}{16}$. The vertex is $(0,0)$, the focus is $\left(0,-\frac{1}{16}\right)$, and the directrix is $y = \frac{1}{16}$.



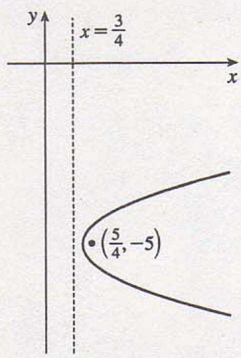
4. $y^2 = 12x$. $4p = 12$, so $p = 3$. The vertex is $(0,0)$, the focus is $(3,0)$, and the directrix is $x = -3$.



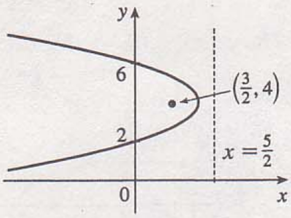
5. $(x+2)^2 = 8(y-3)$. $4p = 8$, so $p = 2$. The vertex is $(-2,3)$, the focus is $(-2,5)$, and the directrix is $y = 1$.



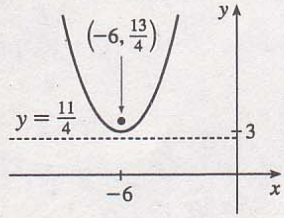
6. $x - 1 = (y+5)^2$. $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(1,-5)$, the focus is $\left(\frac{5}{4},-5\right)$, and the directrix is $x = \frac{3}{4}$.



7. $2x + y^2 - 8y + 12 = 0 \;\Rightarrow$
$(y-4)^2 = -2(x-2) \;\Rightarrow\; p = -\frac{1}{2} \;\Rightarrow$
vertex $(2,4)$, focus $\left(\frac{3}{2},4\right)$, directrix $x = \frac{5}{2}$



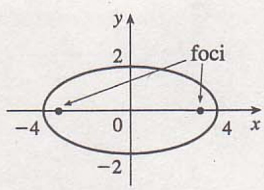
8. $x^2 + 12x - y + 39 = 0 \;\Leftrightarrow\; (x+6)^2 = y - 3$
$\Rightarrow\; p = \frac{1}{4} \;\Rightarrow$ vertex $(-6,3)$, focus $\left(-6,\frac{13}{4}\right)$,
directrix $y = \frac{11}{4}$



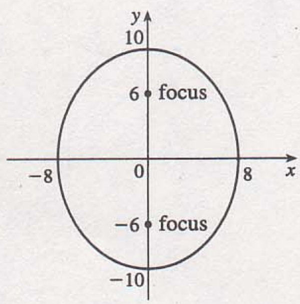
9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1,1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4},0\right)$ while the directrix is $x = \frac{1}{4}$.

10. The vertex is $(2,-2)$, so the equation is of the form $(x-2)^2 = 4p(y+2)$, where $p > 0$. The point $(0,0)$ is on the parabola, so $4 = 4p(2)$ and $4p = 2$. Thus, an equation is $(x-2)^2 = 2(y+2)$. $4p = 2$, so $p = \frac{1}{2}$ and the focus is $\left(2,-\frac{3}{2}\right)$ while the directrix is $y = -\frac{5}{2}$.

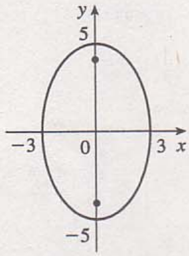
11. $x^2/16 + y^2/4 = 1 \;\Rightarrow\; a = 4, b = 2,$
$c = \sqrt{16-4} = 2\sqrt{3} \;\Rightarrow$ center $(0,0)$, vertices
$(\pm 4, 0)$, foci $\left(\pm 2\sqrt{3}, 0\right)$



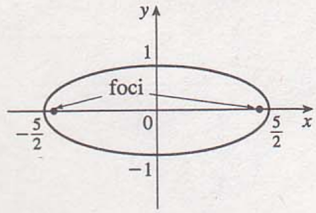
12. $\dfrac{x^2}{64} + \dfrac{y^2}{100} = 1 \;\Rightarrow\; a = 10, b = 8,$
$c = \sqrt{a^2 - b^2} = 6$. The ellipse is centered at $(0,0)$, with vertices at $(0,\pm 10)$. The foci are $(0,\pm 6)$.



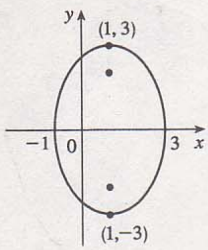
13. $25x^2 + 9y^2 = 225 \quad \Leftrightarrow \quad \frac{1}{9}x^2 + \frac{1}{25}y^2 = 1 \quad \Rightarrow$
$a = 5, b = 3, c = 4 \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(0, \pm 5)$, foci $(0, \pm 4)$



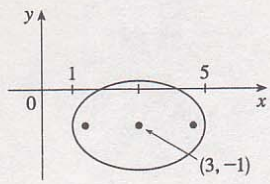
14. $4x^2 + 25y^2 = 25 \quad \Rightarrow \quad \dfrac{x^2}{25/4} + \dfrac{y^2}{1} = 1 \quad \Rightarrow$
$a = \frac{5}{2}, b = 1, c = \sqrt{a^2 - b^2} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$. The ellipse is centered at $(0, 0)$, with vertices at $\left(\pm\frac{5}{2}, 0\right)$. The foci are $\left(\pm\frac{\sqrt{21}}{2}, 0\right)$.



15. $9x^2 - 18x + 4y^2 = 27 \quad \Leftrightarrow$
$\dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1 \quad \Rightarrow \quad a = 3, b = 2, c = \sqrt{5}$
$\Rightarrow \quad$ center $(1, 0)$, vertices $(1, \pm 3)$, foci $\left(1, \pm\sqrt{5}\right)$



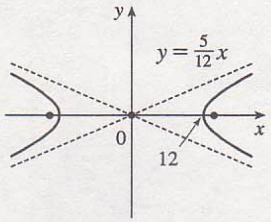
16. $x^2 - 6x + 2y^2 + 4y = -7 \quad \Leftrightarrow$
$\dfrac{(x-3)^2}{4} + \dfrac{(y+1)^2}{2} = 1 \quad \Rightarrow \quad a = 2,$
$b = \sqrt{2} = c \quad \Rightarrow \quad$ center $(3, -1)$, vertices $(1, -1)$ and $(5, -1)$, foci $\left(3 \pm \sqrt{2}, -1\right)$



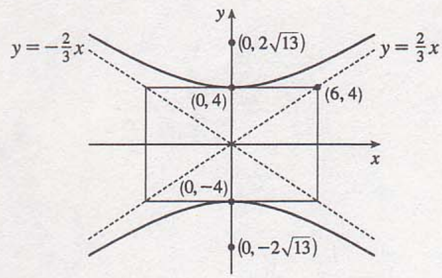
17. The center is $(0, 0)$, $a = 3$, and $b = 2$, so an equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(0, \pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2, 1)$, with $a = 3$ and $b = 2$. An equation is $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$.
$c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(2 \pm \sqrt{5}, 1\right)$.

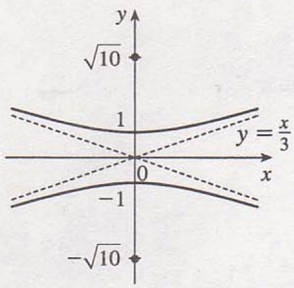
19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1 \quad \Rightarrow \quad a = 12, b = 5,$
$c = \sqrt{144 + 25} = 13 \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(\pm 12, 0)$, foci $(\pm 13, 0)$, asymptotes $y = \pm \frac{5}{12}x$



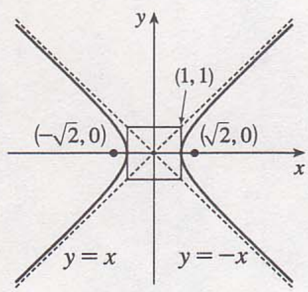
20. $\dfrac{y^2}{16} - \dfrac{x^2}{36} = 1 \quad \Rightarrow \quad a = 4, b = 6,$
$c = \sqrt{a^2 + b^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$. The center is $(0, 0)$, the vertices are $(0, \pm 4)$, the foci are $(0, \pm 2\sqrt{13})$, and the asymptotes are the lines $y = \pm \frac{a}{b}x = \pm \frac{2}{3}x$.



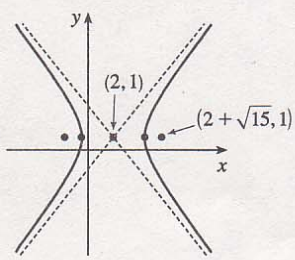
21. $9y^2 - x^2 = 9 \quad \Rightarrow \quad y^2 - \frac{1}{9}x^2 = 1 \quad \Rightarrow \quad a = 1,$
$b = 3, c = \sqrt{10} \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(0, \pm 1)$, foci $(0, \pm\sqrt{10})$, asymptotes $y = \pm \frac{1}{3}x$



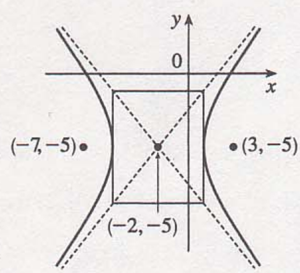
22. $x^2 - y^2 = 1 \quad \Rightarrow \quad a = b = 1, c = \sqrt{2} \quad \Rightarrow$
center $(0, 0)$, vertices $(\pm 1, 0)$, foci $(\pm\sqrt{2}, 0)$, asymptotes $y = \pm x$



23. $2y^2 - 4y - 3x^2 + 12x = -8 \quad \Leftrightarrow$
$\dfrac{(x-2)^2}{6} - \dfrac{(y-1)^2}{9} = 1 \quad \Rightarrow \quad a = \sqrt{6}, b = 3,$
$c = \sqrt{15} \quad \Rightarrow \quad$ center $(2, 1)$, vertices $(2 \pm \sqrt{6}, 1)$, foci $(2 \pm \sqrt{15}, 1)$, asymptotes $y - 1 = \pm \frac{3}{\sqrt{6}}(x - 2)$



24. $16x^2 + 64x - 9y^2 - 90y = 305 \quad \Leftrightarrow$
$\dfrac{(x+2)^2}{9} - \dfrac{(y+5)^2}{16} = 1 \quad \Rightarrow \quad a = 3, b = 4,$
$c = 5 \quad \Rightarrow \quad$ center $(-2, -5)$, vertices $(-5, -5)$ and $(1, -5)$, foci $(-7, -5)$ and $(3, -5)$, asymptotes $y + 5 = \pm \frac{4}{3}(x + 2)$



25. The parabola with vertex $(0,0)$ and focus $(0,-2)$ opens downward and has $p=-2$, so its equation is $x^2=4py=-8y$.

26. The parabola with vertex $(1,0)$ and directrix $x=-5$ opens to the right and has $p=6$, so its equation is $y^2=4p\,(x-1)=24\,(x-1)$.

27. Vertex at $(2,0)$, $p=1$, opens to right $\Rightarrow$ $y^2=4p\,(x-2)=4\,(x-2)$

28. Vertex $(1,2)$, parabola opens down $\Rightarrow$ $p=-3$ $\Rightarrow$ $(x-1)^2=4p\,(y-2)=-12\,(y-2)$ $\Leftrightarrow$ $x^2-2x+12y-23=0$

29. The parabola must have equation $y^2=4px$, so $(-4)^2=4p\,(1)$ $\Rightarrow$ $p=4$ $\Rightarrow$ $y^2=16x$.

30. Vertical axis $\Rightarrow$ $(x-h)^2=4p\,(y-k)$. Substituting $(-2,3)$ and $(0,3)$ gives $(-2-h)^2=4p\,(3-k)$ and $(-h)^2=4p\,(3-k)$ $\Rightarrow$ $(-2-h)^2=(-h)^2$ $\Rightarrow$ $4+4h+h^2=h^2$ $\Rightarrow$ $h=-1$ $\Rightarrow$ $1=4p\,(3-k)$. Substituting $(1,9)$ gives $[1-(-1)]^2=4p\,(9-k)$ $\Rightarrow$ $4=4p\,(9-k)$. Solving for p from these equations gives $p=\dfrac{1}{4\,(3-k)}=\dfrac{1}{9-k}$ $\Rightarrow$ $4\,(3-k)=9-k$ $\Rightarrow$ $k=1$ $\Rightarrow$ $p=\frac{1}{8}$ $\Rightarrow$ $(x+1)^2=\frac{1}{2}\,(y-1)$ $\Rightarrow$ $2x^2+4x-y+3=0$.

31. The ellipse with foci $(\pm 2,0)$ and vertices $(\pm 5,0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=2$, so $b=\sqrt{a^2-c^2}=\sqrt{21}$. An equation is $\dfrac{x^2}{25}+\dfrac{y^2}{21}=1$.

32. The ellipse with foci $(0,\pm 5)$ and vertices $(0,\pm 13)$ has center $(0,0)$ and a vertical major axis, with $c=5$ and $a=13$, so $b=\sqrt{a^2-c^2}=12$. An equation is $\dfrac{x^2}{144}+\dfrac{y^2}{169}=1$.

33. Center $(3,0)$, $c=1$, $a=3$ $\Rightarrow$ $b=\sqrt{8}=2\sqrt{2}$ $\Rightarrow$ $\frac{1}{8}\,(x-3)^2+\frac{1}{9}y^2=1$

34. Center $(0,2)$, $c=1$, $a=3$, major axis horizontal $\Rightarrow$ $b=2\sqrt{2}$ and $\frac{1}{9}x^2+\frac{1}{8}\,(y-2)^2=1$

35. Center $(2,2)$, $c=2$, $a=3$ $\Rightarrow$ $b=\sqrt{5}$ $\Rightarrow$ $\frac{1}{9}\,(x-2)^2+\frac{1}{5}\,(y-2)^2=1$

36. Center $(0,0)$, $c=2$, major axis horizontal $\Rightarrow$ $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and $b^2=a^2-c^2=a^2-4$. Since the ellipse passes through $(2,1)$, we have $2a=|PF_1|+|PF_2|=\sqrt{17}+1$ $\Rightarrow$ $a^2=\frac{9+\sqrt{17}}{2}$ and $b^2=\frac{1+\sqrt{17}}{2}$, so the ellipse has equation $\dfrac{2x^2}{9+\sqrt{17}}+\dfrac{2y^2}{1+\sqrt{17}}=1$.

37. Center $(0,0)$, vertical axis, $c=3$, $a=1$ $\Rightarrow$ $b=\sqrt{8}=2\sqrt{2}$ $\Rightarrow$ $y^2-\frac{1}{8}x^2=1$

38. Center $(0,0)$, horizontal axis, $c=6$, $a=4$ $\Rightarrow$ $b=2\sqrt{5}$ $\Rightarrow$ $\frac{1}{16}x^2-\frac{1}{20}y^2=1$

39. Center $(4,3)$, horizontal axis, $c=3$, $a=2$ $\Rightarrow$ $b=\sqrt{5}$ $\Rightarrow$ $\frac{1}{4}\,(x-4)^2-\frac{1}{5}\,(y-3)^2=1$

40. Center $(2,3)$, vertical axis, $c=5$, $a=3$ $\Rightarrow$ $b=4$ $\Rightarrow$ $\frac{1}{9}\,(y-3)^2-\frac{1}{16}\,(x-2)^2=1$

41. Center $(0,0)$, horizontal axis, $a=3$, $\frac{b}{a}=2$ $\Rightarrow$ $b=6$ $\Rightarrow$ $\frac{1}{9}x^2-\frac{1}{36}y^2=1$

42. Center $(4,2)$, horizontal axis, asymptotes $y-2=\pm\,(x-4)$ $\Rightarrow$ $c=2$, $b/a=1$ $\Rightarrow$ $a=b$ $\Rightarrow$ $c^2=4=a^2+b^2=2a^2$ $\Rightarrow$ $a^2=2$ $\Rightarrow$ $\frac{1}{2}\,(x-4)^2-\frac{1}{2}\,(y-2)^2=1$

43. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a - c$ from it) while the farthest point is the other vertex (at a distance of $a + c$). So for this lunar orbit,
$(a - c) + (a + c) = 2a = (1728 + 110) + (1728 + 314)$, or $a = 1940$; and
$(a + c) - (a - c) = 2c = 314 - 110$, or $c = 102$. Thus, $b^2 = a^2 - c^2 = 3{,}753{,}196$, and the equation is
$\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$.

44. (a) Choose V to be the origin, with x-axis through V and F. Then F is $(p, 0)$, A is $(p, 5)$, so substituting A into the equation $y^2 = 4px$ gives $25 = 4p^2$ so $p = \frac{5}{2}$ and $y^2 = 10x$.

(b) $x = 11 \;\Rightarrow\; y = \sqrt{110} \;\Rightarrow\; |CD| = 2\sqrt{110}$

45. (a) Set up the coordinate system so that A is $(-200, 0)$ and B is $(200, 0)$.
$|PA| - |PB| = (1200)(980) = 1{,}176{,}000 \text{ ft} = \frac{2450}{11} \text{ mi} = 2a \;\Rightarrow\; a = \frac{1225}{11}$, and $c = 200$ so
$b^2 = c^2 - a^2 = \frac{3{,}339{,}375}{121} \;\Rightarrow\; \dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1$.

(b) Due north of $B \;\Rightarrow\; x = 200 \;\Rightarrow\; \dfrac{(121)(200)^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1 \;\Rightarrow\; y = \dfrac{133{,}575}{539} \approx 248$ mi

46. $|PF_1| - |PF_2| = \pm 2a \;\Leftrightarrow\; \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \;\Leftrightarrow$
$\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a \;\Leftrightarrow\; (x+c)^2 + y^2 = (x-c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2}$
$\Leftrightarrow\; 4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2} \;\Leftrightarrow\; c^2x^2 - 2a^2cx + a^4 = a^2\left(x^2 - 2cx + c^2 + y^2\right) \;\Leftrightarrow$
$\left(c^2 - a^2\right)x^2 - a^2y^2 = a^2\left(c^2 - a^2\right) \;\Leftrightarrow\; b^2x^2 - a^2y^2 = a^2b^2$ (where $b^2 = c^2 - a^2$) $\;\Leftrightarrow\; \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

47. The function whose graph is the upper branch of this hyperbola is concave upward. The function is $y = f(x) = a\sqrt{1 + \dfrac{x^2}{b^2}} = \dfrac{a}{b}\sqrt{b^2 + x^2}$, so $y' = \dfrac{a}{b}x\left(b^2 + x^2\right)^{-1/2}$ and
$y'' = \dfrac{a}{b}\left[\left(b^2 + x^2\right)^{-1/2} - x^2\left(b^2 + x^2\right)^{-3/2}\right] = ab\left(b^2 + x^2\right)^{-3/2} > 0$ for all x, and so f is concave upward.

48. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1, 1)$ and $(-1, -1)$ in the distance formula (first equation of that derivation) so
$\sqrt{(x-1)^2 + (y-1)^2} + \sqrt{(x+1)^2 + (y+1)^2} = 4$ will lead to $3x^2 - 2xy + 3y^2 = 8$.

49. (a) ellipse

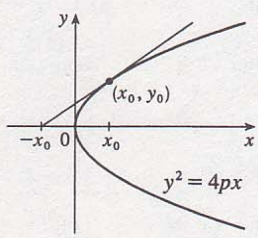
(b) hyperbola

(c) empty graph (no curve)

(d) In case (a), $a^2 = k$, $b^2 = k - 16$, and $c^2 = a^2 - b^2 = 16$, so the foci are at $(\pm 4, 0)$. In case (b), $k - 16 < 0$, so $a^2 = k$, $b^2 = 16 - k$, and $c^2 = a^2 + b^2 = 16$, and so again the foci are at $(\pm 4, 0)$.

50. (a) $y^2 = 4px \;\Rightarrow\; 2yy' = 4p \;\Rightarrow\; y' = \dfrac{2p}{y}$, so the tangent line is

$y - y_0 = \dfrac{2p}{y_0}(x - x_0) \;\Rightarrow\; yy_0 - y_0^2 = 2p(x - x_0) \;\Leftrightarrow$
$yy_0 - 4px_0 = 2px - 2px_0 \;\Rightarrow\; yy_0 = 2p(x + x_0)$.



(b) The x-intercept is $-x_0$.

51. Use the parametrization $x = 2\cos t$, $y = \sin t$, $0 \le t \le 2\pi$ to get

$$L = 4\int_0^{\pi/2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = 4\int_0^{\pi/2} \sqrt{4\sin^2 t + \cos^2 t}\,dt = 4\int_0^{\pi/2} \sqrt{3\sin^2 t + 1}\,dt$$

Using Simpson's Rule with $n = 10$,

$$L \approx \tfrac{4}{3}\left(\tfrac{\pi}{20}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{\pi}{10}\right) + \cdots + 2f\left(\tfrac{2\pi}{5}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right]$$

with $f(t) = \sqrt{3\sin^2 t + 1}$, so $L \approx 9.69$.

52. The length of the major axis is $2a$, so $a = \frac{1}{2}\left(1.18 \times 10^{10}\right) = 5.9 \times 10^9$. The length of the minor axis is $2b$, so $b = \frac{1}{2}\left(1.14 \times 10^{10}\right) = 5.7 \times 10^9$. Therefore the equation of the ellipse is $\dfrac{x^2}{(5.9 \times 10^9)^2} + \dfrac{y^2}{(5.7 \times 10^9)^2} = 1$. Converting into parametric equations, $x = 5.9 \times 10^9 \cos\theta$ and $y = 5.7 \times 10^9 \sin\theta$. So

$$\begin{aligned} L &= 4\int_0^{\pi/2} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = 4\int_0^{\pi/2} \sqrt{3.48 \times 10^{19}\sin^2\theta + 3.249 \times 10^{19}\cos^2\theta}\,d\theta \\ &= 4\sqrt{3.249 \times 10^{19}}\int_0^{\pi/2} \sqrt{1.0714\sin^2\theta + \cos^2\theta}\,d\theta \end{aligned}$$

Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10} = \frac{\pi}{20}$ and $f(\theta) = \sqrt{1.0714\sin^2\theta + \cos^2\theta}$ we get

$$\begin{aligned} L &\approx 4\left(5.7 \times 10^9\right) \cdot S_{10} \\ &= 4\left(5.7 \times 10^9\right)\tfrac{\pi}{20 \cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{\pi}{10}\right) + \cdots + 2f\left(\tfrac{2\pi}{5}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \\ &\approx \tfrac{\pi}{15}\left(5.7 \times 10^9\right)(30.529) \approx 3.64 \times 10^{10}\text{ km} \end{aligned}$$

53. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = -\dfrac{b^2 x}{a^2 y}$ $(y \neq 0)$. Thus, the slope of the tangent line at P is $-\dfrac{b^2 x_1}{a^2 y_1}$. The slope of F_1P is $\dfrac{y_1}{x_1 + c}$ and of F_2P is $\dfrac{y_1}{x_1 - c}$. By the formula from Problems Plus, we have

$$\begin{aligned} \tan\alpha &= \frac{\dfrac{y_1}{x_1 + c} + \dfrac{b^2 x_1}{a^2 y_1}}{1 - \dfrac{b^2 x_1 y_1}{a^2 y_1 (x_1 + c)}} = \frac{a^2 y_1^2 + b^2 x_1 (x_1 + c)}{a^2 y_1 (x_1 + c) - b^2 x_1 y_1} = \frac{a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 + a^2 c y_1} \quad \left(\begin{array}{c}\text{using } b^2 x_1^2 + a^2 y_1^2 = a^2 b^2 \\ \text{and } a^2 - b^2 = c^2\end{array}\right) \\ &= \frac{b^2 (c x_1 + a^2)}{c y_1 (c x_1 + a^2)} = \frac{b^2}{c y_1} \end{aligned}$$

and

$$\tan\beta = \frac{-\dfrac{y_1}{x_1 - c} - \dfrac{b^2 x_1}{a^2 y_1}}{1 - \dfrac{b^2 x_1 y_1}{a^2 y_1 (x_1 - c)}} = \frac{-a^2 y_1^2 - b^2 x_1 (x_1 - c)}{a^2 y_1 (x_1 - c) - b^2 x_1 y_1} = \frac{-a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 - a^2 c y_1} = \frac{b^2 (c x_1 - a^2)}{c y_1 (c x_1 - a^2)} = \frac{b^2}{c y_1}$$

So $\alpha = \beta$.

54. The slopes of the line segments F_1P and F_2P are $\dfrac{y_1}{x_1+c}$ and $\dfrac{y_1}{x_1-c}$, where P is (x_1, y_1). Differentiating implicitly, $\dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{b^2x}{a^2y} \Rightarrow$ the slope of the tangent at P is $\dfrac{b^2x_1}{a^2y_1}$, so by the formula from Problems Plus,

$$\begin{aligned}\tan\alpha &= \frac{\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1+c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{b^2x_1(x_1+c) - a^2y_1^2}{a^2y_1(x_1+c) + b^2x_1y_1} \\ &= \frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)} \quad \left(\begin{array}{c}\text{using } x_1^2/a^2 - y_1^2/b^2 = 1 \\ \text{and } a^2+b^2=c^2\end{array}\right) \quad = \frac{b^2}{cy_1}\end{aligned}$$

and

$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} + \dfrac{y_1}{x_1-c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-b^2x_1(x_1-c) + a^2y_1^2}{a^2y_1(x_1-c) + b^2x_1y_1} = \frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)} = \frac{b^2}{cy_1}$$

So $\alpha = \beta$.

11.7 Conic Sections in Polar Coordinates ET 10.7

1. The directrix $y = 6$ is above the focus at the origin, so we use the form with "$+\,e\sin\theta$" in the denominator.
$r = \dfrac{ed}{1+e\sin\theta} = \dfrac{\frac{7}{4}\cdot 6}{1+\frac{7}{4}\sin\theta} = \dfrac{42}{4+7\sin\theta}$

2. The directrix $x = 4$ is to the right of the focus at the origin, so we use the form with "$+\,e\cos\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{1\cdot 4}{1+1\cos\theta} = \dfrac{4}{1+\cos\theta}$

3. The directrix $x = -5$ is to the left of the focus at the origin, so we use the form with "$-\,e\cos\theta$" in the denominator. $r = \dfrac{ed}{1-e\cos\theta} = \dfrac{\frac{3}{4}\cdot 5}{1-\frac{3}{4}\cos\theta} = \dfrac{15}{4-3\cos\theta}$

4. The directrix $y = -2$ is below the focus at the origin, so we use the form with "$-\,e\sin\theta$" in the denominator.
$r = \dfrac{ed}{1-e\sin\theta} = \dfrac{2\cdot 2}{1-2\sin\theta} = \dfrac{4}{1-2\sin\theta}$

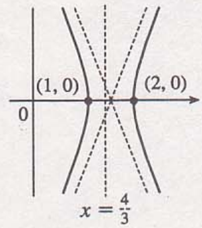
5. $r = 5\sec\theta \iff x = r\cos\theta = 5$, so $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{4\cdot 5}{1+4\cos\theta} = \dfrac{20}{1+4\cos\theta}$

6. $r = 2\csc\theta \iff y = r\sin\theta = 2$, so $r = \dfrac{ed}{1+e\sin\theta} = \dfrac{\frac{3}{5}\cdot 2}{1+\frac{3}{5}\sin\theta} = \dfrac{6}{5+3\sin\theta}$

7. Focus $(0,0)$, vertex $\left(5, \frac{\pi}{2}\right) \Rightarrow$ directrix $y = 10 \Rightarrow r = \dfrac{ed}{1+e\sin\theta} = \dfrac{10}{1+\sin\theta}$

8. The directrix is $x = 4$, so $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{\frac{2}{5}\cdot 4}{1+\frac{2}{5}\cos\theta} = \dfrac{8}{5+2\cos\theta}$

9. $r = \dfrac{4}{1 + 3\cos\theta}$



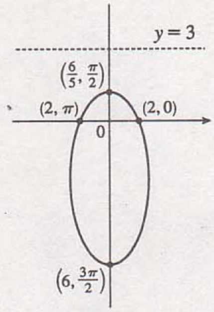
(a) $e = 3$

(b) Since $e = 3 > 1$, the conic is a hyperbola.

(c) $ed = 4 \;\Rightarrow\; d = \frac{4}{3} \;\Rightarrow\;$ directrix $x = \frac{4}{3}$

(d) The vertices are $(1, 0)$ and $(-2, \pi) = (2, 0)$; the center is $\left(\frac{3}{2}, 0\right)$; the asymptotes are parallel to $\theta = \pm\cos^{-1}\left(-\frac{1}{3}\right)$

10. $r = \dfrac{6}{3 + 2\sin\theta} = \dfrac{2}{1 + \frac{2}{3}\sin\theta} = \dfrac{\frac{2}{3}\cdot 3}{1 + \frac{2}{3}\sin\theta}$



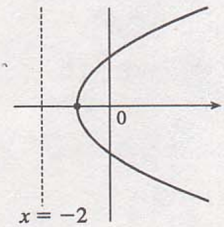
(a) $e = \frac{2}{3}$

(b) Ellipse

(c) $y = 3$

(d) Vertices $\left(\frac{6}{5}, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$; center $\left(\frac{12}{5}, \frac{3\pi}{2}\right)$

11. $r = \dfrac{2}{1 - \cos\theta}$



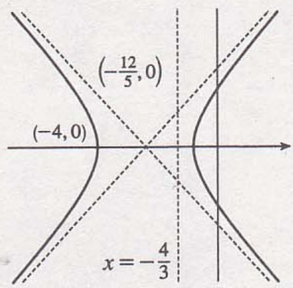
(a) $e = 1$

(b) Parabola

(c) $ed = 2 \;\Rightarrow\; d = 2 \;\Rightarrow\;$ directrix $x = -2$

(d) Vertex $(-1, 0) = (1, \pi)$

12. $r = \dfrac{8}{4 - 6\cos\theta} = \dfrac{2}{1 - \frac{3}{2}\cos\theta} = \dfrac{\frac{3}{2}\cdot\frac{4}{3}}{1 - \frac{3}{2}\cos\theta}$



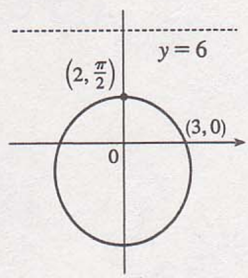
(a) $e = \frac{3}{2}$

(b) Hyperbola

(c) $x = -\frac{4}{3}$

(d) The vertices are $(-4, 0)$ and $\left(\frac{4}{5}, \pi\right) = \left(-\frac{4}{5}, 0\right)$, so the center is $\left(-\frac{12}{5}, 0\right)$. The asymptotes are parallel to $\theta = \pm\cos^{-1}\frac{2}{3}$. [Their slopes are $\pm\tan\left(\cos^{-1}\frac{2}{3}\right) = \pm\frac{\sqrt{5}}{2}$.]

13. $r = \dfrac{3}{1 + \frac{1}{2}\sin\theta}$



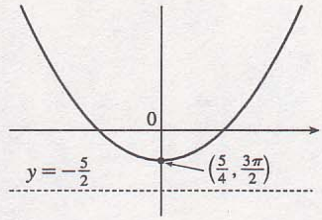
(a) $e = \frac{1}{2}$

(b) Ellipse

(c) $ed = 3 \;\Rightarrow\; d = 6 \;\Rightarrow\;$ directrix $y = 6$

(d) Vertices $\left(2, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$; center $\left(2, \frac{3\pi}{2}\right)$

14. $r = \dfrac{5}{2 - 2\sin\theta} = \dfrac{\frac{5}{2}}{1 - \sin\theta}$



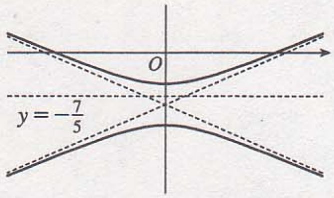
(a) $e = 1$

(b) Parabola

(c) $y = -\frac{5}{2}$

(d) The focus is $(0, 0)$, so the vertex is $\left(\frac{5}{4}, \frac{3\pi}{2}\right)$ and the parabola opens up.

15. $r = \dfrac{7/2}{1 - \frac{5}{2}\sin\theta}$



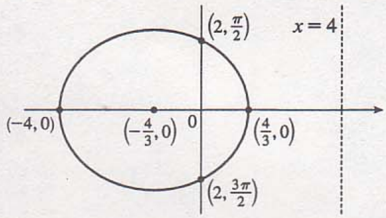
(a) $e = \frac{5}{2}$

(b) Hyperbola

(c) $ed = \frac{7}{2} \;\Rightarrow\; d = \frac{7}{5} \;\Rightarrow\;$ directrix $y = -\frac{7}{5}$

(d) Center $\left(\frac{5}{3}, \frac{3\pi}{2}\right)$; vertices $\left(-\frac{7}{3}, \frac{\pi}{2}\right) = \left(\frac{7}{3}, \frac{3\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$

16. $r = \dfrac{4}{2 + \cos\theta} = \dfrac{2}{1 + \frac{1}{2}\cos\theta} = \dfrac{\frac{1}{2} \cdot 4}{1 + \frac{1}{2}\cos\theta}$



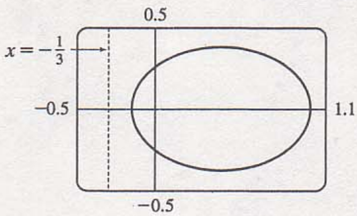
(a) $e = \frac{1}{2}$

(b) Ellipse

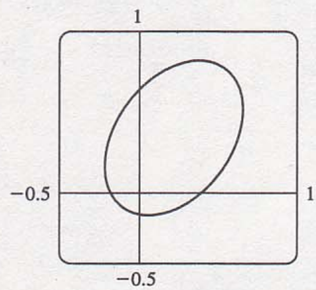
(c) $x = 4$

(d) The vertices are $\left(\frac{4}{3}, 0\right)$ and $(4, \pi) = (-4, 0)$, so the center is $\left(-\frac{4}{3}, 0\right)$.

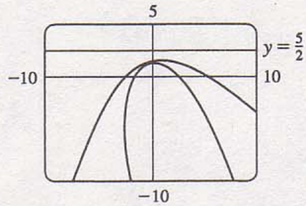
17. (a) The equation is $r = \dfrac{1}{4 - 3\cos\theta} = \dfrac{1/4}{1 - \frac{3}{4}\cos\theta}$, so $e = \frac{3}{4}$ and $ed = \frac{1}{4} \;\Rightarrow\; d = \frac{1}{3}$. The conic is an ellipse, and the equation of its directrix is $x = r\cos\theta = -\frac{1}{3} \;\Rightarrow\; r = -\dfrac{1}{3\cos\theta}$. We must be careful in our choice of parameter values in this equation ($-1 \le \theta \le 1$ works well).



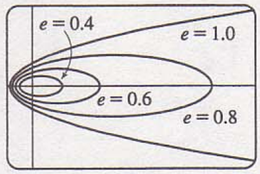
(b) The equation is obtained by replacing θ with $\theta - \frac{\pi}{3}$ in the equation of the original conic (see Example 4), so $r = \dfrac{1}{4 - 3\cos\left(\theta - \frac{\pi}{3}\right)}$.



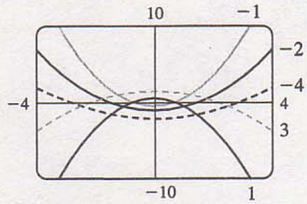
18. $r = \dfrac{5}{2 + 2\sin\theta} = \dfrac{5/2}{1 + \sin\theta}$, so $e = 1$ and $d = \frac{5}{2}$. The equation of the directrix is $y = r\sin\theta = \frac{5}{2} \;\Rightarrow\; r = \dfrac{5}{2\sin\theta}$. If the parabola is rotated about its focus (the origin) through $\frac{\pi}{6}$, its equation is the same as that of the original, with θ replaced by $\theta - \frac{\pi}{6}$ (see Example 4), so $r = \dfrac{5}{2 + 2\sin(\theta - \pi/6)}$. In graphing each of these curves, we must be careful to select parameter ranges which prevent the denominator from vanishing while still showing enough of the curve.



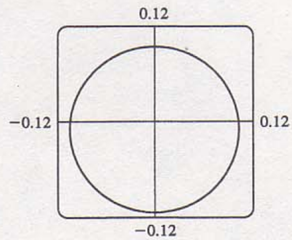
19. For $e < 1$ the curve is an ellipse. It is nearly circular when e is close to 0. As e increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At $e = 1$, the curve becomes a parabola with focus at the origin.



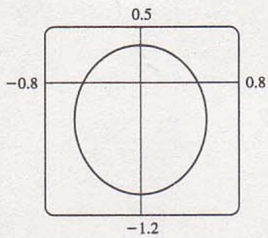
20. (a) The value of d does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d < 0$ it opens upward, for $d > 0$ it opens downward).



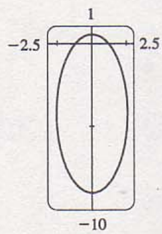
(b) We consider only positive values of e. When $0 < e < 1$, the conic is an ellipse. As $e \to 0^+$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e = 1$ it turns into a parabola. For $e > 1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.



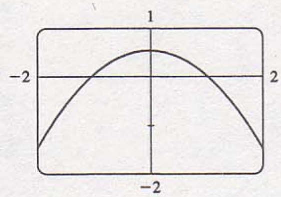
$e = 0.1$



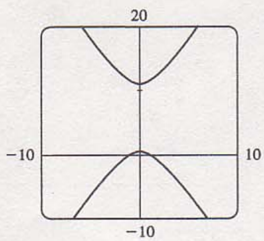
$e = 0.5$



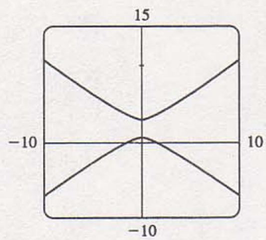
$e = 0.9$



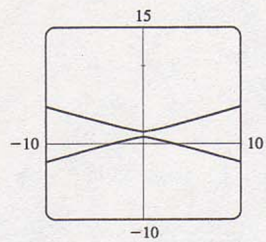
$e = 1$



$e = 1.1$



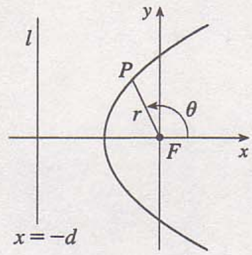
$e = 1.5$



$e = 10$

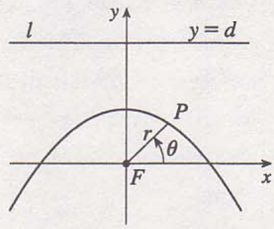
21. $|PF| = e\,|Pl| \;\Rightarrow\; r = e\,[d - r\cos(\pi - \theta)] = e\,(d + r\cos\theta) \;\Rightarrow$

$r\,(1 - e\cos\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\cos\theta}$



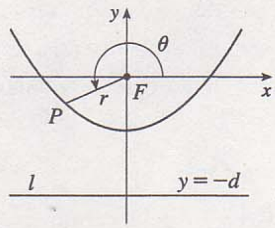
22. $|PF| = e\,|Pl| \;\Rightarrow\; r = e\,[d - r\sin\theta] \;\Rightarrow\; r\,(1 + e\sin\theta) = ed \;\Rightarrow$

$r = \dfrac{ed}{1 + e\sin\theta}$



23. $|PF| = e\,|Pl| \;\Rightarrow\; r = e\,[d - r\sin(\theta - \pi)] = e\,(d + r\sin\theta) \;\Rightarrow$

$r\,(1 - e\sin\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\sin\theta}$



24. The parabolas intersect at the two points where $\dfrac{c}{1+\cos\theta} = \dfrac{d}{1-\cos\theta} \;\Rightarrow\; \cos\theta = \dfrac{c-d}{c+d} \;\Rightarrow\; r = \dfrac{c+d}{2}$.

For the first parabola, $\dfrac{dr}{d\theta} = \dfrac{c\sin\theta}{(1+\cos\theta)^2}$, so

$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{c\sin^2\theta + c\cos\theta\,(1+\cos\theta)}{c\sin\theta\cos\theta - c\sin\theta\,(1+\cos\theta)} = \frac{1+\cos\theta}{-\sin\theta}$$

and similarly for the second, $\dfrac{dy}{dx} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$. Since the product of these slopes is -1, the parabolas intersect at right angles.

25. (a) If the directrix is $x = -d$, then $r = \dfrac{ed}{1 - e\cos\theta}$ [see Figure 2(b)], and, from (4), $a^2 = \dfrac{e^2d^2}{(1-e^2)^2} \;\Rightarrow$

$ed = a\left(1 - e^2\right)$. Therefore, $r = \dfrac{a\left(1-e^2\right)}{1 - e\cos\theta}$.

(b) $e = 0.017$ and the major axis $= 2a = 2.99 \times 10^8 \;\Rightarrow\; a = 1.495 \times 10^8$.

Therefore $r = \dfrac{1.495 \times 10^8\left[1 - (0.017)^2\right]}{1 - 0.017\cos\theta} \approx \dfrac{1.49 \times 10^8}{1 - 0.017\cos\theta}$.

26. (a) At perihelion, $\theta = \pi$, so $r = \dfrac{a\left(1-e^2\right)}{1 - e\cos\pi} = \dfrac{a\left(1-e^2\right)}{1 - e(-1)} = \dfrac{a\,(1-e)\,(1+e)}{1+e} = a\,(1-e)$.

At aphelion, $\theta = 0$, so $r = \dfrac{a\left(1-e^2\right)}{1 - e\cos 0} = \dfrac{a\,(1-e)\,(1+e)}{1-e} = a\,(1+e)$.

(b) At perihelion, $r = a\,(1-e) \approx \left(1.495 \times 10^8\right)(1 - 0.017) \approx 1.47 \times 10^8$ km. At aphelion, $r = a\,(1+e) \approx \left(1.495 \times 10^8\right)(1 + 0.017) \approx 1.52 \times 10^8$ km.

27. Here $2a$ = length of major axis = 36.18 AU $\Rightarrow$ $a = 18.09$ AU and $e = 0.97$. By Exercise 25(a), the equation of the orbit is $r = \dfrac{18.09\left[1-(0.97)^2\right]}{1-0.97\cos\theta} \approx \dfrac{1.07}{1-0.97\cos\theta}$. By Exercise 26(a), the maximum distance from the comet to the sun is $18.09\,(1+0.97) \approx 35.64$ AU or about 3.314 billion miles.

28. Here $2a$ = length of major axis = 356.5 AU $\Rightarrow$ $a = 178.25$ AU and $e = 0.9951$. By Exercise 25(a), the equation of the orbit is $r = \dfrac{178.25\left[1-(0.9951)^2\right]}{1-0.9951\cos\theta} \approx \dfrac{1.7426}{1-0.9951\cos\theta}$. By Exercise 26(a), the minimum distance from the comet to the sun is $178.25\,(1-0.9951) \approx 0.8734$ AU or about 81 million miles.

29. The minimum distance is at perihelion where $4.6 \times 10^7 = r = a\,(1-e) = a\,(1-0.206) = a\,(0.794)$ $\Rightarrow$ $a = 4.6 \times 10^7/0.794$. So the maximum distance, which is at aphelion, is $r = a\,(1+e) = \left(4.6 \times 10^7/0.794\right) \times 10^7\,(1.206) \approx 7.0 \times 10^7$ km.

30. At perihelion, $r = a\,(1-e) = 4.43 \times 10^9$, and at aphelion, $r = a\,(1+e) = 7.37 \times 10^9$. Adding, we get $2a = 11.80 \times 10^9$, so $a = 5.90 \times 10^9$ km. Therefore $1 + e = a\,(1+e)\,/e = \frac{7.37}{5.90} \approx 1.249$ and $e \approx 0.249$.

31. From Exercise 29, we have $e = 0.206$ and $a\,(1-e) = 4.6 \times 10^7$ km. Thus, $a = 4.6 \times 10^7/0.794$. From Exercise 25, we can write the equation of Mercury's orbit as $r = a\dfrac{1-e^2}{1-e\cos\theta}$. So since

$$\frac{dr}{d\theta} = \frac{-a\left(1-e^2\right)e\sin\theta}{(1-e\cos\theta)^2} \quad \Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^2} + \frac{a^2\left(1-e^2\right)^2 e^2\sin^2\theta}{(1-e\cos\theta)^4} = \frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^4}\left(1-2e\cos\theta+e^2\right)$$

the length of the orbit is

$$L = \int_0^{2\pi}\sqrt{r^2+(dr/d\theta)^2}\,d\theta = a\left(1-e^2\right)\int_0^{2\pi}\frac{\sqrt{1+e^2-2e\cos\theta}}{(1-e\cos\theta)^2}\,d\theta \approx 3.6 \times 10^8 \text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a \approx 3.6 \times 10^8$ km.

11 Review ET 10

CONCEPT CHECK

1. (a) A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are continuous functions of a variable t.

(b) Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t, either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y. It may be easier to graph that equation than to work with the original formulas for x and y in terms of t.

2. (a) You can find $\dfrac{dy}{dx}$ as a function of t by calculating $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ (if $dx/dt \neq 0$).

(b) Calculate the area as $\int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$ [or $\int_\beta^\alpha g(t)\,f'(t)\,dt$ if the leftmost point is $(f(\beta), g(\beta))$ rather than $(f(\alpha), g(\alpha))$].

3. (a) $L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$

(b) $S = \int_\alpha^\beta 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$

4. (a) See Figure 5 in Section 11.4 [ET 10.4].

(b) $x = r\cos\theta$, $y = r\sin\theta$

(c) To find a polar representation (r, θ) with $r \geq 0$ and $0 \leq \theta < 2\pi$, first calculate $r = \sqrt{x^2 + y^2}$. Then θ is specified by $\cos\theta = x/r$ and $\sin\theta = y/r$.

5. (a) Calculate $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(d/d\theta)\,(r\sin\theta)}{(d/d\theta)\,(r\cos\theta)} = \dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$, where $r = f\,(\theta)$.

(b) Calculate $A = \int_a^b \frac{1}{2} r^2\,d\theta = \int_a^b \frac{1}{2}\,[f\,(\theta)]^2\;d\theta$

(c) $L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_a^b \sqrt{[f\,(\theta)]^2 + [f'\,(\theta)]^2}\,d\theta$

6. (a) A parabola is a set of points in a plane whose distances from a fixed point F (the focus) and a fixed line ℓ (the directrix) are equal.

(b) $x^2 = 4py$; $y^2 = 4px$

7. (a) An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$.

8. (a) A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant.

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$

(c) $y = \pm\dfrac{\sqrt{c^2 - a^2}}{a}x$

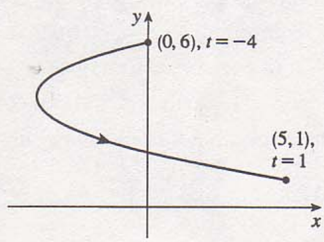
9. (a) If a conic section has focus F and corresponding directrix ℓ, then the eccentricity e is the fixed ratio $|PF|\,/\,|P\ell|$ for points P of the conic section.

(b) $e < 1$ for an ellipse; $e > 1$ for a hyperbola; $e = 1$ for a parabola.

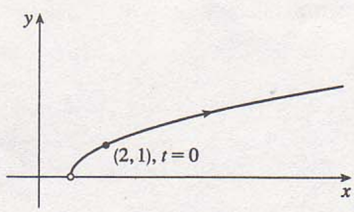
(c) $x = d$: $r = \dfrac{ed}{1 + e\cos\theta}$. $x = -d$: $r = \dfrac{ed}{1 - e\cos\theta}$. $y = d$: $r = \dfrac{ed}{1 + e\sin\theta}$. $y = -d$: $r = \dfrac{ed}{1 - e\sin\theta}$.

EXERCISES

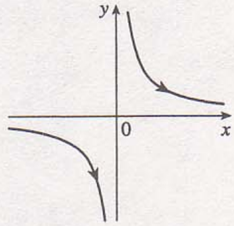
1. $x = t^2 + 4t$, $y = 2 - t$, $-4 \leq t \leq 1$. $t = 2 - y$, so
$x = (2-y)^2 + 4\,(2-y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12$. This is part of a parabola with vertex $(-4, 4)$, opening to the right.



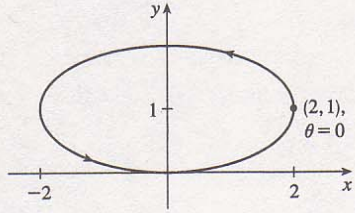
2. $x = 1 + e^{2t}$, $y = e^t$. $x = 1 + y^2$, $y > 0$.



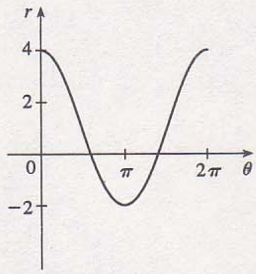
3. $x = \tan\theta$, $y = \cot\theta$. $y = 1/\tan\theta = 1/x$. The whole curve is traced out as θ ranges over the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ [or any open interval of the form $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$, where n is an integer].

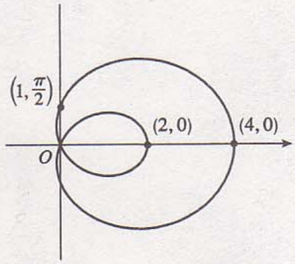


4. $x = 2\cos\theta$, $y = 1 + \sin\theta$, $\cos^2\theta + \sin^2\theta = 1 \;\Rightarrow$
$\left(\frac{x}{2}\right)^2 + (y-1)^2 = 1 \;\Rightarrow\; \frac{x^2}{4} + (y-1)^2 = 1$. This is an ellipse, centered at $(0, 1)$, with semimajor axis of length 2 and semiminor axis of length 1.

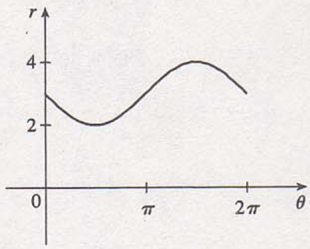


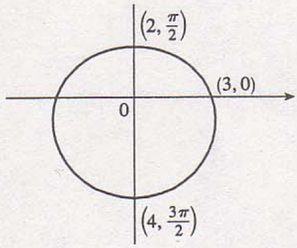
5. $r = 1 + 3\cos\theta$



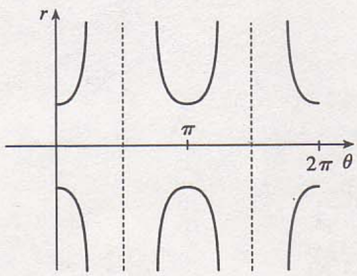


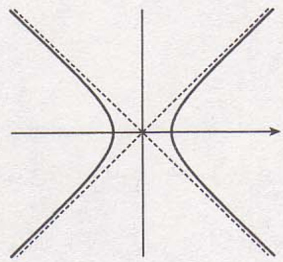
6. $r = 3 - \sin\theta$



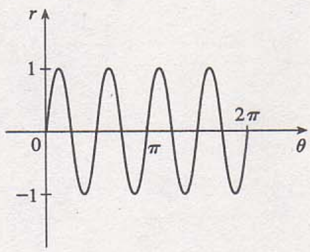


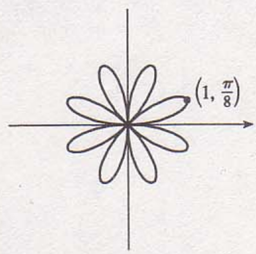
7. $r^2 = \sec 2\theta \;\Rightarrow$
$r^2\cos 2\theta = 1 \;\Rightarrow$
$r^2\left(\cos^2\theta - \sin^2\theta\right) = 1 \;\Rightarrow$
$r^2\cos^2\theta - r^2\sin^2\theta = 1 \;\Rightarrow$
$x^2 - y^2 = 1$, a hyperbola



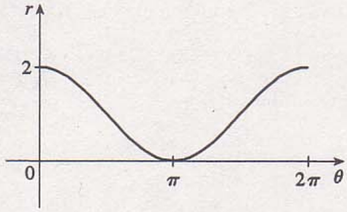


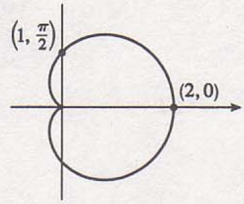
8. $r = \sin 4\theta$. This is an eight-leafed rose.



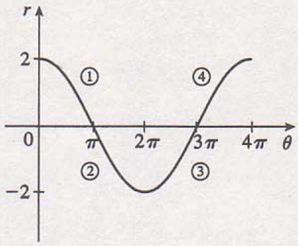


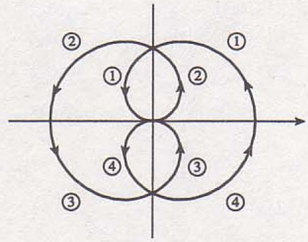
9. $r = 2\cos^2(\theta/2) = 1 + \cos\theta$



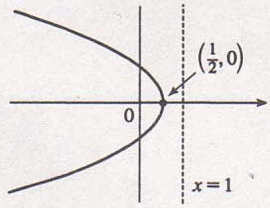


10. $r = 2\cos(\theta/2)$. The curve is symmetric about the pole and both the horizontal and vertical axes.

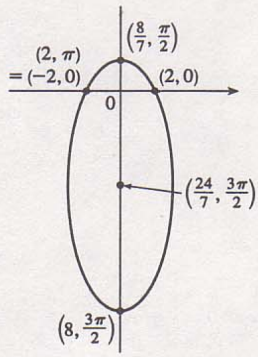




11. $r = \dfrac{1}{1+\cos\theta} \quad\Rightarrow\quad e = 1 \quad\Rightarrow\quad$ parabola; $d = 1 \quad\Rightarrow\quad$ directrix $x = 1$ and vertex $\left(\frac{1}{2}, 0\right)$; y-intercepts are $\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$.



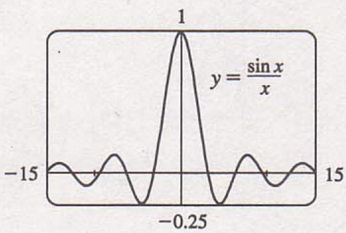
12. $r = \dfrac{8}{4 + 3\sin\theta} = \dfrac{\frac{3}{4}\cdot\frac{8}{3}}{1 + \frac{3}{4}\sin\theta}$. This is an ellipse with focus at the pole, eccentricity $\frac{3}{4}$, and directrix $y = \frac{8}{3}$. The center is $\left(\frac{24}{7}, \frac{3\pi}{2}\right)$.

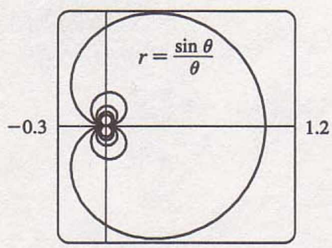


13. $x + y = 2 \quad\Leftrightarrow\quad r\cos\theta + r\sin\theta = 2 \quad\Leftrightarrow\quad r(\cos\theta + \sin\theta) = 2 \quad\Leftrightarrow\quad r = \dfrac{2}{\cos\theta + \sin\theta}$

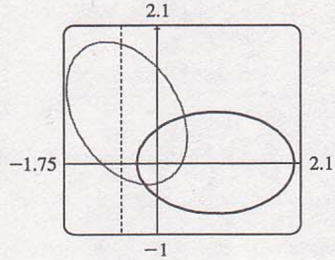
14. $x^2 + y^2 = 2 \quad\Rightarrow\quad r^2 = 2 \quad\Rightarrow\quad r = \sqrt{2}$. ($r = -\sqrt{2}$ gives the same curve.)

15. $r = (\sin\theta)/\theta$. As $\theta \to \pm\infty$, $r \to 0$.





16. $r = \dfrac{2}{4 - 3\cos\theta} = \dfrac{1/2}{1 - \frac{3}{2}\cos\theta} \Rightarrow e = \frac{3}{2}$ and $d = \frac{2}{3}$. The equation of the directrix is $x = r\cos\theta = -\frac{2}{3} \Rightarrow r = -2/\left(3\cos\theta\right)$. To obtain the equation of the rotated ellipse, we replace θ in the original equation with $\theta - \frac{2\pi}{3}$, and get $r = \dfrac{2}{4 - 3\cos\left(\theta - \frac{2\pi}{3}\right)}$.



17. $x = \ln t$, $y = 1 + t^2$; $t = 1$. $\dfrac{dy}{dt} = 2t$ and $\dfrac{dx}{dt} = \dfrac{1}{t}$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$. When $t = 1$, $\dfrac{dy}{dx} = 2$.

18. $x = te^t$, $y = 1 + \sqrt{1+t}$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1/\left(2\sqrt{1+t}\right)}{(1+t)\,e^t} = \dfrac{1}{2}$ when $t = 0$.

19. $\dfrac{dy}{dx} = \dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta} = \dfrac{\frac{1}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}} = \dfrac{4+\pi}{4-\pi}$ when $\theta = \frac{\pi}{4}$.

20. $\dfrac{dy}{dx} = \dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{-2\cos\theta\sin\theta + (3 - 2\sin\theta)\cos\theta}{-2\cos^2\theta - (3 - 2\sin\theta)\sin\theta} = \dfrac{3\cos\theta - 2\sin 2\theta}{-3\sin\theta - 2\cos 2\theta} = 0$ when $\theta = \frac{\pi}{2}$.

21. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t\cos t + \sin t}{-t\sin t + \cos t}$. $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$, where

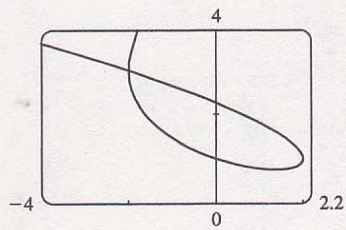
$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(-t\sin t + \cos t)\,(-t\sin t + 2\cos t) - (t\cos t + \sin t)\,(-t\cos t - 2\sin t)}{(-t\sin t + \cos t)^2} = \frac{t^2+2}{(-t\sin t + \cos t)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{t^2+2}{(-t\sin t + \cos t)^3}.$$

22. $x = 1 + t^2$, $y = t - t^3$. $\dfrac{dy}{dt} = 1 - 3t^2$ and $\dfrac{dx}{dt} = 2t$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 3t^2}{2t} = \frac{1}{2}t^{-1} - \frac{3}{2}t$.

$$\frac{d^2y}{dx^2} = \frac{d\left(dy/dx\right)/dt}{dx/dt} = \frac{-\frac{1}{2}t^{-2} - \frac{3}{2}}{2t} = -\tfrac{1}{4}t^{-3} - \tfrac{3}{4}t^{-1} = -\frac{1}{4t^3}\left(1 + 3t^2\right) = -\frac{3t^2+1}{4t^3}.$$

23. We graph the curve for $-2.2 \le t \le 1.2$. By zooming in or using a cursor, we find that the lowest point is about $(1.4, 0.75)$. To find the exact values, we find the t-value at which $dy/dt = 2t + 1 = 0 \Leftrightarrow t = -\frac{1}{2} \Leftrightarrow (x, y) = \left(\frac{11}{8}, \frac{3}{4}\right)$.



24. We estimate the coordinates of the point of intersection to be $(-2, 3)$. In fact this is exact, since both $t = -2$ and $t = 1$ give the point $(-2, 3)$. So the area enclosed by the loop is

$$\int_{t=-2}^{t=1} y\,dx = \int_{-2}^{1}\left(t^2 + t + 1\right)\left(3t^2 - 3\right)dt = \int_{-2}^{1}\left(3t^4 + 3t^3 - 3t - 3\right)dt$$

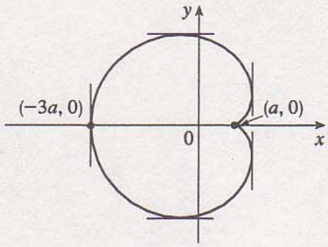
$$= \left[\tfrac{3}{5}t^5 + \tfrac{3}{4}t^4 - \tfrac{3}{2}t^2 - 3t\right]_{-2}^{1} = \left(\tfrac{3}{5} + \tfrac{3}{4} - \tfrac{3}{2} - 3\right) - \left[-\tfrac{96}{5} + 12 - 6 - (-6)\right] = \tfrac{81}{20}$$

25. $\dfrac{dx}{dt} = -2a\sin t + 2a\sin 2t = 2a\sin t\,(2\cos t - 1) = 0 \quad\Leftrightarrow\quad \sin t = 0 \text{ or } \cos t = \frac{1}{2} \quad\Rightarrow\quad t = 0, \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3}.$

$\dfrac{dy}{dt} = 2a\cos t - 2a\cos 2t = 2a\left(1 + \cos t - 2\cos^2 t\right) = 2a\,(1 - \cos t)\,(1 + 2\cos t) = 0 \quad\Rightarrow\quad t = 0, \frac{2\pi}{3},$ or $\frac{4\pi}{3}$.

Thus the graph has vertical tangents where $t = \frac{\pi}{3}$, π and $\frac{5\pi}{3}$, and horizontal tangents where $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. To determine what the slope is where $t = 0$, we use l'Hospital's Rule to evaluate $\displaystyle\lim_{t\to 0} \frac{dy/dt}{dx/dt} = 0$, so there is a horizontal tangent there.

t	x	y
0	a	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
π	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



26. From Exercise 25, $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t \quad\Rightarrow$

$$\begin{aligned} A &= 2\int_\pi^0 (2a\sin t - a\sin 2t)\,(-2a\sin t + 2a\sin 2t)\,dt = 4a^2\int_0^\pi \left(2\sin^2 t + \sin^2 2t - 3\sin t\sin 2t\right)dt \\ &= 4a^2\int_0^\pi \left[(1 - \cos 2t) + \tfrac{1}{2}(1 - \cos 4t) - 6\sin^2 t\cos t\right]dt = 4a^2\left[t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}t - \tfrac{1}{8}\sin 4t - 2\sin^3 t\right]_0^\pi \\ &= 4a^2\left(\frac{3}{2}\right)\pi = 6\pi a^2 \end{aligned}$$

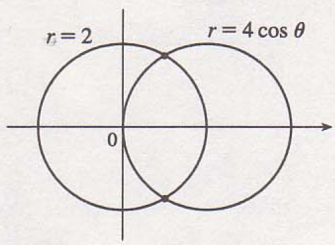
27. This curve has 10 "petals". For instance, for $-\frac{\pi}{10} \le \theta \le \frac{\pi}{10}$, there are two petals, one with $r > 0$ and one with $r < 0$.

$$A = 10\int_{-\pi/10}^{\pi/10} \tfrac{1}{2}r^2\,d\theta = 5\int_{-\pi/10}^{\pi/10} 9\cos 5\theta\,d\theta = 90\int_0^{\pi/10}\cos 5\theta\,d\theta = \left[18\sin 5\theta\right]_0^{\pi/10} = 18$$

28. $r = 1 - 3\sin\theta$. The inner loop is traced out as θ goes from $\alpha = \sin^{-1}\frac{1}{3}$ to $\pi - \alpha$, so

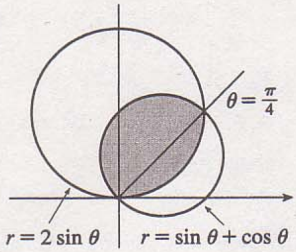
$$\begin{aligned} A &= \int_\alpha^{\pi-\alpha} \tfrac{1}{2}r^2\,d\theta = \int_\alpha^{\pi/2} (1 - 3\sin\theta)^2\,d\theta = \int_\alpha^{\pi/2}\left[1 - 6\sin\theta + \tfrac{9}{2}(1 - \cos 2\theta)\right]d\theta \\ &= \left[\tfrac{11}{2}\theta + 6\cos\theta - \tfrac{9}{4}\sin 2\theta\right]_\alpha^{\pi/2} = \tfrac{11}{4}\pi - \tfrac{11}{2}\sin^{-1}\tfrac{1}{3} - 3\sqrt{2} \end{aligned}$$

29. The curves intersect where $4\cos\theta = 2$; that is, at $\left(2, \frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$.



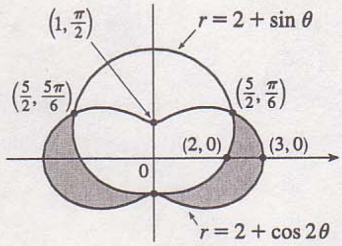
30. The two curves clearly both contain the pole. For other points of intersection, $\cot\theta = 2\cos(\theta + 2n\pi)$ or $-2\cos(\theta + \pi + 2n\pi)$, both of which reduce to $\cot\theta = 2\cos\theta \quad\Leftrightarrow\quad \cos\theta = 2\sin\theta\cos\theta \quad\Leftrightarrow$ $\cos\theta\,(1 - 2\sin\theta) = 0 \quad\Rightarrow\quad \cos\theta = 0 \text{ or } \sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2} \quad\Rightarrow$ intersection points are $\left(0, \frac{\pi}{2}\right)$, $\left(\sqrt{3}, \frac{\pi}{6}\right)$, and $\left(\sqrt{3}, \frac{11\pi}{6}\right)$.

31. The curves intersect where $2\sin\theta = \sin\theta + \cos\theta \;\Rightarrow\; \sin\theta = \cos\theta \;\Rightarrow$ $\theta = \frac{\pi}{4}$, and also at the origin (at which $\theta = \frac{3\pi}{4}$ on the second curve).



$$A = \int_0^{\pi/4} \tfrac{1}{2}\left(2\sin\theta\right)^2\,d\theta + \int_{\pi/4}^{3\pi/4} \tfrac{1}{2}\left(\sin\theta + \cos\theta\right)^2\,d\theta$$
$$= \int_0^{\pi/4} \left(1 - \cos 2\theta\right)d\theta + \tfrac{1}{2}\int_{\pi/4}^{3\pi/4} \left(1 + \sin 2\theta\right)d\theta$$
$$= \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} + \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\cos 2\theta\right]_{\pi/4}^{3\pi/4} = \tfrac{1}{2}\left(\pi - 1\right)$$

32. $A = 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}\left[\left(2 + \cos 2\theta\right)^2 - \left(2 + \sin\theta\right)^2\right]d\theta$



$$= \int_{-\pi/2}^{\pi/6} \left[4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta\right]d\theta$$
$$= \left[2\sin 2\theta + \tfrac{1}{2}\theta + \tfrac{1}{8}\sin 4\theta + 4\cos\theta - \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/6}$$
$$= \tfrac{51}{16}\sqrt{3}$$

33. $x = 3t^2$, $y = 2t^3$.

$$L = \int_0^2 \sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}\,dt = \int_0^2 \sqrt{\left(6t\right)^2 + \left(6t^2\right)^2}\,dt = 6\int_0^2 t\sqrt{1 + t^2}\,dt$$
$$= \left[2\left(1 + t^2\right)^{3/2}\right]_0^2 = 2\left(5\sqrt{5} - 1\right)$$

34.
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left[-\sin t + \frac{\frac{1}{2}\sec^2\left(t/2\right)}{\tan\left(t/2\right)}\right]^2 + \cos^2 t = \left[-\sin t + \frac{1}{2\sin\left(t/2\right)\cos\left(t/2\right)}\right]^2 + \cos^2 t$$
$$= \left(-\sin t + \frac{1}{\sin t}\right)^2 + \cos^2 t = \csc^2 t - 1 = \cot^2 t \;\Rightarrow$$

$$L = \int_{\pi/2}^{3\pi/4} \left|\cot t\right|dt = -\int_{\pi/2}^{3\pi/4} \cot t\,dt = \left[-\ln\left|\sin t\right|\right]_{\pi/2}^{3\pi/4} = \ln\sqrt{2}$$

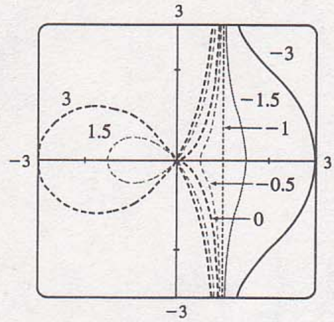
35.
$$L = \int_\pi^{2\pi} \sqrt{r^2 + \left(dr/d\theta\right)^2}\,d\theta = \int_\pi^{2\pi} \sqrt{\left(1/\theta\right)^2 + \left(-1/\theta^2\right)^2}\,d\theta$$
$$\overset{24}{=} \int_\pi^{2\pi} \frac{\sqrt{\theta^2 + 1}}{\theta^2}\,d\theta = \left[-\frac{\sqrt{\theta^2 + 1}}{\theta} + \ln\left|\theta + \sqrt{\theta^2 + 1}\right|\right]_\pi^{2\pi}$$
$$= \frac{\sqrt{\pi^2 + 1}}{\pi} - \frac{\sqrt{4\pi^2 + 1}}{2\pi} + \ln\left|\frac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right|$$

36.
$$L = \int_0^\pi \sqrt{r^2 + \left(dr/d\theta\right)^2}\,d\theta = \int_0^\pi \sqrt{\sin^6\left(\tfrac{1}{3}\theta\right) + \sin^4\left(\tfrac{1}{3}\theta\right)\cos^2\left(\tfrac{1}{3}\theta\right)}\,d\theta$$
$$= \int_0^\pi \sin^2\left(\tfrac{1}{3}\theta\right)d\theta = \left[\tfrac{1}{2}\left(\theta - \tfrac{3}{2}\sin\left(\tfrac{2}{3}\theta\right)\right)\right]_0^\pi = \tfrac{1}{2}\pi - \tfrac{3}{8}\sqrt{3}$$

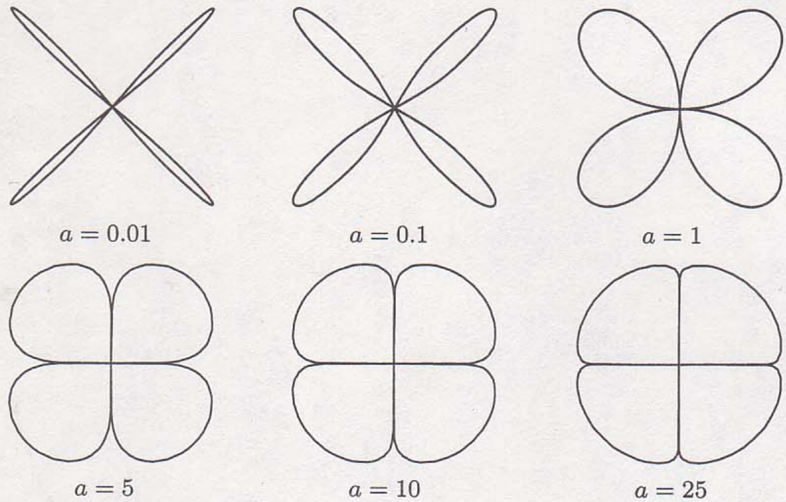
37.
$$S = \int_1^4 2\pi y\sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}\,dt = \int_1^4 2\pi\left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^{-2}\right)\sqrt{\left(2/\sqrt{t}\right)^2 + \left(t^2 - t^{-3}\right)^2}\,dt$$
$$= 2\pi\int_1^4 \left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^{-2}\right)\sqrt{\left(t^2 + t^{-3}\right)^2}\,dt = 2\pi\int_1^4 \left(\tfrac{1}{3}t^5 + \tfrac{5}{6} + \tfrac{1}{2}t^{-5}\right)dt$$
$$= 2\pi\left[\tfrac{1}{18}t^6 + \tfrac{5}{6}t - \tfrac{1}{8}t^{-4}\right]_1^4 = \tfrac{471{,}295}{1024}\pi$$

38. From Exercise 34, we find that $S = \int_{\pi/2}^{3\pi/4} 2\pi \sin t\,|\cot t|\,dt = -2\pi \int_{\pi/2}^{3\pi/4} \cos t\,dt = \pi\left(2 - \sqrt{2}\right)$.

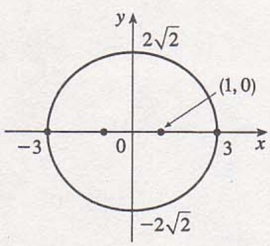
39. For all c except -1, the curve is asymptotic to the line $x = 1$. For $c < -1$, the curve bulges to the right near $y = 0$. As c increases, the bulge becomes smaller, until at $c = -1$ the curve is the straight line $x = 1$. As c continues to increase, the curve bulges to the left, until at $c = 0$ there is a cusp at the origin. For $c > 0$, there is a loop to the left of the origin, whose size and roundness increase as c increases. Note that the x-intercept of the curve is always $-c$.



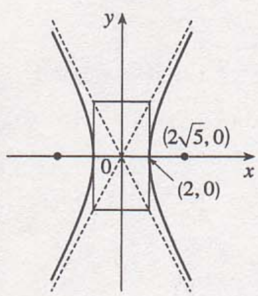
40. For a close to 0, the graph consists of four thin petals. As a increases, the petals get fatter, until as $a \to \infty$, each petal occupies almost its entire quarter-circle.



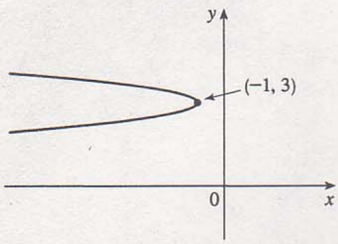
41. Ellipse, center $(0, 0)$, $a = 3$, $b = 2\sqrt{2}$, $c = 1$ $\Rightarrow$ foci $(\pm 1, 0)$, vertices $(\pm 3, 0)$.



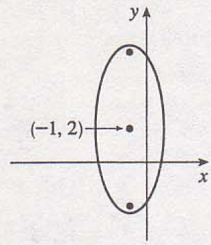
42. $x^2/4 - y^2/16 = 1$ is a hyperbola with center $(0, 0)$, vertices $(\pm 2, 0)$, $a = 2$, $b = 4$, $c = \sqrt{16 + 4} = 2\sqrt{5}$, foci $(\pm 2\sqrt{5}, 0)$ and asymptotes $y = \pm 2x$.



43. $6(y^2 - 6y + 9) = -(x+1) \quad \Leftrightarrow$
$(y-3)^2 = -\frac{1}{6}(x+1)$, a parabola with vertex $(-1,3)$, opening to the left, $p = -\frac{1}{24} \quad \Rightarrow \quad$ focus $\left(-\frac{25}{24}, 3\right)$ and directrix $x = -\frac{23}{24}$.



44. $25(x+1)^2 + 4(y-2)^2 = 100 \quad \Leftrightarrow$
$\frac{1}{4}(x+1)^2 + \frac{1}{25}(y-2)^2 = 1$ is an ellipse centered at $(-1,2)$ with foci on the line $x = -1$, vertices $(-1,7)$ and $(-1,-3)$; $a = 5$, $b = 2 \quad \Rightarrow c = \sqrt{21} \quad \Rightarrow \quad$ foci $\left(-1, 2 \pm \sqrt{21}\right)$.



45. The parabola opens upward with vertex $(0,4)$ and $p = 2$, so its equation is $(x-0)^2 = 4 \cdot 2(y-4) \quad \Leftrightarrow$
$x^2 = 8(y-4)$.

46. Center is $(0,0)$, and $c = 5$, $a = 2 \Rightarrow b = \sqrt{21}$; foci on y-axis $\quad \Rightarrow \quad$ equation of the hyperbola is $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1$.

47. The hyperbola has center $(0,0)$ and foci on the x-axis. $c = 3$ and $b/a = \frac{1}{2}$ (from the asymptotes)
$\Rightarrow \quad 9 = c^2 = a^2 + b^2 = (2b)^2 + b^2 = 5b^2 \quad \Rightarrow \quad b = \frac{3}{\sqrt{5}} \quad \Rightarrow \quad a = \frac{6}{\sqrt{5}} \quad \Rightarrow \quad$ the equation is
$\dfrac{x^2}{36/5} - \dfrac{y^2}{9/5} = 1 \quad \Leftrightarrow \quad 5x^2 - 20y^2 = 36$.

48. Center is $(3,0)$, and $a = \frac{8}{2} = 4$, $c = 2 \quad \Leftrightarrow \quad b = \sqrt{4^2 - 2^2} = 2\sqrt{3} \quad \Rightarrow \quad$ the equation of the ellipse is
$\dfrac{(x-3)^2}{12} + \dfrac{y^2}{16} = 1$.

49. $x^2 = -y + 100$ has its vertex at $(0,100)$, so one of the vertices of the ellipse is $(0,100)$. Another form of the equation of a parabola is $x^2 = 4p(y-100)$ so $4p(y-100) = -y + 100 \quad \Rightarrow \quad 4py - 4p(100) = 100 - y \quad \Rightarrow$
$4p = \dfrac{100 - y}{y - 100} \quad \Rightarrow \quad p = -\frac{1}{4}$. Therefore the shared focus is found at $\left(0, \frac{399}{4}\right)$ so $2c = \frac{399}{4} - 0 \quad \Rightarrow \quad c = \frac{399}{8}$
and the center of the ellipse is $\left(0, \frac{399}{8}\right)$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \frac{401^2 - 399^2}{8^2} = 25$. So the
equation of the ellipse is $\dfrac{x^2}{b^2} + \dfrac{\left(y - \frac{399}{8}\right)^2}{a^2} = 1 \quad \Rightarrow \quad \dfrac{x^2}{25} + \dfrac{\left(y - \frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2} = 1$ or $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$.

50. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \;\Rightarrow\; \dfrac{2x}{a^2}+\dfrac{2y}{b^2}\dfrac{dy}{dx}=0 \;\Rightarrow\; \dfrac{dy}{dx}=-\dfrac{b^2}{a^2}\dfrac{x}{y}$. Therefore $\dfrac{dy}{dx}=m \;\Leftrightarrow\; y=-\dfrac{b^2}{a^2}\dfrac{x}{m}$.

Combining this condition with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, we find that $x=\pm\dfrac{a^2m}{\sqrt{a^2m^2+b^2}}$. In other words, the two points on the ellipse where the tangent has slope m are $\left(\pm\dfrac{a^2m}{\sqrt{a^2m^2+b^2}},\mp\dfrac{b^2}{\sqrt{a^2m^2+b^2}}\right)$. The tangent lines at these points have the equations $y\pm\dfrac{b^2}{\sqrt{a^2m^2+b^2}}=m\left(x\mp\dfrac{a^2m}{\sqrt{a^2m^2+b^2}}\right)$ or

$y=mx\mp\dfrac{a^2m^2}{\sqrt{a^2m^2+b^2}}\mp\dfrac{b^2}{\sqrt{a^2m^2+b^2}}=mx\pm\sqrt{a^2m^2+b^2}$.

51. Directrix $x=4 \;\Rightarrow\; d=4$, so $e=\frac{1}{3} \;\Rightarrow\; r=\dfrac{ed}{1+e\cos\theta}=\dfrac{4}{3+\cos\theta}$.

52. See the end of the proof of Theorem 11.7.1 [ET 10.7.1]. If $e>1$, then $1-e^2<0$ and Equations 11.7.4 [ET 10.7.4] become $a^2=\dfrac{e^2d^2}{(e^2-1)^2}$ and $b^2=\dfrac{e^2d^2}{e^2-1}$, so $\dfrac{b^2}{a^2}=e^2-1$. The asymptotes $y=\pm\dfrac{b}{a}x$ have slopes $\pm\dfrac{b}{a}=\pm\sqrt{e^2-1}$, so the angles they make with the polar axis are $\pm\tan^{-1}\left[\sqrt{e^2-1}\right]=\cos^{-1}(\pm 1/e)$.

53. In polar coordinates, an equation for the circle is $r=2a\sin\theta$. Thus, the coordinates of Q are $x=r\cos\theta=2a\sin\theta\cos\theta$ and $y=r\sin\theta=2a\sin^2\theta$. The coordinates of R are $x=2a\cot\theta$ and $y=2a$. Since P is the midpoint of QR, we use the midpoint formula to get $x=a\left(\sin\theta\cos\theta+\cot\theta\right)$ and $y=a\left(1+\sin^2\theta\right)$.

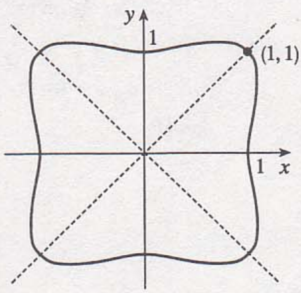
Problems Plus

1. $x = \int_1^t \frac{\cos u}{u}\,du$, $y = \int_1^t \frac{\sin u}{u}\,du$, so by FTC1, we have $\frac{dx}{dt} = \frac{\cos t}{t}$ and $\frac{dy}{dt} = \frac{\sin t}{t}$. Vertical tangent lines occur when $\frac{dx}{dt} = 0 \Leftrightarrow \cos t = 0$. The parameter value corresponding to $(x, y) = (0, 0)$ is $t = 1$, so the nearest vertical tangent occurs when $t = \frac{\pi}{2}$. Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\,dt = \int_1^{\pi/2} \frac{dt}{t} = [\ln t]_1^{\pi/2} = \ln \frac{\pi}{2}$$

2. (a) The curve $x^4 + y^4 = x^2 + y^2$ is symmetric about both axes and about the line $y = x$ (since interchanging x and y does not change the equation) so we need only consider $y \geq x \geq 0$ to begin with. Implicit differentiation gives $4x^3 + 4y^3y' = 2x + 2yy' \Rightarrow y' = \frac{x(1 - 2x^2)}{y(2y^2 - 1)} \Rightarrow y' = 0$ when $x = 0$ and when $x = \pm\frac{1}{\sqrt{2}}$. If $x = 0$, then $y^4 = y^2 \Rightarrow y^2(y^2 - 1) = 0 \Rightarrow y = 0$ or ± 1. The point $(0, 0)$ can't be a highest or lowest point because it is isolated. [If $-1 < x < 1$ and $-1 < y < 1$, then $x^4 < x^2$ and $y^4 < y^2 \Rightarrow$ $x^4 + y^4 < x^2 + y^2$, except for $(0, 0)$.] If $x = \frac{1}{\sqrt{2}}$, then $x^2 = \frac{1}{2}$, $x^4 = \frac{1}{4}$, so $\frac{1}{4} + y^4 = \frac{1}{2} + y^2 \Rightarrow$ $4y^4 - 4y^2 - 1 = 0 \Rightarrow y^2 = \frac{4 \pm \sqrt{16+16}}{8} = \frac{1 \pm \sqrt{2}}{2}$. But $y^2 > 0$, so $y^2 = \frac{1+\sqrt{2}}{2} \Rightarrow$ $y = \pm\sqrt{\frac{1}{2}(1 + \sqrt{2})}$. Near the point $(0, 1)$, the denominator of y' is positive and the numerator changes from negative to positive as x increases through 0, so $(0, 1)$ is a local minimum point. At $\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$, y' changes from positive to negative, so that point gives a maximum. By symmetry, the highest points on the curve are $\left(\pm\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$ and the lowest points are $\left(\pm\frac{1}{\sqrt{2}}, -\sqrt{\frac{1+\sqrt{2}}{2}}\right)$.

(b) We use the information from part (a), together with symmetry with respect to the axes and the lines $y = \pm x$, to sketch the curve.



(c) In polar coordinates, $x^4 + y^4 = x^2 + y^2$ becomes $r^4 \cos^4\theta + r^4 \sin^4\theta = r^2$ or $r^2 = 1/(\cos^4\theta + \sin^4\theta)$. By the symmetry shown in part (b), the area enclosed by the curve is $A = 8 \cdot \frac{1}{2}\int_0^{\pi/4} r^2\,d\theta = 4\int_0^{\pi/4} \frac{d\theta}{\cos^4\theta + \sin^4\theta}$. (If we have a CAS, this can be evaluated to give $\sqrt{2}\pi$).

The usual Weierstrass substitution $t = \tan(\theta/2)$ leads to a complicated integrand, so we first simplify:

$$\begin{aligned}\cos^4\theta + \sin^4\theta &= \left(1 - \sin^2\theta\right)^2 + \sin^4\theta = 1 - 2\sin^2\theta + 2\sin^4\theta = 1 - 2\sin^2\theta\left(1 - \sin^2\theta\right) \\ &= 1 - 2\sin^2\theta\cos^2\theta = 1 - \tfrac{1}{2}\sin^2 2\theta\end{aligned}$$

[continued]

Then we substitute $t = \tan 2\theta$, which gives $\theta = \frac{1}{2}\tan^{-1} t \;\Rightarrow\; d\theta = \dfrac{dt}{2\,(1+t^2)}$ and $\sin 2\theta = \dfrac{t}{\sqrt{1+t^2}}$.
Also, $\theta \to \frac{\pi}{4} \;\Rightarrow\; t \to \infty$, so we get the following improper integral:

$$A = 4\int_0^{\pi/4} \frac{d\theta}{1 - \frac{1}{2}\sin^2 2\theta} = 4\int_0^{\infty} \frac{1}{1 - \frac{1}{2}\left[t^2/(1+t^2)\right]}\,\frac{dt}{2\,(1+t^2)} = 4\int_0^{\infty} \frac{dt}{t^2+2}$$
$$= \lim_{x\to\infty} 4\left[\tfrac{1}{\sqrt{2}}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}t\right)\right]_0^x = 2\sqrt{2}\lim_{x\to\infty}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}x\right) = 2\sqrt{2}\cdot\tfrac{\pi}{2} = \sqrt{2}\pi$$

3. (a) If $\tan\theta = \sqrt{\dfrac{y}{C-y}}$, then $\tan^2\theta = \dfrac{y}{C-y}$, so $C\tan^2\theta - y\tan^2\theta = y$ and

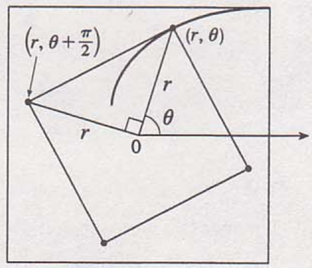
$$y = \frac{C\tan^2\theta}{1+\tan^2\theta} = \frac{C\tan^2\theta}{\sec^2\theta} = C\tan^2\theta\cos^2\theta = C\sin^2\theta = \frac{C}{2}\,(1-\cos 2\theta).\ \text{ Now}$$

$$dx = \sqrt{\frac{y}{C-y}}\,dy = \tan\theta\cdot\frac{C}{2}\cdot 2\sin 2\theta\,d\theta = C\tan\theta\cdot 2\sin\theta\cos\theta\,d\theta = 2C\sin^2\theta\,d\theta = C\,(1-\cos 2\theta)\,d\theta$$

Thus, $x = C\left(\theta - \frac{1}{2}\sin 2\theta\right) + K$ for some constant K. When $\theta = 0$, we have $y = 0$. We require that $x = 0$ when $\theta = 0$ so that the curve passes through the origin when $\theta = 0$. This yields $K = 0$. We now have $x = \frac{1}{2}C\,(2\theta - \sin 2\theta)$, $y = \frac{1}{2}C\,(1-\cos 2\theta)$.

(b) Setting $\phi = 2\theta$ and $r = \frac{1}{2}C$, we get $x = r\,(\phi - \sin\phi)$, $y = r\,(1-\cos\phi)$. Comparison with Equations 11.1.1 [ET 10.1.1] shows that the curve is a cycloid.

4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are (r,θ), then the polar coordinates of the bug that it is crawling toward must be $\left(r, \theta + \frac{\pi}{2}\right)$. (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be $\frac{\pi}{2}$.) The Cartesian coordinates of the first bug are $(r\cos\theta, r\sin\theta)$ and for the second bug we have



$x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin\theta$, $y = r\sin\left(\theta+\frac{\pi}{2}\right) = r\cos\theta$. So the slope of the line joining the bugs is $\dfrac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta+\cos\theta}$. This must be equal to the slope of the tangent line at (r,θ), so by Equation 11.4.3 [ET 10.4.3] we have $\dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$. Solving for $\dfrac{dr}{d\theta}$, we get

$$\frac{dr}{d\theta}\sin^2\theta + \frac{dr}{d\theta}\sin\theta\cos\theta + r\sin\theta\cos\theta + r\cos^2\theta = \frac{dr}{d\theta}\sin\theta\cos\theta - \frac{dr}{d\theta}\cos^2\theta - r\sin^2\theta + r\sin\theta\cos\theta$$

$\Rightarrow\; \dfrac{dr}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) + r\left(\cos^2\theta + \sin^2\theta\right) = 0 \;\Rightarrow\; \dfrac{dr}{d\theta} = -r$. Solving this differential equation as a separable equation (as in Section 10.3 [ET 9.3]), or using Theorem 10.4.2 [ET 9.4.2] with $k = -1$, we get $r = Ce^{-\theta}$. To determine C we use the fact that, at its starting position, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\sqrt{2}}a$, so $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \;\Rightarrow\; C = \frac{1}{\sqrt{2}}ae^{\pi/4}$. Therefore, a polar equation of the bug's path is $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$ or $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$.

(b) The distance traveled by this bug is $L = \int_{\pi/4}^{\infty} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$, where $\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{2}}e^{\pi/4}\left(-e^{-\theta}\right)$ and so

$$r^2 + (dr/d\theta)^2 = \tfrac{1}{2}a^2 e^{\pi/2} e^{-2\theta} + \tfrac{1}{2}a^2 e^{\pi/2} e^{-2\theta} = a^2 e^{\pi/2} e^{-2\theta}$$

Thus

$$L = \int_{\pi/4}^{\infty} a e^{\pi/4} e^{-\theta}\,d\theta = a e^{\pi/4} \lim_{t\to\infty} \int_{\pi/4}^{t} e^{-\theta}\,d\theta = a e^{\pi/4} \lim_{t\to\infty} \left[-e^{-\theta}\right]_{\pi/4}^{t} = a e^{\pi/4} \lim_{t\to\infty} \left[e^{-\pi/4} - e^{-t}\right]$$

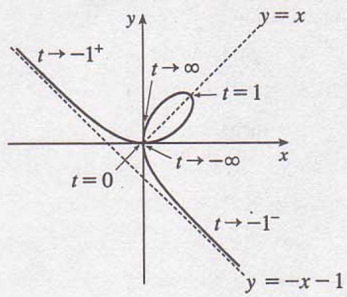
$$= a e^{\pi/4} e^{-\pi/4} = a$$

5. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\dfrac{3t_1}{1+t_1^3} = a$ and $\dfrac{3t_1^2}{1+t_1^3} = b$. If $t_1 = 0$, the point is $(0,0)$, which lies on the line $y = x$. If $t_1 \neq 0$, then the point corresponding to $t = \dfrac{1}{t_1}$ is given by $x = \dfrac{3\,(1/t_1)}{1+(1/t_1)^3} = \dfrac{3t_1^2}{t_1^3+1} = b$, $y = \dfrac{3\,(1/t_1)^2}{1+(1/t_1)^3} = \dfrac{3t_1}{t_1^3+1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when $\dfrac{3t}{1+t^3} = \dfrac{3t^2}{1+t^3} \Rightarrow t = t^2 \Rightarrow t = 0$ or 1, so the points are $(0,0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(b) $\dfrac{dy}{dt} = \dfrac{(1+t^3)(6t) - 3t^2(3t^2)}{(1+t^3)^2} = \dfrac{6t - 3t^4}{(1+t^3)^2} = 0$ when $6t - 3t^4 = 3t\left(2 - t^3\right) = 0 \Rightarrow t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0,0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0,0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$. Also $y - (-x-1) = y + x + 1 = \dfrac{3t + 3t^2 + (1+t^3)}{1+t^3} = \dfrac{(t+1)^3}{1+t^3} = \dfrac{(t+1)^2}{t^2 - t + 1} \to 0$ as $t \to -1$. So $y = -x - 1$ is a slant asymptote.

(d) $\dfrac{dx}{dt} = \dfrac{(1+t^3)(3) - 3t(3t^2)}{(1+t^3)^2} = \dfrac{3 - 6t^3}{(1+t^3)^2}$ and from part (b) we have $\dfrac{dy}{dt} = \dfrac{6t - 3t^4}{(1+t^3)^2}$. So $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t\left(2 - t^3\right)}{1 - 2t^3}$. Also $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{2\left(1+t^3\right)^4}{3\left(1-2t^3\right)^3} > 0 \Leftrightarrow t < \frac{1}{\sqrt[3]{2}}$. So the curve is concave upward there and has a minimum point at $(0,0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



(e) $x^3+y^3=\left(\dfrac{3t}{1+t^3}\right)^3+\left(\dfrac{3t^2}{1+t^3}\right)^3=\dfrac{27t^3+27t^6}{(1+t^3)^3}=\dfrac{27t^3\,(1+t^3)}{(1+t^3)^3}=\dfrac{27t^3}{(1+t^3)^2}$ and

$3xy=3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right)=\dfrac{27t^3}{(1+t^3)^2}$, so $x^3+y^3=3xy$.

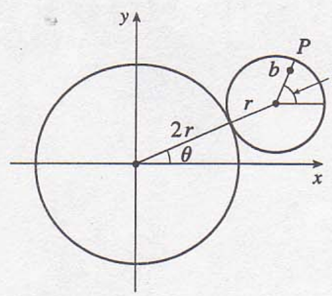
(f) We start with the equation from part (e) and substitute $x=r\cos\theta$, $y=r\sin\theta$. Then $x^3+y^3=3xy \Rightarrow$ $r^3\cos^3\theta+r^3\sin^3\theta=3r^2\cos\theta\sin\theta$. For $r\neq 0$, this gives $r=\dfrac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r=\dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1+\dfrac{\sin^3\theta}{\cos^3\theta}}=\dfrac{3\sec\theta\tan\theta}{1+\tan^3\theta}$.

(g) The loop corresponds to $\theta\in\left(0,\frac{\pi}{2}\right)$, so its area is

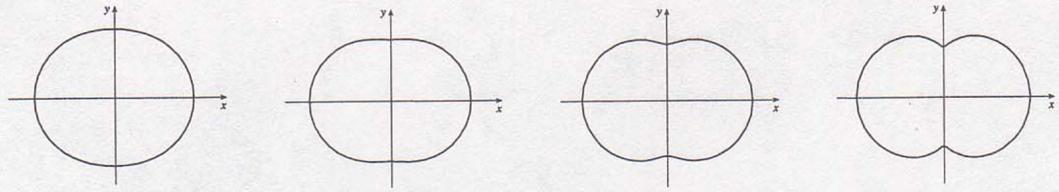
$$A=\int_0^{\pi/2}\frac{r^2}{2}d\theta=\frac{1}{2}\int_0^{\pi/2}\left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2d\theta=\frac{9}{2}\int_0^{\pi/2}\frac{\sec^2\theta\tan^2\theta}{(1+\tan^3\theta)^2}d\theta$$
$$=\frac{9}{2}\int_0^{\infty}\frac{u^2du}{(1+u^3)^2}\ (\text{put } u=\tan\theta)\ =\lim_{b\to\infty}\tfrac{9}{2}\left[-\tfrac{1}{3}\left(1+u^3\right)^{-1}\right]_0^b=\tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y=-x-1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y=-x-1 \Rightarrow r\sin\theta=-r\cos\theta-1 \Rightarrow r=-\dfrac{1}{\sin\theta+\cos\theta}$, the area in the fourth quadrant is $\dfrac{1}{2}\displaystyle\int_{-\pi/2}^{-\pi/4}\left[\left(-\frac{1}{\sin\theta+\cos\theta}\right)^2-\left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2\right]d\theta=\dfrac{1}{2}$ (using a CAS). Therefore, the total area is $\frac{1}{2}+2\left(\frac{1}{2}\right)=\frac{3}{2}$.

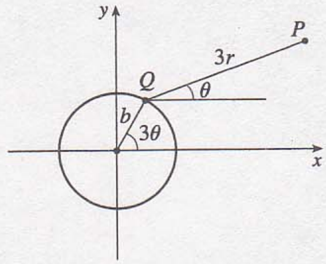
6. (a) Since the smaller circle rolls without slipping around C, the amount of arc traversed on C ($2r\theta$ in the figure) must equal the amount of arc of the smaller circle that has been in contact with C. Since the smaller circle has radius r, it must have turned through an angle of $2r\theta/r=2\theta$. In addition to turning through an angle 2θ, the little circle has rolled through an angle θ against C. Thus, P has turned through an angle of 3θ as shown in the figure. (If the little circle had turned through an angle of 2θ with its center pinned to the x-axis, then P would have turned only 2θ instead of 3θ. The movement of the little circle around C adds θ to the angle.) From the figure, we see that the center of the small circle has coordinates $(3r\cos\theta, 3r\sin\theta)$. Thus, P has coordinates (x,y), where $x=3r\cos\theta+b\cos 3\theta$ and $y=3r\sin\theta+b\sin 3\theta$.



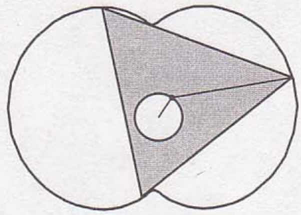
(b)



$b = \frac{1}{5}r$ $\qquad$ $b = \frac{2}{5}r$ $\qquad$ $b = \frac{3}{5}r$ $\qquad$ $b = \frac{4}{5}r$



(c) The diagram gives an alternate description of point P on the epitrochoid. Q moves around a circle of radius b, and P rotates one-third as fast with respect to Q at a distance of $3r$. Place an equilateral triangle with sides of length $3\sqrt{3}r$ so that its centroid is at Q and one vertex is at P. (The distance from the centroid to a vertex is $\frac{1}{\sqrt{3}}$ times the length of a side of the equilateral triangle.)



As θ increases by $\frac{2\pi}{3}$, the point Q travels once around the circle of radius b, returning to its original position. At the same time, P (and the rest of the triangle) rotate through an angle of $\frac{2\pi}{3}$ about Q, so P's position is occupied by another vertex. In this way, we see that the epitrochoid traced out by P is simultaneously traced out by the other two vertices as well. The whole equilateral triangle sits inside the epitrochoid (touching it only with its vertices) and each vertex traces out the curve once while the centroid moves around the circle three times.

(d) We view the epitrochoid as being traced out in the same way as in part (c), by a rotor for which the distance from its center to each vertex is $3r$, so it has radius $6r$. To show that the rotor fits inside the epitrochoid, it suffices to show that for any position of the tracing point P, there are no points on the opposite side of the rotor which are outside the epitrochoid. But the most likely case of intersection is when P is on the y-axis, so as long as the diameter of the rotor (which is $3\sqrt{3}r$) is less than the distance between the y-intercepts, the rotor will fit. The y-intercepts occur when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2} \quad \Rightarrow \quad y = \pm(3r - b)$, so the distance between the intercepts is $6r - 2b$, and the rotor will fit if $3\sqrt{3}r \leq 6r - 2b \quad \Leftrightarrow \quad b \leq \frac{3(2-\sqrt{3})}{2}r$.

12 Infinite Sequences and Series ET 11

12.1 Sequences ET 11.1

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n = 1 - (0.2)^n$, so the sequence is $\{0.8, 0.96, 0.992, 0.9984, 0.99968, \dots\}$.

4. $a_n = \dfrac{n+1}{3n-1}$, so the sequence is $\left\{\dfrac{2}{2}, \dfrac{3}{5}, \dfrac{4}{8}, \dfrac{5}{11}, \dfrac{6}{14}, \dots\right\} = \left\{1, \dfrac{3}{5}, \dfrac{1}{2}, \dfrac{5}{11}, \dfrac{3}{7}, \dots\right\}$.

5. $a_n = \dfrac{3(-1)^n}{n!}$, so the sequence is $\left\{\dfrac{-3}{1}, \dfrac{3}{2}, \dfrac{-3}{6}, \dfrac{3}{24}, \dfrac{-3}{120}, \dots\right\} = \left\{-3, \dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{8}, -\dfrac{1}{40}, \dots\right\}$.

6. $a_n = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)$, so the sequence is
$\{2, 2\cdot 4, 2\cdot 4\cdot 6, 2\cdot 4\cdot 6\cdot 8, 2\cdot 4\cdot 6\cdot 8\cdot 10, \dots\} = \{2, 8, 48, 384, 3840, \dots\}$.

7. $a_n = \sin \dfrac{n\pi}{2}$, so the sequence is $\{1, 0, -1, 0, 1, \dots\}$.

8. $a_1 = 1$, $a_{n+1} = \dfrac{1}{1+a_n}$, so the sequence is

$$\left\{1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{2}}, \frac{1}{1+\frac{2}{3}}, \frac{1}{1+\frac{3}{5}}, \dots\right\} = \left\{1, \frac{1}{2}, \frac{1}{\frac{3}{2}}, \frac{1}{\frac{5}{3}}, \frac{1}{\frac{8}{5}}\right\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \dots\right\}$$

9. The numerators are all 1 and the denominators are powers of 2, so $a_n = \dfrac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2, so $a_n = \dfrac{1}{2n}$.

11. $\{2, 7, 12, 17, \dots\}$. Each term is larger than the preceding one by 5, so
$a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

12. $\left\{-\dfrac{1}{4}, \dfrac{2}{9}, -\dfrac{3}{16}, \dfrac{4}{25}, \dots\right\}$. The numerator of the nth term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

13. $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \dots\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

14. $\{0, 2, 0, 2, 0, 2, \ldots\}$. One is halfway between 0 and 2, so we can think of alternately subtracting and adding 1 (from 1 and to 1) to obtain the given sequence: $a_n = 1 - (-1)^{n-1}$.

15. $a_n = n\,(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

16. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}} = \dfrac{1}{1/\sqrt{n}+1}$, so $a_n \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$, so $\lim\limits_{n\to\infty} a_n = \frac{1}{3}\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\cdot 0 = 0$ by (7) with $r = \frac{2}{3}$. Converges

20. $a_n = \dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$ and the sequence diverges.

21. $a_n = \dfrac{(-1)^{n-1}\,n}{n^2+1} = \dfrac{(-1)^{n-1}}{n+1/n}$, so $0 \le |a_n| = \dfrac{1}{n+1/n} \le \dfrac{1}{n} \to 0$ as $n \to \infty$, so $a_n \to 0$ by the Squeeze Theorem and Theorem 5. Converges

22. $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, \ldots\}$. This sequence oscillates among 1, 0, and -1, so the sequence diverges.

23. $a_n = 2 + \cos n\pi$, so

$$\begin{aligned}\{a_n\} &= \{2+\cos\pi, 2+\cos 2\pi, 2+\cos 3\pi, 2+\cos 4\pi, \ldots\} = \{2-1, 2+1, 2-1, 2+1, \ldots\}\\ &= \{1, 3, 1, 3, \ldots\}\end{aligned}$$

This sequence oscillates between 1 and 3, so it diverges.

24. $2n \to \infty$ as $n \to \infty$, so since $\lim\limits_{x\to\infty} \arctan x = \frac{\pi}{2}$, we have $\lim\limits_{n\to\infty} \arctan 2n = \frac{\pi}{2}$. Convergent

25. $0 < \dfrac{3+(-1)^n}{n^2} \le \dfrac{4}{n^2}$ and $\lim\limits_{n\to\infty} \dfrac{4}{n^2} = 0$, so $\left\{\dfrac{3+(-1)^n}{n^2}\right\}$ converges to 0 by the Squeeze Theorem.

26. $\lim\limits_{n\to\infty} \dfrac{n!}{(n+2)!} = \lim\limits_{n\to\infty} \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{1\cdot 2\cdot 3\cdot\cdots\cdot n\,(n+1)\,(n+2)} = \lim\limits_{n\to\infty} \dfrac{1}{(n+2)\,(n+1)} = 0$. Convergent

27. $\lim\limits_{x\to\infty} \dfrac{\ln(x^2)}{x} = \lim\limits_{x\to\infty} \dfrac{2\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2/x}{1} = 0$, so by Theorem 2, $\left\{\dfrac{\ln(n^2)}{n}\right\}$ converges to 0.

28. $\lim\limits_{n\to\infty} \sin\left(\dfrac{1}{n}\right) = \sin 0 = 0$ since $\dfrac{1}{n} \to 0$ as $n \to \infty$, so by Theorem 5, $\left\{(-1)^n \sin\left(\dfrac{1}{n}\right)\right\}$ converges to 0.

29. $b_n = \sqrt{n+2} - \sqrt{n} = \left(\sqrt{n+2} - \sqrt{n}\right)\dfrac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = \dfrac{2}{\sqrt{n+2}+\sqrt{n}} < \dfrac{2}{2\sqrt{n}} = \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$. So by the Squeeze Theorem with $a_n = 0$ and $c_n = 1/\sqrt{n}$, $\left\{\sqrt{n+2} - \sqrt{n}\right\}$ converges to 0.

30. $\lim\limits_{x\to\infty} \dfrac{\ln(2+e^x)}{3x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{e^x/(2+e^x)}{3} = \lim\limits_{x\to\infty} \dfrac{1}{6e^{-x}+3} = \dfrac{1}{3}$, so by Theorem 2, $\lim\limits_{n\to\infty} \dfrac{\ln(2+e^n)}{3n} = \dfrac{1}{3}$. Convergent

31. $\lim\limits_{x\to\infty} \dfrac{x}{2^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{(\ln 2)\,2^x} = 0$, so by Theorem 2, $\{n2^{-n}\}$ converges to 0.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1 + \dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Convergent

33. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\displaystyle\lim_{n\to\infty} \frac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

34. $y = (1+3x)^{1/x} \;\Rightarrow\; \ln(y) = \dfrac{1}{x}\ln(1+3x) \;\Rightarrow\; \displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+3x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3/(1+3x)}{1} = 0$

$\Rightarrow \displaystyle\lim_{x\to\infty} y = e^0 = 1$, so by Theorem 2, $\left\{(1+3n)^{1/n}\right\}$ converges to 1.

35. The series converges, since

$$a_n = \frac{1+2+3+\cdots+n}{n^2} = \frac{n(n+1)/2}{n^2} \text{ [sum of the first } n \text{ positive integers]} = \frac{n+1}{2n} = \frac{1+1/n}{2} \to \frac{1}{2}$$

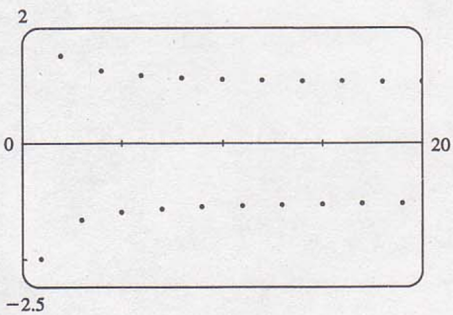
as $n \to \infty$.

36. $0 \le |a_n| = \dfrac{n\,|\cos n|}{n^2+1} \le \dfrac{n}{n^2+1} = \dfrac{1}{n+1/n} \to 0$ as $n\to\infty$, so by the Squeeze Theorem and Theorem 5, $\{a_n\}$ converges to 0.

37. $a_n = \dfrac{1}{2}\cdot\dfrac{2}{2}\cdot\dfrac{3}{2}\cdot\cdots\cdot\dfrac{(n-1)}{2}\cdot\dfrac{n}{2} \ge \dfrac{1}{2}\cdot\dfrac{n}{2} = \dfrac{n}{4} \to \infty$ as $n\to\infty$, so $\{a_n\}$ diverges.

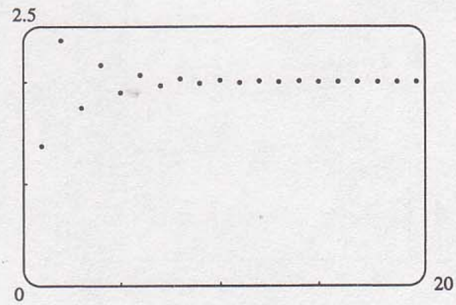
38. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{3}\cdot\cdots\cdot\dfrac{3}{(n-1)}\cdot\dfrac{3}{n} \le 3\cdot\dfrac{3}{2}\cdot\dfrac{3}{n} = \dfrac{27}{2n} \to 0$ as $n\to\infty$, so by the Squeeze Theorem and Theorem 5, $\{(-3)^n/n\}$ converges to 0.

39.



From the graph, we see that the sequence $\left\{(-1)^n \dfrac{n+1}{n}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

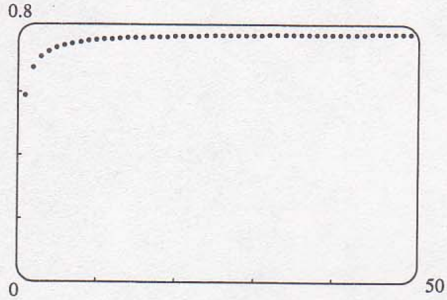
40.



From the graph, it appears that the sequence converges to 2.

$\left\{\left(-\frac{2}{\pi}\right)^n\right\}$ converges to 0 by (7), and hence $\left\{2+\left(-\frac{2}{\pi}\right)^n\right\}$ converges to $2+0=2$.

41.

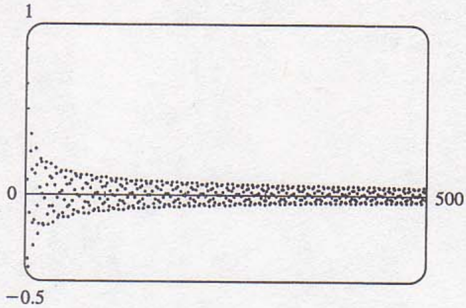


From the graph, it appears that the sequence converges to about 0.78.

$$\lim_{n\to\infty} \frac{2n}{2n+1} = \lim_{n\to\infty} \frac{2}{2+1/n} = 1, \text{ so}$$

$$\lim_{n\to\infty} \arctan\left(\frac{2n}{2n+1}\right) = \arctan 1 = \frac{\pi}{4}.$$

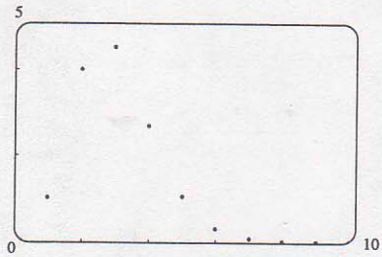
42.



From the graph, it appears that the sequence converges (slowly) to 0.

$0 \leq \dfrac{|\sin n|}{\sqrt{n}} \leq \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 5, $\left\{\dfrac{\sin n}{\sqrt{n}}\right\}$ converges to 0.

43.



From the graph, it appears that the sequence converges to 0.

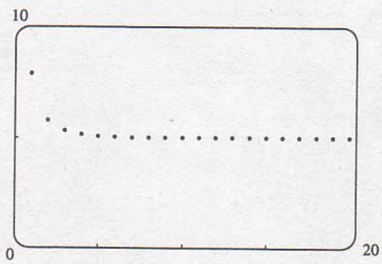
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdot \cdots \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\leq \frac{n^2}{(n-1)(n-2)(n-3)} \quad (\text{for } n \geq 4)$$

$$= \frac{1/n}{(1-1/n)(1-2/n)(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\{n^3/n!\}$ converges to 0.

44.



From the graph, it appears that the sequence converges to 5.

$$5 = \sqrt[n]{5^n} \leq \sqrt[n]{3^n + 5^n} \leq \sqrt[n]{5^n + 5^n} = \sqrt[n]{2}\,\sqrt[n]{5^n}$$

$$= \sqrt[n]{2} \cdot 5 \to 5 \text{ as } n \to \infty$$

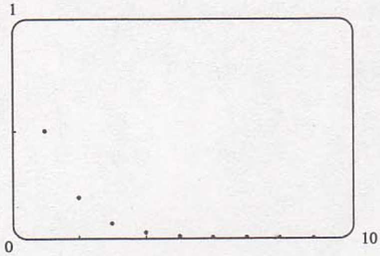
Hence, $a_n \to 5$ by the Squeeze Theorem.

Alternate Solution: Let $y = (3^x + 5^x)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(3^x + 5^x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5$$

so $\lim\limits_{x\to\infty} y = e^{\ln 5} = 5$, and so $\left\{\sqrt[n]{3^n + 5^n}\right\}$ converges to 5.

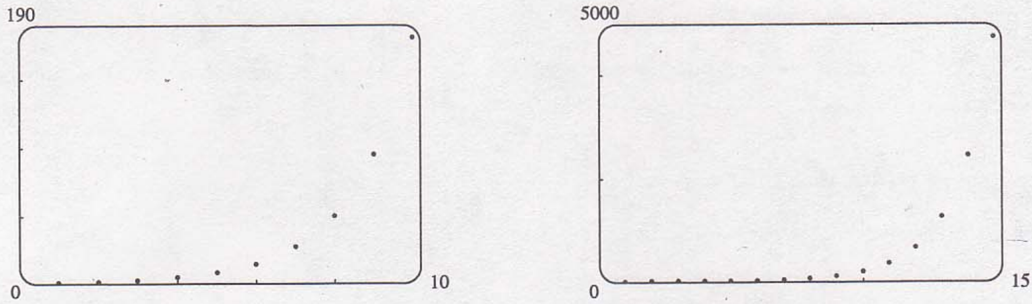
45. 

From the graph, it appears that the sequence approaches 0.

$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdot \cdots \cdot \frac{2n-1}{2n}$$

$$\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdot \cdots \cdot (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{ \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} \right\}$ converges to 0.

46. 

From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\dfrac{3}{2}\right)^{n-1}$ for all n. This is clearly true for $n = 1$, so let $P(n)$ be the statement that the above is true for n. We must show it is then true for $n+1$. $a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis). But $\dfrac{2n+1}{n+1} \geq \dfrac{3}{2}$ [since $2\,(2n+1) \geq 3\,(n+1) \;\Leftrightarrow\; 4n+2 \geq 3n+3 \;\Leftrightarrow\; n \geq 1$], and so we get that $a_{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^n$ which is $P\,(n+1)$. Thus, we have proved our first assertion, so since $\left\{ \left(\frac{3}{2}\right)^{n-1} \right\}$ diverges (by Equation 7), so does the given sequence $\{a_n\}$.

47. (a) $a_n = 1000\,(1.06)^n \quad\Rightarrow\quad a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (7) with $r = 1.06 > 1$.

48. $a_{n+1} = \begin{cases} \frac{1}{2} a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

49. If $|r| \geq 1$, then $\{r^n\}$ diverges by (7), so $\{nr^n\}$ diverges also, since $|nr^n| = n\,|r^n| \geq |r^n|$. If $|r| < 1$ then $\lim\limits_{x\to\infty} xr^x = \lim\limits_{x\to\infty} \dfrac{x}{r^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{(-\ln r)\,r^{-x}} = \lim\limits_{x\to\infty} \dfrac{r^x}{-\ln r} = 0$, so $\lim\limits_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges whenever $|r| < 1$.

50. (a) Let $\lim_{n\to\infty} a_n = L$. By Definition 1, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n + 1 > N \iff n > N - 1$. It follows that $\lim_{n\to\infty} a_{n+1} = L$ and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

(b) If $L = \lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow$ $L = \frac{-1+\sqrt{5}}{2}$ (since L has to be non-negative if it exists).

51. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent, that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

52. $a_n = 1/5^n$ defines a decreasing geometric sequence since $a_{n+1} = \frac{1}{5}a_n < a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

53. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

54. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

$f'(x) = \dfrac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$, and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \geq 1$.

55. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $|a_n| \leq 1$ for all $n \geq 1$.

56. $a_n = 3 + (-1)^n/n$ defines a sequence that is not monotonic. The first few terms are 2, 3.5, $2.\overline{6}$, 3.25, and 2.8, showing that the sequence is neither increasing nor decreasing. The sequence is bounded since $2 \leq a_n \leq 3.5$ for all $n \geq 1$.

57. $a_n = \dfrac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2+1}$,

$f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

58. $a_n = \dfrac{\sqrt{n}}{n+2}$ defines a sequence that is neither increasing nor decreasing since $a_1 < a_2$ and $a_2 > a_3$. ($a_1 = \frac{1}{3} = 0.\overline{3}$, $a_2 = \frac{\sqrt{2}}{4} \approx 0.354$, and $a_3 = \frac{\sqrt{3}}{5} \approx 0.346$.) But the sequence $\{a_n \mid n \geq 2\}$ obtained by omitting the first term a_1 *is* decreasing. To see this, note that if $f(x) = \dfrac{\sqrt{x}}{x+2}$ for $x \geq 0$, then

$f'(x) = \dfrac{\frac{x+2}{2\sqrt{x}} - \sqrt{x}}{(x+2)^2} = \dfrac{(x+2) - 2x}{2\sqrt{x}(x+2)^2} = \dfrac{2-x}{2\sqrt{x}(x+2)^2} \leq 0$ for $x \geq 2$. The sequence is bounded since $a_n > 0$ for all $n \geq 1$ and $a_n \leq a_2 = \frac{\sqrt{2}}{4}$ for all $n \geq 1$.

59. $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, ..., so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2$.

Alternate Solution: Let $L = \lim\limits_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.) So L must satisfy $L = \sqrt{2 \cdot L} \;\Rightarrow\; L^2 = 2L \;\Rightarrow\; L(L-2) = 0$ ($L \neq 0$ since the sequence increases), so $L = 2$.

60. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \;\Leftrightarrow\; \sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} \;\Leftrightarrow\; 2 + a_{n+1} \geq 2 + a_n \;\Leftrightarrow\; a_{n+1} \geq a_n$ which is the induction hypothesis. $a_{n+1} \leq 3 \;\Leftrightarrow\; \sqrt{2 + a_n} \leq 3 \;\Leftrightarrow\; 2 + a_n \leq 9 \;\Leftrightarrow\; a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n, and so $a_1 \leq a_n \leq 3$ (the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim\limits_{n\to\infty} a_n$ exists.

(b) If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so $L = \sqrt{2 + L} \;\Rightarrow\; L^2 - L - 2 = 0 \;\Rightarrow\; (L+1)(L-2) = 0 \;\Rightarrow\; L = 2$ (since L can't be negative).

61. We show by induction that $\{a_n\}$ is increasing and bounded above by 3.
Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1} > a_n \;\Rightarrow\; \dfrac{1}{a_{n+1}} < \dfrac{1}{a_n} \;\Rightarrow\; -\dfrac{1}{a_{n+1}} > -\dfrac{1}{a_n}$.
Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \;\Leftrightarrow\; P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem. If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \;\Rightarrow\; L^2 - 3L + 1 = 0 \;\Rightarrow\; L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3 + \sqrt{5}}{2}$.

62. We use induction. Let P_n be the statement that $0 < a_{n+1} \leq a_n \leq 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \leq a_n \;\Rightarrow\; -a_{n+1} \geq -a_n \;\Rightarrow\; 3 - a_{n+1} \geq 3 - a_n \;\Rightarrow\; a_{n+2} = \dfrac{1}{3 - a_{n+1}} \leq \dfrac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ (since $3 - a_{n+1}$ is positive) and $a_{n+1} \leq 2$ by the induction hypothesis, so P_{n+1} is true.
To find the limit, we use the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} \;\Rightarrow\; L = \frac{1}{3-L} \;\Rightarrow\; L^2 - 3L + 1 = 0 \;\Rightarrow\; L = \frac{3 \pm \sqrt{5}}{2}$. But $L \leq 2$, so we must have $L = \frac{3 - \sqrt{5}}{2}$.

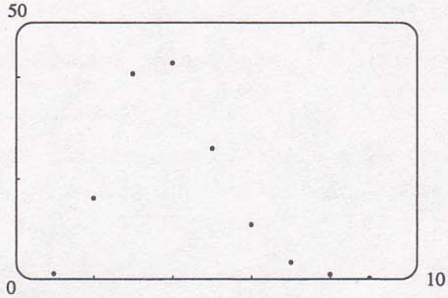
63. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

(b) $a_n = \dfrac{f_{n+1}}{f_n} \;\Rightarrow\; a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $L = \lim\limits_{n\to\infty} a_n$, then $L = \lim\limits_{n\to\infty} a_{n-1}$ and $L = \lim\limits_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \dfrac{1}{L} \;\Rightarrow\; L^2 - L - 1 = 0 \;\Rightarrow\; L = \frac{1 + \sqrt{5}}{2}$ (since L must be positive).

64. (a) If f is continuous, then $f(L) = f\left(\lim\limits_{n\to\infty} a_n\right) = \lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} a_{n+1} = L$ by Exercise 50(a).

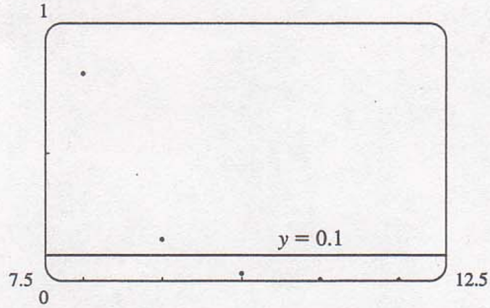
(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.

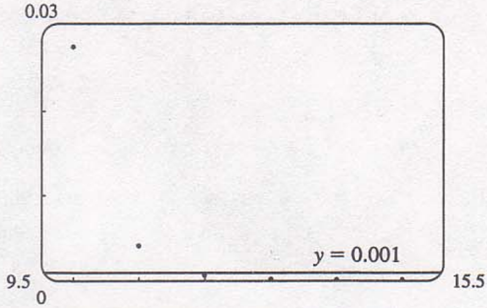
65. (a)



From the graph, it appears that the sequence $\left\{\dfrac{n^5}{n!}\right\}$ converges to 0, that is, $\displaystyle\lim_{n\to\infty}\frac{n^5}{n!}=0$.

(b)





From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon = 0.1$ is 9, since $n^5/n! < 0.1$ whenever $n \geq 10$, but $9^5/9! > 0.1$. From the second graph, it seems that for $\varepsilon = 0.001$, the smallest possible value for N is 11.

66. Let $\varepsilon > 0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n > N$ then $n > \ln(\varepsilon)/\ln|r| \;\Rightarrow\; n\ln|r| < \ln\varepsilon \;\;$ [since $|r| < 1 \;\Rightarrow\; \ln|r| < 0$] $\;\Rightarrow\; \ln(|r|^n) < \ln\varepsilon \;\Rightarrow\; |r|^n < \varepsilon \;\Rightarrow\; |r^n - 0| < \varepsilon$, and so by Definition 1, $\displaystyle\lim_{n\to\infty} r^n = 0$.

67. If $\displaystyle\lim_{n\to\infty}|a_n| = 0$ then $\displaystyle\lim_{n\to\infty} -|a_n| = 0$, and since $-|a_n| \leq a_n \leq |a_n|$, we have that $\displaystyle\lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

68. (a)
$$\frac{b^{n+1}-a^{n+1}}{b-a} = b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n$$
$$< b^n + b^{n-1}b + b^{n-2}b^2 + b^{n-3}b^3 + \cdots + bb^{n-1} + b^n = (n+1)\,b^n$$

(b) Since $b - a > 0$, we have $b^{n+1} - a^{n+1} < (n+1)\,b^n\,(b-a) \;\Rightarrow\; b^{n+1} - (n+1)\,b^n\,(b-a) < a^{n+1} \;\Rightarrow\; b^n\left[(n+1)\,a - nb\right] < a^{n+1}$.

(c) With this substitution, $(n+1)\,a - nb = 1$, and so $b^n = \left(1+\dfrac{1}{n}\right)^n < a^{n+1} = \left(1+\dfrac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1+\dfrac{1}{2n}\right)^n\left(\dfrac{1}{2}\right) < 1 \;\Rightarrow\; \left(1+\dfrac{1}{2n}\right)^n < 2 \;\Rightarrow\; \left(1+\dfrac{1}{2n}\right)^{2n} < 4$.

(e) $a_n < a_{2n}$ since $\{a_n\}$ is increasing, so $a_n < a_{2n} < 4$.

(f) Since $\{a_n\}$ is increasing and bounded above by 4, $a_1 \leq a_n \leq 4$, and so $\{a_n\}$ is bounded and monotonic, and hence has a limit by Theorem 10.

69. (a) First we show that $a > a_1 > b_1 > b$.

$a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$ (since $a > b$) $\Rightarrow$ $a_1 > b_1$. Also $a - a_1 = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) > 0$ and $b - b_1 = b - \sqrt{ab} = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right) < 0$, so $a > a_1 > b_1 > b$. In the same way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is, $a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$\begin{aligned} a_{k+2} - b_{k+2} &= \tfrac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \tfrac{1}{2}\left(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1}\right) \\ &= \tfrac{1}{2}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2 > 0 \\ a_{k+1} - a_{k+2} &= a_{k+1} - \tfrac{1}{2}(a_{k+1} + b_{k+1}) = \tfrac{1}{2}(a_{k+1} - b_{k+1}) > 0 \end{aligned}$$

and $b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}\left(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}\right) < 0 \Rightarrow$ $a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$, so the assertion is true for $n = k+1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{a_n + b_n}{2} \Rightarrow \alpha = \dfrac{\alpha + \beta}{2} \Rightarrow$ $2\alpha = \alpha + \beta \Rightarrow \alpha = \beta$.

70. (a) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim\limits_{n\to\infty} a_{2n+1} = L$, there exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then $n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim\limits_{n\to\infty} a_n = L$.

(b) $a_1 = 1$, $a_2 = 1 + \frac{1}{1+1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{5/2} = \frac{7}{5} = 1.4$, $a_4 = 1 + \frac{1}{12/5} = \frac{17}{12} = 1.41\overline{6}$, $a_5 = 1 + \frac{1}{29/12} = \frac{41}{29} \approx 1.413793$, $a_6 = 1 + \frac{1}{70/29} = \frac{99}{70} \approx 1.414286$, $a_7 = 1 + \frac{1}{169/70} = \frac{239}{169} \approx 1.414201$, $a_8 = 1 + \frac{1}{408/169} = \frac{577}{408} \approx 1.414216$. Notice that $a_1 < a_3 < a_5 < a_7$ and $a_2 > a_4 > a_6 > a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2} > a_{2n}$ and $a_{2n-1} < a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2} > a_{2k}$. Then $1 + a_{2k-2} > 1 + a_{2k} \Rightarrow$ $\dfrac{1}{1 + a_{2k-2}} < \dfrac{1}{1 + a_{2k}} \Rightarrow 1 + \dfrac{1}{1 + a_{2k-2}} < 1 + \dfrac{1}{1 + a_{2k}} \Rightarrow a_{2k-1} < a_{2k+1} \Rightarrow$ $1 + a_{2k-1} < 1 + a_{2k+1} \Rightarrow \dfrac{1}{1 + a_{2k-1}} > \dfrac{1}{1 + a_{2k+1}} \Rightarrow 1 + \dfrac{1}{1 + a_{2k-1}} > 1 + \dfrac{1}{1 + a_{2k+1}} \Rightarrow$ $a_{2k} > a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2, so both $\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and therefore convergent by Theorem 10. Let $\lim\limits_{n\to\infty} a_{2n} = L$. Then $\lim\limits_{n\to\infty} a_{2n+2} = L$ also. We have $a_{n+2} = 1 + \dfrac{1}{1 + 1 + 1/(1 + a_n)} = 1 + \dfrac{1}{(3 + 2a_n)/(1 + a_n)} = \dfrac{4 + 3a_n}{3 + 2a_n}$, so $a_{2n+2} = \dfrac{4 + 3a_{2n}}{3 + 2a_{2n}}$. Taking limits of both sides, we get $L = \dfrac{4 + 3L}{3 + 2L} \Rightarrow 3L + 2L^2 = 4 + 3L \Rightarrow L^2 = 2 \Rightarrow L = \sqrt{2}$ (since $L > 0$). Thus, $\lim\limits_{n\to\infty} a_{2n} = \sqrt{2}$. Similarly we find that $\lim\limits_{n\to\infty} a_{2n+1} = \sqrt{2}$. So, by part (a), $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

Laboratory Project □ Logistic Sequences

1. To write such a program in Maple it is best to calculate all the points first and then graph them. One possible sequence of commands [taking $p_0 = \frac{1}{2}$ and $k = 1.5$ for the difference equation] is

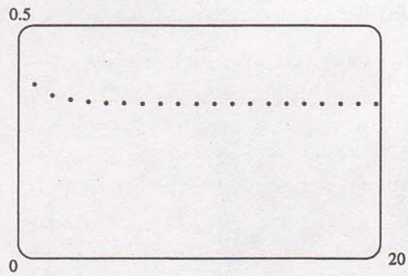
```
p(0):=1/2;k:=1.5;
for j from 1 to 20  do p(j):=k*p(j-1)*(1-p(j-1)) od;
plot({[t,p(t)] $t=0..20},t=0..20,p=0..0.5,style=point);
```

In Mathematica, we can use the following program:

```
p[0]=1/2
k=1.5
p[j_]:=k*p[j-1]*(1-p[j-1])
P=Table[p[t],{t,20}]
ListPlot[P]
```

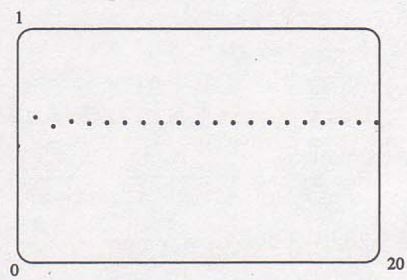
With $p_0 = \frac{1}{2}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.5	7	0.3338465076	14	0.3333373303
1	0.375	8	0.3335895255	15	0.3333353318
2	0.3515625	9	0.3334613309	16	0.3333343326
3	0.3419494629	10	0.3333973076	17	0.3333338329
4	0.3375300416	11	0.3333653143	18	0.3333335831
5	0.3354052689	12	0.3333493223	19	0.3333334582
6	0.3343628617	13	0.3333413274	20	0.3333333958



With $p_0 = \frac{1}{2}$ and $k = 2.5$:

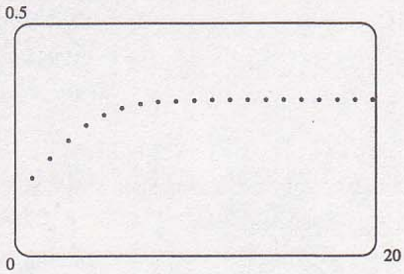
n	p_n	n	p_n	n	p_n
0	0.5	7	0.6004164790	14	0.5999967417
1	0.625	8	0.5997913269	15	0.6000016291
2	0.5859375	9	0.6001042277	16	0.5999991854
3	0.6065368651	10	0.5999478590	17	0.6000004073
4	0.5966247409	11	0.6000260637	18	0.5999997964
5	0.6016591486	12	0.5999869664	19	0.6000001018
6	0.5991635437	13	0.6000065164	20	0.5999999491



Both of these sequences seem to converge (the first to about $\frac{1}{3}$, the second to about 0.60).

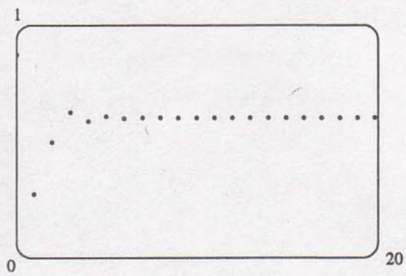
With $p_0 = \frac{7}{8}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.3239166554	14	0.3332554829
1	0.1640625	8	0.3284919837	15	0.3332943990
2	0.2057189941	9	0.3308775005	16	0.3333138639
3	0.2450980344	10	0.3320963702	17	0.3333235980
4	0.2775374819	11	0.3327125567	18	0.3333284655
5	0.3007656421	12	0.3330223670	19	0.3333308994
6	0.3154585059	13	0.3331777051	20	0.3333321164



With $p_0 = \frac{7}{8}$ and $k = 2.5$:

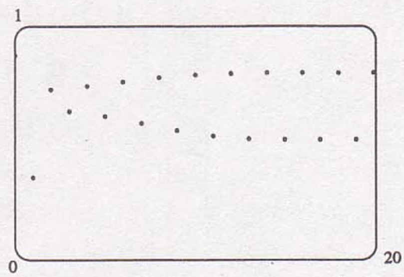
n	p_n	n	p_n	n	p_n
0	0.875	7	0.6016572368	14	0.5999869815
1	0.2734375	8	0.5991645155	15	0.6000065088
2	0.4966735840	9	0.6004159972	16	0.5999967455
3	0.6249723374	10	0.5997915688	17	0.6000016272
4	0.5859547872	11	0.6001041070	18	0.5999991864
5	0.6065294364	12	0.5999479194	19	0.6000004068
6	0.5966286980	13	0.6000260335	20	0.5999997966



The limit of the sequence seems to depend on k, but not on p_0.

2. With $p_0 = \frac{7}{8}$ and $k = 3.2$:

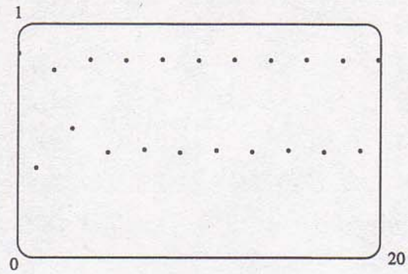
n	p_n	n	p_n	n	p_n
0	0.875	7	0.5830728495	14	0.7990633827
1	0.35	8	0.7779164854	15	0.5137954979
2	0.728	9	0.5528397669	16	0.7993909896
3	0.6336512	10	0.7910654689	17	0.5131681132
4	0.7428395416	11	0.5288988570	18	0.7994451225
5	0.6112926626	12	0.7973275394	19	0.5130643795
6	0.7603646184	13	0.5171082698	20	0.7994538304



It seems that eventually the terms fluctuate between two values (about 0.5 and 0.8 in this case).

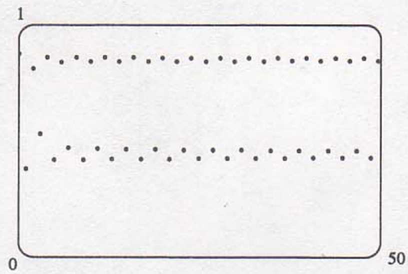
3. With $p_0 = \frac{7}{8}$ and $k = 3.42$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4523028596	14	0.8442074951
1	0.3740625	8	0.8472194412	15	0.4498025048
2	0.8007579316	9	0.4426802161	16	0.8463823232
3	0.5456427596	10	0.8437633929	17	0.4446659586
4	0.8478752457	11	0.4508474156	18	0.8445284520
5	0.4411212220	12	0.8467373602	19	0.4490464985
6	0.8431438501	13	0.4438243545	20	0.8461207931



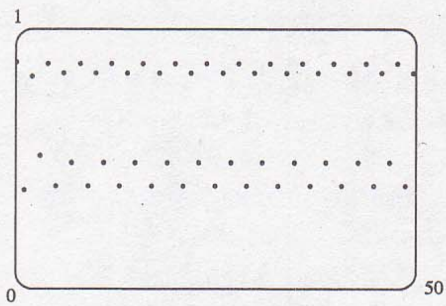
With $p_0 = \frac{7}{8}$ and $k = 3.45$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4670259170	14	0.8403376122
1	0.37734375	8	0.8587488490	15	0.4628875685
2	0.8105962830	9	0.4184824586	16	0.8577482026
3	0.5296783241	10	0.8395743720	17	0.4209559716
4	0.8594612299	11	0.4646778983	18	0.8409445432
5	0.4167173034	12	0.8581956045	19	0.4614610237
6	0.8385707740	13	0.4198508858	20	0.8573758782

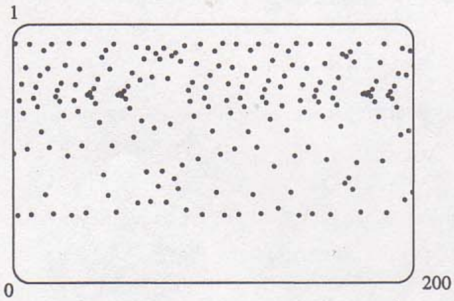


From the graphs above, it seems that for k between 3.4 and 3.5, the terms eventually fluctuate between four values. In the graph below, the pattern followed by the terms is $0.395, 0.832, 0.487, 0.869, 0.395, \ldots$. Note that even for $k = 3.42$ (as in the first graph), there are four distinct "branches; even after 1000 terms, the first and third terms in the pattern differ by about 2×10^{-9}, while the first and fifth terms differ by only 2×10^{-10}.

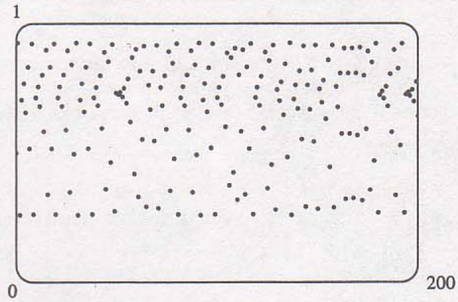
With $p_0 = \frac{7}{8}$ and $k = 3.48$:



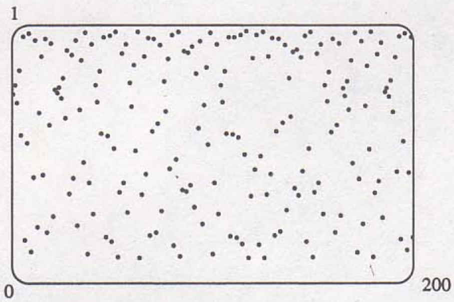
4.



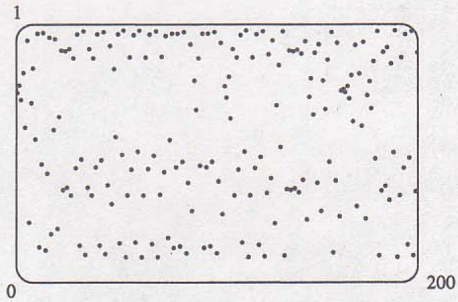
$p_0 = 0.5, k = 3.7$



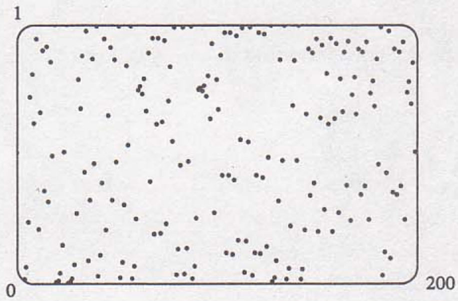
$p_0 = 0.501, k = 3.7$



$p_0 = 0.75, k = 3.9$



$p_0 = 0.749, k = 3.9$



$p_0 = 0.5, k = 3.999$

From the graphs, it seems that if p_0 is changed by 0.001, the whole graph changes completely. (Note, however, that this might be partially due to accumulated round-off error in the CAS. These graphs were generated by Maple with 100-digit accuracy, and different degrees of accuracy give different graphs.) There seem to be some some fleeting patterns in these graphs, but on the whole they are certainly very chaotic. As k increases, the graph spreads out vertically, with more extreme values close to 0 or 1.

12.2 Series ET 11.2

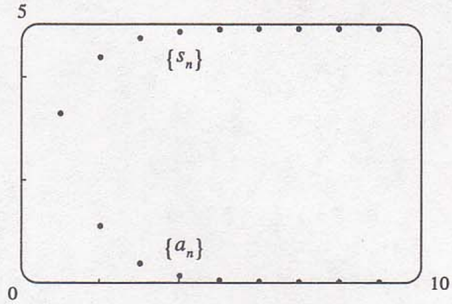
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

   (b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

3.

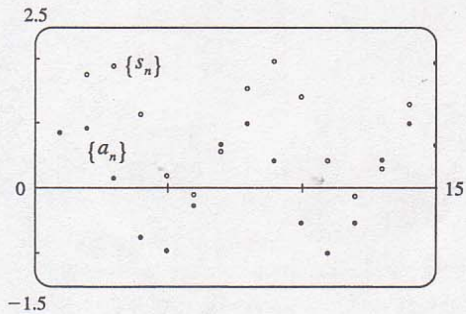
n	s_n
1	3.33333
2	4.44444
3	4.81481
4	4.93827
5	4.97942
6	4.99314
7	4.99771
8	4.99924
9	4.99975
10	4.99992
11	4.99997
12	4.99999



From the graph, it seems that the series converges. In fact, it is a geometric series with $a = \frac{10}{3}$ and $r = \frac{1}{3}$, so its sum is $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3^n} = \frac{10/3}{1 - 1/3} = 5$. Note that the dot corresponding to $n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

4.

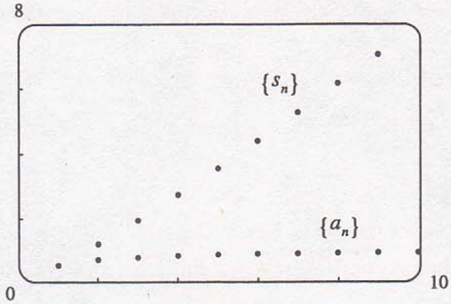
n	s_n
1	0.8415
2	1.7508
3	1.8919
4	1.1351
5	0.1762
6	−0.1033
7	0.5537
8	1.5431
9	1.9552
10	1.4112
11	0.4112
12	−0.1254



The series diverges,since its terms do not approach 0.

5.

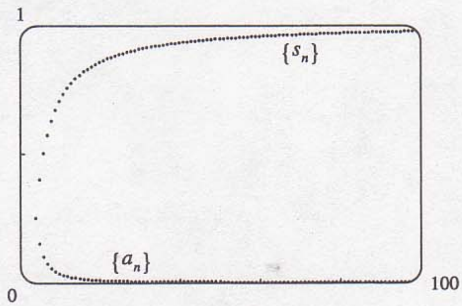
n	s_n
1	0.50000
2	1.16667
3	1.91667
4	2.71667
5	3.55000
6	4.40714
7	5.28214
8	6.17103
9	7.07103
10	7.98012



The series diverges,since its terms do not approach 0.

6.

n	s_n
4	0.25000
5	0.40000
6	0.50000
7	0.57143
8	0.62500
9	0.66667
10	0.70000
11	0.72727
12	0.75000
13	0.76923
...	...
99	0.96970
100	0.97000



From the graph, it seems that the series converges to about 1. To find the sum, we proceed as in Example 6: since $\dfrac{3}{i\,(i-1)} = \dfrac{3}{i-1} - \dfrac{3}{i}$, the partial sums are

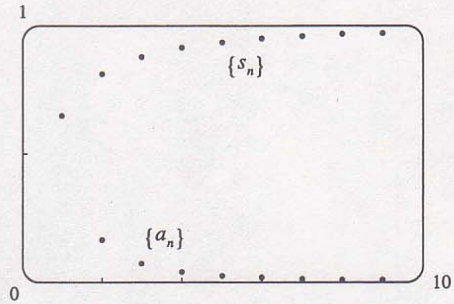
$$s_n = \sum_{i=4}^{n} \left(\frac{3}{i-1} - \frac{3}{i} \right) = \left(\frac{3}{3} - \frac{3}{4} \right) + \left(\frac{3}{4} - \frac{3}{5} \right) + \cdots$$

$$+ \left(\frac{3}{n-2} - \frac{3}{n-1} \right) + \left(\frac{3}{n-1} - \frac{3}{n} \right) = 1 - \frac{3}{n}$$

and so the sum is $\lim\limits_{n\to\infty} s_n = 1$.

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



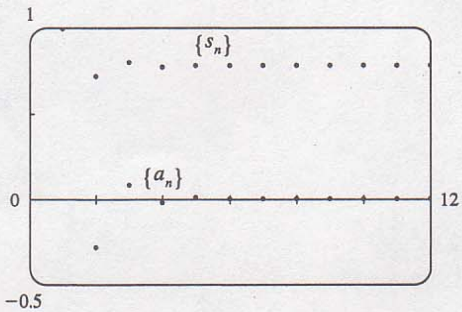
From the graph, it seems that the series converges to 1. To find the sum, we write

$$s_n = \sum_{i=1}^{n}\left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}}\right) = \left(1 - \frac{1}{2^{1.5}}\right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right)$$
$$+ \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}}\right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right) = 1 - \frac{1}{(n+1)^{1.5}}$$

So the sum is $\lim_{n\to\infty} s_n = 1$.

8.

n	s_n
1	1.000000
2	0.714286
3	0.795918
4	0.772595
5	0.779259
6	0.777355
7	0.777899
8	0.777743
9	0.777788
10	0.777775
11	0.777779
12	0.777778



From the graph, it seems that the series converges to about 0.8. In fact, it is a geometric series with $a = 1$ and $r = -\frac{2}{7}$, so its sum is

$$\sum_{n=1}^{\infty}\left(-\frac{2}{7}\right)^{n-1} = \frac{1}{1-(-2/7)} = \frac{7}{9}.$$

9. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (12.1.1 [ET 11.1.1]).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).

10. (a) Both $\sum\limits_{i=1}^{n} a_i$ and $\sum\limits_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

(b) $\sum\limits_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum\limits_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \cdots$ is a geometric series with $a = 4$ and $r = \frac{2}{5}$. Since $|r| = \frac{2}{5} < 1$, the series converges to $\frac{4}{1-2/5} = \frac{4}{3/5} = \frac{20}{3}$.

12. $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ is a geometric series with $a = 1$ and $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series diverges.

13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

15. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.

16. $\sum\limits_{n=1}^{\infty} \dfrac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

17. $\sum\limits_{n=1}^{\infty} \dfrac{(-3)^{n-1}}{4^n} = \dfrac{1}{4}\sum\limits_{n=1}^{\infty}\left(-\dfrac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

18. $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{e^2}\right)^n \quad \Rightarrow \quad a = \dfrac{1}{e^2} = |r| < 1$, so the series converges to $\dfrac{1/e^2}{1-1/e^2} = \dfrac{1}{e^2-1}$.

19. For $\sum_{n=1}^{\infty} 3^{-n}8^{n+1} = \sum_{n=1}^{\infty} 8\left(\frac{8}{3}\right)^n$, $a = \frac{64}{3}$ and $r = \frac{8}{3} > 1$, so the series diverges.

20. $\sum_{n=0}^{\infty} 4\left(\frac{4}{5}\right)^n \quad \Rightarrow \quad a = 4$, $|r| = \frac{4}{5} < 1$, so the series converges to $\frac{4}{1-4/5} = 20$.

21. $\sum\limits_{n=1}^{\infty} \dfrac{n}{n+5}$ diverges since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

22. $\sum_{n=1}^{\infty}(3/n) = 3\sum_{n=1}^{\infty}(1/n)$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty}(1/n)$, which diverges. [If $\sum_{n=1}^{\infty}(3/n)$ were to converge, then $\sum_{n=1}^{\infty}(1/n)$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Converges. $s_n = \sum\limits_{i=1}^{n} \dfrac{1}{i(i+2)} = \sum\limits_{i=1}^{n}\left(\dfrac{1/2}{i} - \dfrac{1/2}{i+2}\right)$ (using partial fractions) $= \dfrac{1}{2}\sum\limits_{i=1}^{n}\left(\dfrac{1}{i} - \dfrac{1}{i+2}\right)$. The latter sum is a telescoping series:

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

Thus, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+2)} = \dfrac{1}{2}\lim\limits_{n\to\infty}\left(1 + \dfrac{1}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}\right) = \dfrac{1}{2}\left(1 + \dfrac{1}{2}\right) = \dfrac{3}{4}$.

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty}\left(1+\frac{1}{n^2+2n}\right) = 1 \neq 0.$$

25. $\sum_{n=1}^{\infty}\left[2(0.1)^n + (0.2)^n\right] = 2\sum_{n=1}^{\infty}(0.1)^n + \sum_{n=1}^{\infty}(0.2)^n$. These are convergent geometric series and so by Theorem 8, their sum is also convergent. $2\left(\frac{0.1}{1-0.1}\right) + \frac{0.2}{1-0.2} = \frac{2}{9} + \frac{1}{4} = \frac{17}{36}$

26. Converges. $s_n = \displaystyle\sum_{i=1}^{n} \frac{2}{i^2+4i+3} = \sum_{i=1}^{n}\left(\frac{1}{i+1} - \frac{1}{i+3}\right)$ (using partial fractions). The latter sum is

$$\left(\tfrac{1}{2}-\tfrac{1}{4}\right)+\left(\tfrac{1}{3}-\tfrac{1}{5}\right)+\left(\tfrac{1}{4}-\tfrac{1}{6}\right)+\left(\tfrac{1}{5}-\tfrac{1}{7}\right)+\cdots+\left(\tfrac{1}{n}-\tfrac{1}{n+2}\right)+\left(\tfrac{1}{n+1}-\tfrac{1}{n+3}\right) = \tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{n+2}-\tfrac{1}{n+3}$$

(telescoping series). Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right) = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}$.

27. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{\sqrt{1+n^2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}} = 1 \neq 0$, so the series diverges by the Test for Divergence.

28. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}+\frac{2}{3^{n-1}}\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} + 2\sum_{n=1}^{\infty}\frac{1}{3^{n-1}} = \frac{1}{1-1/2} + 2\left(\frac{1}{1-1/3}\right) = 5$

29. Converges. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{3^n}{6^n}+\frac{2^n}{6^n}\right) = \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

30. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty}\left(\frac{1}{2+5/n}\right) = \ln\tfrac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\arctan n = \tfrac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{5+2^{-n}} = \frac{1}{5} \neq 0$, so the series diverges by the Test for Divergence.

33. $s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + [\ln n - \ln(n+1)] = \ln 1 - \ln(n+1) = -\ln(n+1)$ (telescoping series). Thus, $\lim\limits_{n\to\infty} s_n = -\infty$, so the series is divergent.

34. $s_n = \displaystyle\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)} = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1}{i+1} + \frac{1/2}{i+2}\right) = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1/2}{i+1}\right) + \sum_{i=1}^{n}\left(-\frac{1/2}{i+1} + \frac{1/2}{i+2}\right)$, both of which are clearly telescoping sums, so

$$s_n = \left[\frac{1}{2} - \frac{1}{2(n+1)}\right] + \left[-\frac{1}{4} + \frac{1}{2(n+2)}\right] = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} s_n = \frac{1}{4}$.

35. $0.\overline{2} = \dfrac{2}{10} + \dfrac{2}{10^2} + \cdots = \dfrac{2/10}{1-1/10} = \dfrac{2}{9}$

36. $0.\overline{73} = \dfrac{73}{10^2} + \dfrac{73}{10^4} + \cdots = \dfrac{73/10^2}{1-1/10^2} = \dfrac{73/100}{99/100} = \dfrac{73}{99}$

37. $3.\overline{417} = 3 + \dfrac{417}{10^3} + \dfrac{417}{10^6} + \cdots = 3 + \dfrac{417/10^3}{1-1/10^3} = 3 + \dfrac{417}{999} = \dfrac{3414}{999} = \dfrac{1138}{333}$

38. $6.2\overline{54} = 6.2 + \dfrac{54}{10^3} + \dfrac{54}{10^5} + \cdots = 6.2 + \dfrac{54/10^3}{1 - 1/10^2} = \dfrac{62}{10} + \dfrac{54}{990} = \dfrac{6192}{990} = \dfrac{344}{55}$

39. $0.123\overline{456} = \dfrac{123}{1000} + \dfrac{0.000456}{1 - 0.001} = \dfrac{123}{1000} + \dfrac{456}{999{,}000} = \dfrac{123{,}333}{999{,}000} = \dfrac{41{,}111}{333{,}000}$

40. $5.\overline{6021} = 5 + \dfrac{6021}{10^4} + \dfrac{6021}{8^4} + \cdots = 5 + \dfrac{6021/10^4}{1 - 1/10^4} = 5 + \dfrac{6021}{9999} = \dfrac{56{,}016}{9999} = \dfrac{6224}{1111}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$. In that case, the sum of the series is $\dfrac{x/3}{1 - x/3} = \dfrac{x}{3 - x}$.

42. $\sum_{n=1}^{\infty} (x-4)^n$ is a geometric series with $r = x - 4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x - 4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)} = \dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4\,|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1 - 4x}$.

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ is a geometric series with $r = \dfrac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x + 3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is $\dfrac{1}{1 - (x+3)/2} = \dfrac{2}{2 - (x+3)} = -\dfrac{2}{x+1}$.

45. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$ is geometric with $r = \dfrac{1}{x}$, so it converges whenever $\left|\dfrac{1}{x}\right| < 1$ $\Leftrightarrow$ $|x| > 1$ $\Leftrightarrow$ $x > 1$ or $x < -1$, and the sum is $\dfrac{1}{1 - 1/x} = \dfrac{x}{x-1}$.

46. $\sum_{n=0}^{\infty} \tan^n x$ is geometric and converges when $|\tan x| < 1$ $\Leftrightarrow$ $-1 < \tan x < 1$ $\Leftrightarrow$ $n\pi - \frac{\pi}{4} < x < n\pi + \frac{\pi}{4}$ (n any integer). On these intervals the sum is $\dfrac{1}{1 - \tan x}$.

47. After defining f, We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)} = -\dfrac{1/4}{4n+1} + \dfrac{1/4}{4n-3}$. So the nth partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(-\frac{1/4}{4k+1} + \frac{1/4}{4k-3}\right) = \frac{1}{4}\left(\frac{1}{4k-3} - \frac{1}{4k+1}\right) \\ &= \frac{1}{4}\left[\left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \cdots + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)\right] = \frac{1}{4}\left(1 - \frac{1}{4n+1}\right) \end{aligned}$$

The series converges to $\displaystyle\lim_{n\to\infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using `sum(f,1..infinity);` (in Maple), `Sum[f,{n,1,Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and `Simplify` (in Derive).

48. See Exercise 47 for specific CAS commands. $\dfrac{n^2+3n+1}{(n^2+n)^2}=\dfrac{1}{n^2}+\dfrac{1}{n}-\dfrac{1}{(n+1)^2}-\dfrac{1}{n+1}$. So the nth partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n}\left(\frac{1}{k^2}+\frac{1}{k}-\frac{1}{(k+1)^2}-\frac{1}{k+1}\right) \\ &= \left(1+1-\frac{1}{2^2}-\frac{1}{2}\right)+\left(\frac{1}{2^2}+\frac{1}{2}-\frac{1}{3^2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n^2}+\frac{1}{n}-\frac{1}{(n+1)^2}-\frac{1}{n+1}\right) \\ &= 1+1-\frac{1}{(n+1)^2}-\frac{1}{n+1} \end{aligned}$$

The series converges to $\lim\limits_{n\to\infty} s_n = 2$.

49. For $n=1$, $a_1=0$ since $s_1=0$. For $n>1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1}-\frac{(n-1)-1}{(n-1)+1}=\frac{(n-1)\,n-(n+1)\,(n-2)}{(n+1)\,n}=\frac{2}{n\,(n+1)}$$

Also, $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\dfrac{1-1/n}{1+1/n}=1.$

50. $a_1=s_1=\frac{5}{2}$. For $n\neq 1$,

$$a_n = s_n - s_{n-1} = \left(3-n2^{-n}\right)-\left[3-(n-1)\,2^{-(n-1)}\right]=-\frac{n}{2^n}+\frac{n-1}{2^{n-1}}\cdot\frac{2}{2}=\frac{2\,(n-1)}{2^n}-\frac{n}{2^n}=\frac{n-2}{2^n}$$

Also, $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(3-\dfrac{n}{2^n}\right)=3$ because $\lim\limits_{x\to\infty}\dfrac{x}{2^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1}{2^x\ln 2}=0.$

51. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is $S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \dfrac{D\,(1-c^n)}{1-c}$ by (3).

(b) $$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\frac{D\,(1-c^n)}{1-c}=\frac{D}{1-c}\lim_{n\to\infty}(1-c^n)=\frac{D}{1-c}\quad \left(\text{since } 0<c<1 \;\Rightarrow\; \lim_{n\to\infty} c^n=0\right) \\ &= \frac{D}{s}\quad (\text{since } c+s=1) \;=\; kD\quad (\text{since } k=1/s)\end{aligned}$$

If $c=0.8$, then $s=1-c=0.2$ and the multiplier is $k=1/s=5$.

52. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$\begin{aligned} H+2rH+2r^2H+2r^3H+\cdots &= H\left(1+2r+2r^2+2r^3+\cdots\right) \\ &= H\left[1+2r\left(1+r+r^2+\cdots\right)\right]=H\left[1+2r\left(\frac{1}{1-r}\right)\right]=H\left(\frac{1+r}{1-r}\right)\text{ meters}\end{aligned}$$

(b) From Example 3 in Section 2.1 [ET 2.1], we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t=\sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\begin{aligned} \sqrt{\frac{2H}{g}}+2\sqrt{\frac{2H}{g}r}+2\sqrt{\frac{2H}{g}r^2}+2\sqrt{\frac{2H}{g}r^3}+\cdots &= \sqrt{\frac{2H}{g}}\left[1+2\sqrt{r}+2\sqrt{r}^2+2\sqrt{r}^3+\cdots\right] \\ &= \sqrt{\frac{2H}{g}}\left(1+2\sqrt{r}\left[1+\sqrt{r}+\sqrt{r}^2+\cdots\right]\right)=\sqrt{\frac{2H}{g}}\left[1+2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right]=\sqrt{\frac{2H}{g}}\frac{1+\sqrt{r}}{1-\sqrt{r}}\end{aligned}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$: $\dfrac{d^2y}{dt^2} = -g \;\Rightarrow\; v = \dfrac{dy}{dt} = v_0 - gt \;\Rightarrow\; y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
...	...	...	...	...	...

The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots = \sqrt{\frac{2H}{g}}\left(1 + 2k + 2k^2 + 2k^3 + \cdots\right)$$

$$= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(1 + k + k^2 + \cdots\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+k}{1-k}$$

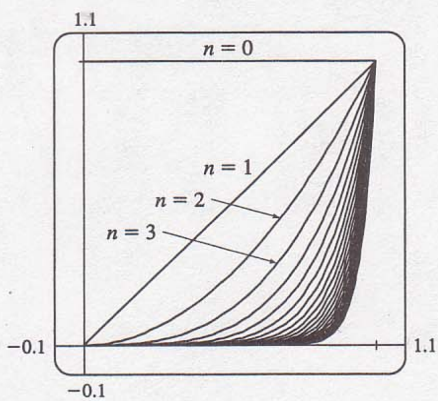
Another Method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right| < 1 \;\Leftrightarrow\; |1+c| > 1 \;\Leftrightarrow\; 1+c > 1$ or $1+c < -1 \;\Leftrightarrow\; c > 0$ or $c < -2$. We calculate the sum of the series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \;\Leftrightarrow\; \left(\dfrac{1}{1+c}\right)^2 = 2 - 2\left(\dfrac{1}{1+c}\right) \;\Leftrightarrow$ $1 = 2(1+c)^2 - 2(1+c) = 0 \;\Leftrightarrow\; 2c^2 + 2c - 1 = 0 \;\Leftrightarrow\; c = \frac{-2\pm\sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

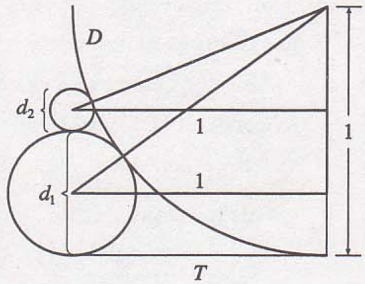
54. The area between $y = x^{n-1}$ and $y = x^n$ for $0 \le x \le 1$ is

$$\int_0^1 \left(x^{n-1} - x^n\right)dx = \left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$



We can see from the diagram that as $n \to \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1. So $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$.



55. Let d_n be the diameter of C_n. We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write $1^2 + \left(1 - \frac{1}{2}d_1\right)^2 = \left(1 + \frac{1}{2}d_1\right)^2 \quad \Leftrightarrow$
$1 = \left(1 + \frac{1}{2}d_1\right)^2 - \left(1 - \frac{1}{2}d_1\right)^2 = 2d_1$ (difference of squares)
$\Rightarrow \quad d_1 = \frac{1}{2}$. Similarly,

$$1 = \left(1 + \tfrac{1}{2}d_2\right)^2 - \left(1 - d_1 - \tfrac{1}{2}d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$$
$$= (2 - d_1)(d_1 + d_2) \quad \Leftrightarrow$$

$d_2 = \dfrac{1}{2 - d_1} - d_1 = \dfrac{(1 - d_1)^2}{2 - d_1}$, $1 = \left(1 + \frac{1}{2}d_3\right)^2 - \left(1 - d_1 - d_2 - \frac{1}{2}d_3\right)^2 \quad \Leftrightarrow \quad d_3 = \dfrac{[1 - (d_1 + d_2)]^2}{2 - (d_1 + d_2)}$, and in general, $d_{n+1} = \dfrac{\left(1 - \sum_{i=1}^n d_i\right)^2}{2 - \sum_{i=1}^n d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\dfrac{1}{6} = \dfrac{1}{2 \cdot 3}$ and $\dfrac{1}{12} = \dfrac{1}{3 \cdot 4}$ respectively, so we suspect that in general, $d_n = \dfrac{1}{n(n+1)}$. To prove this, we use induction: assume that for all $k \le n$, $d_k = \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Then

$\displaystyle\sum_{i=1}^n d_i = 1 - \frac{1}{n+1} = \frac{n}{n+1}$ (telescoping sum). Substituting this into our formula for d_{n+1}, we get

$$d_{n+1} = \frac{\left[1 - \dfrac{n}{n+1}\right]^2}{2 - \left(\dfrac{n}{n+1}\right)} = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{n+2}{n+1}} = \frac{1}{(n+1)(n+2)},$$ and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^n d_i$ of the diameters of the circles approach 1 as $n \to \infty$; that is,
$\displaystyle\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n(n+1)} = 1$, which is what we wanted to prove.

56. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta, \ldots$. Therefore,
$\displaystyle |CD| + |DE| + |EF| + |FG| + \cdots = b\sum_{n=1}^\infty \sin^n\theta = b\left(\frac{\sin\theta}{1 - \sin\theta}\right)$ since this is a geometric series with $r = \sin\theta$ and $|\sin\theta| < 1$ (because $0 < \theta < \frac{\pi}{2}$).

57. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

58. If $\sum\limits_{n=1}^\infty a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim\limits_{n\to\infty} \dfrac{1}{a_n} \ne 0$, and so $\sum\limits_{n=1}^\infty \dfrac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^\infty ca_n = \lim\limits_{n\to\infty} \sum_{i=1}^n ca_i = \lim\limits_{n\to\infty} c\sum_{i=1}^n a_i = c\lim\limits_{n\to\infty} \sum_{i=1}^n a_i = c\sum_{n=1}^\infty a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum (1/c)(ca_n) = \sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum (a_n + b_n)$ converges. Then by Theorem 8(iii), so would $\sum [(a_n + b_n) - a_n] = \sum b_n$, a contradiction.

62. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum(-n)$, which both diverge, yet $\sum(a_n + b_n) = \sum 0$, which converges with sum 0.

63. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \leq 1000$ for all n. So by Theorem 12.1.10 [ET 11.1.10], the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

64. (a) $$\text{RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{(f_{n-1} + f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{1}{f_{n-1} f_{n+1}} = \text{LHS}$$

(b) $$\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right) \quad \text{[from part (a)]}$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}\right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}\right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}}\right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1\cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

(c) $$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_n f_{n+1}}\right) \text{ (as above) } = \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}\right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

65. (a) At the first step, only the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, which have a total length of $2 \cdot \left(\frac{1}{3}\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the nth step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1} \cdot \left(\frac{1}{3}\right)^n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum\limits_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1 - 2/3} = 1$ (geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$). Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n, \left(\frac{2}{3}\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1 - \left(\frac{2}{3}\right)^n, 1 - \left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot \left(\frac{1}{9}\right)^2$; at the third step, $(8)^2 \cdot \left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is $\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1 - 8/9} = 1$.

66. (a)

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650
a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}, \frac{8}{3}$, 2, 3, 667, and 334. Note that the limits appear to be "weighted" more toward a_2. In general, we guess that the limit is $\dfrac{a_1 + 2a_2}{3}$.

(b) $a_{n+1} - a_n = \frac{1}{2}(a_n + a_{n-1}) - a_n = -\frac{1}{2}(a_n - a_{n-1}) = -\frac{1}{2}\left[\frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1}\right]$

$= -\frac{1}{2}\left[-\frac{1}{2}(a_{n-1} - a_{n-2})\right] = \cdots = \left(-\frac{1}{2}\right)^{n-1}(a_2 - a_1)$

Note that we have used the formula $a_k = \frac{1}{2}(a_{k-1} + a_{k-2})$ a total of $n-1$ times in this calculation, once for each k between 3 and $n+1$. Now we can write

$$\begin{aligned} a_n &= a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) \\ &= a_1 + \textstyle\sum_{k=1}^{n-1}(a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1}\left(-\frac{1}{2}\right)^{k-1}(a_2 - a_1) \end{aligned}$$

and so

$$\begin{aligned} \lim_{n\to\infty} a_n &= a_1 + (a_2 - a_1)\textstyle\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1} = a_1 + (a_2 - a_1)\left[\frac{1}{1-(-1/2)}\right] \\ &= a_1 + \tfrac{2}{3}(a_2 - a_1) = \frac{a_1 + 2a_2}{3} \end{aligned}$$

67. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!} \Rightarrow s_1 = \frac{1}{1\cdot 2} = \frac{1}{2}, s_2 = \frac{1}{2} + \frac{2}{1\cdot 2\cdot 3} = \frac{5}{6}, s_3 = \frac{5}{6} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{23}{24}$,

$s_4 = \dfrac{23}{24} + \dfrac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \dfrac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n = \dfrac{(n+1)! - 1}{(n+1)!}$.

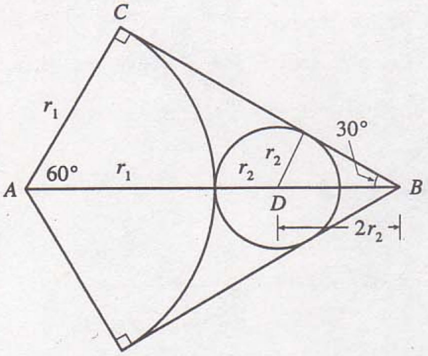
(b) For $n = 1$, $s_1 = \dfrac{1}{2} = \dfrac{2! - 1}{2!}$, so the formula holds for $n = 1$. Assume $s_k = \dfrac{(k+1)! - 1}{(k+1)!}$. Then

$$\begin{aligned} s_{k+1} &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+1)!\,(k+2)} \\ &= \frac{(k+2)! - (k+2) + k + 1}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!} \end{aligned}$$

Thus, the formula is true for $n = k+1$. So by induction, the guess is correct.

(c) $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{(n+1)! - 1}{(n+1)!} = \lim_{n\to\infty}\left[1 - \frac{1}{(n+1)!}\right] = 1$ and so $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n+1)!} = 1$.

68.



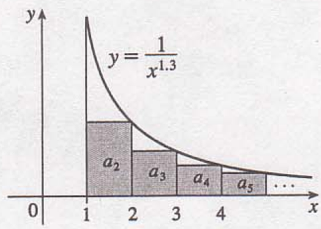
Let r_1 = radius of the large circle, r_2 = radius of next circle, and so on. From the figure we have $\angle BAC = 60°$ and $\cos 60° = r_1/|AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$. Therefore, $2r_1 = r_1 + r_2 + 2r_2 = r_1 + 3r_2 \Rightarrow$ $r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3}r_n$, so the total area is

$$\begin{aligned} A &= \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots \\ &= \pi r_1^2 + 3\pi r_2^2\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right) \\ &= \pi r_1^2 + 3\pi r_2^2 \cdot \frac{1}{1 - 1/9} = \pi r_1^2 + \tfrac{27}{8}\pi r_2^2 \end{aligned}$$

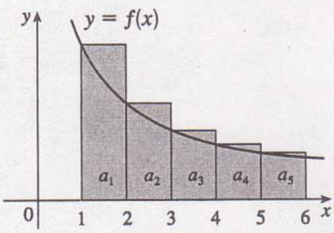
Since the sides of the triangle have length 1, $|BC| = \frac{1}{2}$ and $\tan 30° = \dfrac{r_1}{1/2}$. Thus, $r_1 = \dfrac{\tan 30°}{2} = \dfrac{1}{2\sqrt{3}} \Rightarrow$ $r_2 = \frac{1}{6\sqrt{3}}$, so $A = \pi\left(\frac{1}{2\sqrt{3}}\right)^2 + \frac{27\pi}{8}\left(\frac{1}{6\sqrt{3}}\right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

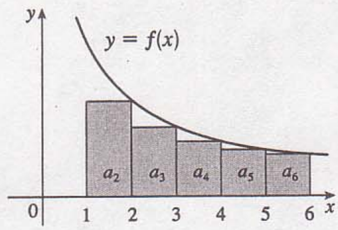
12.3 The Integral Test and Estimates of Sums — ET 11.3

1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$, $a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (8.8.2 [ET 7.8.2]) with $p = 1.3 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)\,dx < \sum_{i=1}^5 a_i$. From the second figure, we see that $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^5 a_i$.





3. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{b\to\infty}\int_1^b x^{-4}\,dx = \lim_{b\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{3b^3} + \frac{1}{3}\right) = \frac{1}{3}, \text{ so } \sum_{n=1}^{\infty}\frac{1}{n^4} \text{ converges.}$$

4. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} x^{-1/4}\,dx = \lim\limits_{b\to\infty}\int_1^b x^{-1/4}\,dx = \lim\limits_{b\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^b = \lim\limits_{b\to\infty}\left(\frac{4}{3}b^{3/4} - \frac{4}{3}\right) = \infty$, so $\sum_{n=1}^{\infty} 1/\sqrt[4]{n}$ diverges.

5. The function $f(x) = 1/(3x+1)$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty}\frac{dx}{3x+1} = \lim_{b\to\infty}\int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3x+1)\right]_1^b = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3b+1) - \tfrac{1}{3}\ln 4\right] = \infty$$

so the improper integral diverges, and so does the series $\sum_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x) = e^{-x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. $\int_1^\infty e^{-x}\,dx = \lim\limits_{b\to\infty} \int_1^b e^{-x}\,dx = \lim\limits_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim\limits_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = e^{-1}$, so $\sum_{n=1}^\infty e^{-n}$ converges. *Note:* This is a geometric series, with first term $a = e^{-1}$ and ratio $r = e^{-1}$. Since $|r| < 1$, the series converges to $e^{-1}/\left(1 - e^{-1}\right) = 1/(e-1)$.

7. $f(x) = xe^{-x}$ is continuous and positive on $[1, \infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0$ for $x > 1$, so f is decreasing on $[1, \infty)$. Thus, the Integral Test applies.

$\int_1^\infty xe^{-x}\,dx = \lim\limits_{b\to\infty} \int_1^b xe^{-x}\,dx = \lim\limits_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_1^b$ (by parts) $= \lim\limits_{b\to\infty} \left[-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right] = 2/e$

since $\lim\limits_{b\to\infty} be^{-b} = \lim\limits_{b\to\infty} \left(b/e^b\right) \overset{\text{H}}{=} \lim\limits_{b\to\infty} \left(1/e^b\right) = 0$ and $\lim\limits_{b\to\infty} e^{-b} = 0$. Thus, $\sum_{n=1}^\infty ne^{-n}$ converges.

8. $\dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{11} + \dfrac{1}{15} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{4n-1}$. The function $f(x) = \dfrac{1}{4x-1}$ is positive, continuous, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{dx}{4x-1} = \lim_{b\to\infty} \int_1^b \frac{dx}{4x-1} = \lim_{b\to\infty} \left[\tfrac{1}{4}\ln(4x-1)\right]_1^b = \lim_{b\to\infty} \left[\tfrac{1}{4}\ln(4b-1) - \tfrac{1}{4}\ln 3\right] = \infty$$

so the improper integral diverges, and so does the series.

9. $\sum_{n=5}^\infty \left(1/n^{1.0001}\right)$ is a p-series, $p = 1.0001 > 1$, so it converges.

10. $\sum_{n=1}^\infty n^{-0.99} = \sum_{n=1}^\infty \left(1/n^{0.99}\right)$ which diverges since $p = 0.99 < 1$.

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^\infty \left(1/n^3\right)$. This is a p-series with $p = 3 > 1$, so it converges by (1).

12. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

13. $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3} = 5\sum_{n=1}^\infty \frac{1}{n^3} - 2\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ by Theorem 12.2.8 [ET 11.2.8], since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ both converge by (1) (with $p = 3$ and $p = \frac{5}{2}$). Thus, $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3}$ converges.

14. $f(x) = \dfrac{1}{x^2 - 1}$ is positive, continuous, and decreasing on $[2, \infty)$, so applying the Integral Test,

$$\int_2^\infty \frac{dx}{x^2-1} = \int_2^\infty \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right) dx = \lim_{t\to\infty} \left[\ln\left(\frac{x-1}{x+1}\right)^{1/2}\right]_2^t = \ln\sqrt{3} \;\Rightarrow\; \sum_{n=2}^\infty \frac{1}{n^2-1} \text{ converges.}$$

15. $f(x) = xe^{-x^2}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = e^{-x^2}\left(1 - 2x^2\right) < 0$ for $x > 1$, f is decreasing as well. Thus, we can use the Integral Test.

$\int_1^\infty xe^{-x^2}\,dx = \lim\limits_{t\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_1^t = 0 - \left(-\frac{1}{2}e^{-1}\right) = 1/(2e)$. Since the integral converges, the series converges.

16. $f(x) = \dfrac{x}{2^x}$ is positive and continuous on $[1, \infty)$, and since $f'(x) = \dfrac{1 - x\ln 2}{2^x} < 0$ when $x > \dfrac{1}{\ln 2} \approx 1.44$, f is eventually decreasing, so we can apply the Integral Test. Integrating by parts, we get

$$\int_1^\infty \frac{x}{2^x}dx = \lim_{t\to\infty} \left(-\frac{1}{\ln 2}\left[\frac{x}{2^x} + \frac{1}{2^x\ln 2}\right]_1^t\right) = \frac{1}{2\ln 2} + \frac{1}{2(\ln 2)^2}, \text{ since } \lim_{t\to\infty} \frac{t}{2^t} = 0 \text{ by l'Hospital's Rule,}$$

and so $\displaystyle\sum_{n=1}^\infty \frac{n}{2^n}$ converges.

17. $f(x) = \dfrac{x}{x^2+1}$ is continuous and positive on $[1,\infty)$, and since $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$, f is also decreasing. Using the Integral Test, $\displaystyle\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{2}\right]_1^t = \infty$, so the series diverges.

18. The function $f(x) = \dfrac{1}{2x^2+3x+1} = \dfrac{1}{(2x+1)(x+1)}$ is positive, continuous, and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty}\int_1^b\left(\frac{2}{2x+1} - \frac{1}{x+1}\right)dx \text{ (partial fractions) } = \lim_{b\to\infty}\left[\ln(2x+1) - \ln(x+1)\right]_1^b$$
$$= \lim_{b\to\infty}\left[\ln\left(\frac{2x+1}{x+1}\right)\right]_1^b = \lim_{b\to\infty}\left(\ln\frac{2b+1}{b+1} - \ln\tfrac{3}{2}\right) = \ln 2 - \ln\tfrac{3}{2} = \ln\tfrac{4}{3}$$

so the series converges.

19. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2,\infty)$, and also decreasing since $f'(x) = -\dfrac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the series diverges.

20. $f(x) = \dfrac{1}{4x^2+1}$ is continuous, positive and decreasing on $[1,\infty)$, so applying the Integral Test,
$\displaystyle\int_1^\infty \frac{dx}{4x^2+1} = \lim_{t\to\infty}\left[\frac{\arctan 2x}{2}\right]_1^t = \frac{\pi}{4} - \frac{\arctan 2}{2} < \infty$, so the series converges.

21. $f(x) = \dfrac{\arctan x}{1+x^2}$ is continuous and positive on $[1,\infty)$. $f'(x) = \dfrac{1-2x\arctan x}{(1+x^2)^2} < 0$ for $x > 1$, since $2x\arctan x \geq \frac{\pi}{2} > 1$ for $x \geq 1$. So f is decreasing and we can use the Integral Test.
$\displaystyle\int_1^\infty \frac{\arctan x}{1+x^2}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{2}\arctan^2 x\right]_1^t = \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{3\pi^2}{32}$, so the series converges.

22. $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive for $x \geq 2$, and $f'(x) = \dfrac{1-2\ln x}{x^3} < 0$ for $x \geq 2$ so f is decreasing.
$\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (by parts) $= 1$ (by l'Hospital's Rule). Thus, $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2} = \sum_{n=2}^\infty \frac{\ln n}{n^2}$ converges by the Integral Test.

23. $f(x) = \dfrac{1}{x^2+2x+2}$ is continuous and positive on $[1,\infty)$, and $f'(x) = -\dfrac{2x+2}{(x^2+2x+2)^2} < 0$ for $x \geq 1$, so f is decreasing and we can use the Integral Test.
$\displaystyle\int_1^\infty \frac{1}{x^2+2x+2}\,dx = \int_1^\infty \frac{1}{(x+1)^2+1}\,dx = \lim_{t\to\infty}\left[\arctan(x+1)\right]_1^t = \frac{\pi}{2} - \arctan 2$, so the series converges as well.

24. $f(x) = \dfrac{1}{x \ln x \ln(\ln x)}$ is positive and continuous on $[3, \infty)$, and is decreasing since x, $\ln x$, and $\ln(\ln x)$ are all increasing; so we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x \ln x \ln(\ln x)} = \lim_{t\to\infty} [\ln(\ln(\ln x))]_3^t$ which diverges, and hence $\displaystyle\sum_{n=3}^\infty \frac{1}{n \ln n \ln(\ln n)}$ diverges.

25. We have already shown (in Exercise 19) that when $p = 1$ the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$ diverges, so assume that $p \neq 1$. $f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2, \infty)$, and $f'(x) = -\dfrac{p + \ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t \quad (\text{for } p \neq 1) \;= \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p}\right] - \frac{(\ln 2)^{1-p}}{1-p}$$

This limit exists whenever $1 - p < 0 \;\Leftrightarrow\; p > 1$, so the series converges for $p > 1$.

26. As in Exercise 24, we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x \ln x (\ln\ln x)^p} = \lim_{t\to\infty}\left[\frac{(\ln\ln x)^{-p+1}}{-p+1}\right]_3^t$ (for $p \neq 1$; if $p = 1$ see Exercise 24) and $\displaystyle\lim_{t\to\infty}\frac{(\ln\ln t)^{-p+1}}{-p+1}$ exists whenever $-p + 1 < 0 \;\Leftrightarrow\; p > 1$, so the series converges for $p > 1$.

27. Clearly the series cannot converge if $p \geq -\frac{1}{2}$, because then $\displaystyle\lim_{n\to\infty} n(1+n^2)^p \neq 0$. Also, if $p = -1$ the series diverges (see Exercise 17). So assume $p < -\frac{1}{2}$, $p \neq -1$. Then $f(x) = x(1+x^2)^p$ is continuous, positive, and eventually decreasing on $[1, \infty)$, and we can use the Integral Test.

$\displaystyle\int_1^\infty x(1+x^2)^p\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+x^2)^{p+1}}{p+1}\right]_1^t = \lim_{t\to\infty}\frac{1}{2}\cdot\frac{(1+t^2)^{p+1}}{p+1} - \frac{2^p}{p+1}$. This limit exists and is finite $\;\Leftrightarrow\; p + 1 < 0 \;\Leftrightarrow\; p < -1$, so the series converges whenever $p < -1$.

28. If $p \leq 0$, $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \dfrac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$\displaystyle\int_1^\infty \frac{\ln x}{x^p}dx = \lim_{t\to\infty}\left[\frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2}\right]_1^t$ (for $p \neq 1$) $= \dfrac{1}{(1-p)^2}\left[\displaystyle\lim_{t\to\infty} t^{1-p}[(1-p)\ln t - 1] + 1\right]$

which exists whenever $1 - p < 0 \;\Leftrightarrow\; p > 1$. Since we have already done the case $p = 1$ in Exercise 25 (set $p = -1$ in that exercise), $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^p}$ converges $\;\Leftrightarrow\; p > 1$.

29. Since this is a p-series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of numbers x such that the series is convergent.

30. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 1$, and so the Integral Test applies. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.

$R_{10} \leq \displaystyle\int_{10}^\infty \frac{1}{x^4}\,dx = \lim_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3(10)^3}\right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx \;\Rightarrow\; s_{10} + \frac{1}{3\,(11)^3} \le s \le s_{10} + \frac{1}{3\,(10)^3} \;\Rightarrow$
$1.082037 + 0.000250 = 1.082287 \le s \le 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) $R_n \le \int_n^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \;\Rightarrow\; \frac{1}{3n^3} < \frac{1}{10^5} \;\Rightarrow\; 3n^3 > 10^5 \;\Rightarrow$
$n > \sqrt[3]{(10)^5/3} \approx 32.2$, that is, for $n > 32$.

31. (a) $f(x) = \frac{1}{x^2}$ is positive and continuous and $f'(x) = -\frac{2}{x^3}$ is negative for $x > 1$, and so the Integral Test applies. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} \approx s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.549768$.

$R_{10} \le \int_{10}^{\infty} \frac{1}{x^2}\,dx = \lim\limits_{t\to\infty} \left[\frac{-1}{x}\right]_{10}^{t} = \lim\limits_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{10}\right) = \frac{1}{10}$, so the error is at most 0.1.

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^2}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^2}\,dx \;\Rightarrow\; s_{10} + \frac{1}{11} \le s \le s_{10} + \frac{1}{10} \;\Rightarrow$
$1.549768 + 0.090909 = 1.640677 \le s \le 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

(c) $R_n \le \int_n^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}$. So $R_n < 0.001$ if $\frac{1}{n} < \frac{1}{1000} \;\Leftrightarrow\; n > 1000$.

32. $f(x) = 1/x^5$ is positive and continuous and $f'(x) = -5/x^6$ is negative for $x > 1$, and so the Integral Test applies. Using (2), $R_n \le \int_n^{\infty} x^{-5}\,dx = \lim\limits_{t\to\infty} \left[\frac{-1}{4x^4}\right]_n^t = \frac{1}{4n^4}$. If we take $n = 5$, then $s_5 \approx 1.036662$ and $R_5 \le 0.0004$. So $s \approx s_5 \approx 1.037$.

33. $f(x) = x^{-3/2}$ is positive and continuous and $f'(x) = -\frac{3}{2}x^{-5/2}$ is negative for $x > 1$, so the Integral Test applies. From the end of Example 6, we see that the error is at most half the length of the interval. From (3), the interval is $\left(s_n + \int_{n+1}^{\infty} f(x)\,dx,\, s_n + \int_n^{\infty} f(x)\,dx\right)$, so its length is $\int_n^{\infty} f(x)\,dx - \int_{n+1}^{\infty} f(x)\,dx$. Thus, we need n such that

$$0.01 > \tfrac{1}{2}\left(\int_n^{\infty} x^{-3/2}\,dx - \int_{n+1}^{\infty} x^{-3/2}\,dx\right) = \frac{1}{2}\left(\lim_{t\to\infty}\left[\frac{-2}{\sqrt{x}}\right]_n^t - \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x}}\right]_{n+1}^t\right) = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

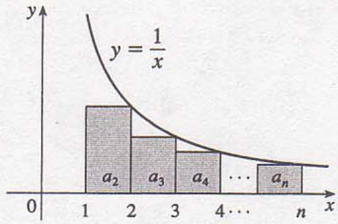
$\Leftrightarrow\; n > 13.08$. Again from the end of Example 6, we approximate s by the midpoint of this interval. In general, the midpoint is $\frac{1}{2}\left[\left(s_n + \int_{n+1}^{\infty} f(x)\,dx\right) + \left(s_n + \int_n^{\infty} f(x)\,dx\right)\right] = s_n + \frac{1}{2}\left(\int_{n+1}^{\infty} f(x)\,dx + \int_n^{\infty} f(x)\,dx\right)$. So using $n = 14$, we have $s \approx s_{14} + \frac{1}{2}\left(\int_{14}^{\infty} x^{-3/2}\,dx + \int_{15}^{\infty} x^{-3/2}\,dx\right) = 2.0872 + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} \approx 2.6127$. Any larger value of n will also work. For instance, $s \approx s_{30} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{31}} \approx 2.6124$.

34. $f(x) = \frac{1}{x\,(\ln x)^2}$ is positive and continuous and $f'(x) = -\frac{\ln x + 2}{x^2\,(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies. Using (2), we need $0.01 > \int_n^{\infty} \frac{dx}{x\,(\ln x)^2} = \lim\limits_{t\to\infty}\left[\frac{-1}{\ln x}\right]_n^t = \frac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to take this many terms, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.

35. (a) From the figure, $a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx$, so with

$f(x) = \dfrac{1}{x}$,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \leq \int_1^n \frac{1}{x}\,dx = \ln n$$



Thus, $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} \leq 1 + \ln n$.

(b) By part (a), $s_{10^6} \leq 1 + \ln 10^6 \approx 14.82 < 15$ and $s_{10^9} \leq 1 + \ln 10^9 \approx 21.72 < 22$.

36. (a) $f(x) = \left(\dfrac{\ln x}{x}\right)^2$ is continuous and positive for $x > 1$, and since $f'(x) = \dfrac{2\ln x\,(1 - \ln x)}{x^3} < 0$ for $x > e$, we can apply the Integral Test. Using a CAS, we get $\displaystyle\int_1^\infty \left(\frac{\ln x}{x}\right)^2 dx = 2$, so the series also converges.

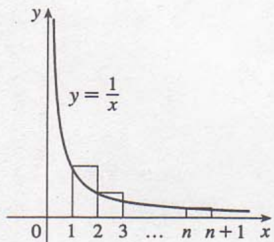
(b) Since the Integral Test applies, the error in $s \approx s_n$ is $R_n \leq \displaystyle\int_n^\infty \left(\frac{\ln x}{x}\right)^2 dx = \frac{(\ln n)^2 + 2\ln n + 2}{n}$.

(c) By graphing the functions $y_1 = \dfrac{(\ln x)^2 + 2\ln x + 2}{x}$ and $y_2 = 0.05$, we see that $y_1 < y_2$ for $n \geq 1373$.

(d) Using the CAS to sum the first 1373 terms, we get $s_{1373} \approx 1.94$.

37. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \dfrac{1}{n^{-\ln b}}$. This is a p-series, which converges for all b such that $-\ln b > 1 \;\Leftrightarrow\; \ln b < -1 \;\Leftrightarrow\; b < e^{-1} \;\Leftrightarrow\; b < 1/e$.

38. (a) The sum of the areas of the n rectangles in the graph to the right is $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. Now $\displaystyle\int_1^{n+1} \frac{dx}{x}$ is less than this sum because the rectangles extend above the curve $y = 1/x$, so

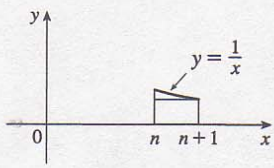


$\displaystyle\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and since

$\ln n < \ln(n+1)$, $0 < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} - \ln n = t_n$.

(b) The area under $f(x) = 1/x$ between $x = n$ and $x = n + 1$ is

$\displaystyle\int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n$, and this is clearly greater than the area of the inscribed rectangle in the figure to the right



$\left[\text{which is } \dfrac{1}{n+1}\right]$, so

$t_n - t_{n+1} = [\ln(n+1) - \ln n] - \dfrac{1}{n+1} > 0$, and so $t_n > t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.

(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n. Thus, $0 < t_n \leq t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by Theorem 12.1.10 [ET 11.1.10].

12.4 The Comparison Tests ET 11.4

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See Note 2 on page 757 [ET 723].)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. $\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

4. $\dfrac{2}{n^3+4} < \dfrac{2}{n^3}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3+4}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3} = 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a constant multiple of a convergent p-series ($p = 3 > 1$).

5. $\dfrac{5}{2+3^n} < \dfrac{5}{3^n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{5}{2+3^n}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{5}{3^n} = 5\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$, which converges because $\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$ is a convergent geometric series with $r = \frac{1}{3}$.

6. $\dfrac{1}{n-\sqrt{n}} > \dfrac{1}{n}$ for all $n \geq 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n-\sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$.

7. $\dfrac{n+1}{n^2} > \dfrac{n}{n^2} = \dfrac{1}{n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^2}$ diverges by comparison with the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

8. $\dfrac{4+3^n}{2^n} > \dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

9. $\dfrac{3}{n2^n} \leq \dfrac{3}{2^n}$. $\sum\limits_{n=1}^{\infty} \dfrac{3}{2^n}$ is a geometric series with $|r| = \frac{1}{2} < 1$, and hence converges, so $\sum\limits_{n=1}^{\infty} \dfrac{3}{n2^n}$ converges also, by the Comparison Test.

10. $\dfrac{\sin^2 n}{n\sqrt{n}} \leq \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{\sin^2 n}{n\sqrt{n}}$ converges by the Comparison Test.

11. $\dfrac{1}{\sqrt{n(n+1)(n+2)}} < \dfrac{1}{\sqrt{n \cdot n \cdot n}} = \dfrac{1}{n^{3/2}}$ and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$), so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n(n+1)(n+2)}}$ by the Comparison Test.

12. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt[3]{n(n+1)(n+2)}}$ and $b_n = \dfrac{1}{n}$.

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt[3]{n(n+1)(n+2)}} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{1(1+1/n)(1+2/n)}} = 1 > 0$, so since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$.

13. If $a_n = \dfrac{n^2+1}{n^3-1}$ and $b_n = \dfrac{1}{n}$, then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+n}{n^3-1} = \lim_{n\to\infty} \frac{1+1/n^2}{1-1/n^3} = 1$, so $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$.

14. $\dfrac{n}{(n+1)2^n} < \dfrac{1}{2^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$ converges by the Comparison Test.

15. $\dfrac{3+\cos n}{3^n} \le \dfrac{4}{3^n}$ since $\cos n \le 1$. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3^n}$ is a geometric series with $|r| = \frac{1}{3} < 1$ so it converges, and so $\displaystyle\sum_{n=1}^{\infty} \frac{3+\cos n}{3^n}$ converges by the Comparison Test.

16. $\dfrac{5n}{2n^2-5} > \dfrac{5n}{2n^2} = \dfrac{5}{2}\left(\dfrac{1}{n}\right)$ and since $\displaystyle\frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) so does $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n^2-5}$ by the Comparison Test.

17. $\dfrac{n}{\sqrt{n^5+4}} < \dfrac{n}{\sqrt{n^5}} = \dfrac{1}{n^{3/2}}$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$) so $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$ converges by the Comparison Test.

18. $\dfrac{\arctan n}{n^4} < \dfrac{\pi/2}{n^4}$ and $\displaystyle\frac{\pi}{2}\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges ($p = 4 > 1$) so $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^4}$ converges by the Comparison Test.

19. $\dfrac{2^n}{1+3^n} < \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series ($|r| = \frac{2}{3} < 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges by the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \dfrac{1+2^n}{1+3^n}$ and $b_n = \dfrac{2^n}{3^n}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(1/2)^n+1}{(1/3)^n+1} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} < 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ also converges.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{1+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \le 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

22. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^2-4}$ and $b_n = \dfrac{1}{n^2}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2-4} = 1 > 0$. Since $\displaystyle\sum_{n=3}^{\infty} b_n$ converges ($p = 2 > 1$), $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ also converges.

23. Let $a_n = \dfrac{n^2+1}{n^4+1}$ and $b_n = \dfrac{1}{n^2}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^4+n^2}{n^4+1} = 1$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), so is $\sum\limits_{n=1}^{\infty} \dfrac{n^2+1}{n^4+1}$ by the Limit Comparison Test.

24. If $a_n = \dfrac{n^2-5n}{n^3+n+1}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3-5n^2}{n^3+n+1} = \lim\limits_{n\to\infty} \dfrac{1-5/n}{1+1/n^2+1/n^3} = 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$. (Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim\limits_{n\to\infty} \dfrac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}} \cdot \frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty} \frac{1+5/n}{\left[(n^7+n^2)/n^7\right]^{1/3}} = \lim_{n\to\infty} \frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1,$$

so $\sum\limits_{n=1}^{\infty} \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$.

27. Let $a_n = \dfrac{n+1}{n2^n}$ and $b_n = \dfrac{1}{2^n}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n2^n}$ converges by the Limit Comparison Test.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ and $b_n = \dfrac{1}{3^n}$.

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2n^2+7n}{n^2+5n-1} = 2 > 0$, and since $\sum\limits_{n=1}^{\infty} b_n$ is a convergent geometric series ($|r| = \frac{1}{3} < 1$), $\sum\limits_{n=1}^{\infty} \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^{n-1}}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$) so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!}$ converges by the Comparison Test.

30. $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)\,n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdots\cdot 1$ for $n \geq 2$, so since $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^2}$ converges ($p = 2 > 1$), $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges also by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges.

32. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^{1+1/n}}$ and $b_n = \dfrac{1}{n}$. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{n^{1+1/n}} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/n}} = 1$ (since $\lim\limits_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series) $\Rightarrow$ $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$ diverges.

33. $\sum\limits_{n=1}^{10} \dfrac{1}{n^4+n^2} = \dfrac{1}{2} + \dfrac{1}{20} + \dfrac{1}{90} + \cdots + \dfrac{1}{10{,}100} \approx 0.567975$. Now $\dfrac{1}{n^4+n^2} < \dfrac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \dfrac{1}{n^4} \le \displaystyle\int_{10}^{\infty} \frac{dx}{x^4} = \lim_{t\to\infty} \left[-\frac{x^{-3}}{3}\right]_{10}^{t} = \frac{1}{3000} = 0.000\overline{3}$.

34. $\sum\limits_{n=1}^{10} \dfrac{1+\cos n}{n^5} = 1 + \cos 1 + \dfrac{1+\cos 2}{32} + \dfrac{1+\cos 3}{243} + \cdots + \dfrac{1+\cos 10}{100{,}000} \approx 1.55972$. Now $\dfrac{1+\cos n}{n^5} \le \dfrac{2}{n^5}$, so as in Example 5, $R_{10} \le T_{10} \le \displaystyle\int_{10}^{\infty} \frac{2}{x^5}\,dx = 2 \lim_{t\to\infty} \left[-\tfrac{1}{4}x^{-4}\right]_{10}^{t} = 0.00005$.

35. $\sum\limits_{n=1}^{10} \dfrac{1}{1+2^n} = \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{9} + \cdots + \dfrac{1}{1025} \approx 0.76352$. Now $\dfrac{1}{1+2^n} < \dfrac{1}{2^n}$, so the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \dfrac{1}{2^n} = \dfrac{1/2^{11}}{1-1/2}$ (geometric series) ≈ 0.00098.

36. $\sum\limits_{n=1}^{10} \dfrac{n}{(n+1)\,3^n} = \dfrac{1}{6} + \dfrac{2}{27} + \dfrac{3}{108} + \cdots + \dfrac{10}{649{,}539} \approx 0.283597$. Now $\dfrac{n}{(n+1)\,3^n} < \dfrac{n}{n\cdot 3^n} = \dfrac{1}{3^n}$, so the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \dfrac{1}{3^n} = \dfrac{1/3^{11}}{1-1/3} \approx 0.0000085$.

37. Since $\dfrac{d_n}{10^n} \le \dfrac{9}{10^n}$ for each n, and since $\sum\limits_{n=1}^{\infty} \dfrac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1d_2d_3\ldots = \sum\limits_{n=1}^{\infty} \dfrac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\lim\limits_{n\to\infty} \dfrac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n$ $\Rightarrow$ $\dfrac{1}{n^p \ln n} \ge \dfrac{1}{n \ln n}$ and $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges (Exercise 12.3.19 [ET 11.3.19]), so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \dfrac{1}{n^p \ln n}$ and $b_n = \dfrac{1}{n^p}$. $\sum\limits_{n=2}^{\infty} b_n$ converges, and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

39. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. (a) Since $\lim\limits_{n\to\infty} (a_n/b_n) = 0$, there is a number $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.

(b) If $a_n = \dfrac{\ln n}{n^3}$ and $b_n = \dfrac{1}{n^2}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^3}$ converges by part (a).

41. (a) We wish to prove that if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$. So suppose on the contrary that $\sum a_n$ converges. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, we have that $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, so by the extension of the Limit Comparison Test proved in Exercise 40(a), if $\sum a_n$ converges, so must $\sum b_n$. But this contradicts our hypothesis, so $\sum a_n$ must diverge.

(b) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$ for $n \ge 2$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$, so by part (a), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

42. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

43. $\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n}$. Since $\lim_{n\to\infty} na_n > 0$ we know that either both series converge or both series diverge, and we also know that $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges (p-series with $p = 1$). Therefore, $\sum a_n$ must be divergent.

44. First we observe that, by l'Hospital's Rule, $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ by Theorem 12.2.6 [ET 11.2.6]. Therefore, $\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = 1$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln(1+a_n)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n \to \infty$, so $\lim_{n\to\infty} \frac{\sin a_n}{a_n} = 1$ by Theorem 3.5.2 [ET 3.4.2]. Thus, $\sum \sin a_n$ converges by the Limit Comparison Test.

46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n \to \infty$, so for some integer N, $a_n \le 1$ for all $n \ge N$. But then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} a_n b_n \le \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty} b_n$ converges. So $\sum a_n b_n$ converges by the Comparison Test.

12.5 Alternating Series — ET 11.5

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges if $0 < b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}, that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$. Here $a_n = (-1)^n \frac{n}{n+2}$. Since $\lim_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7} - \frac{4}{8} + \frac{4}{9} - \frac{4}{10} + \frac{4}{11} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4}{n+6}$. $b_n = \frac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

4. $\sum\limits_{n=2}^{\infty} (-1)^n \dfrac{1}{\ln n}$. $b_n = \dfrac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0$, so the series converges by the Alternating Series Test.

5. $b_n = \dfrac{1}{\sqrt{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n = \dfrac{1}{3n-1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

7. $a_n = (-1)^n \dfrac{2n}{4n+1}$, so $|a_n| = \dfrac{2n}{4n+1} \to \dfrac{1}{2}$ as $n \to \infty$. Therefore, $\lim\limits_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist) and the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{2n}{4n+1}$ diverges by the Test for Divergence.

8. $b_n = \dfrac{2n}{4n^2+1} > 0$, $\{b_n\}$ is decreasing [since

$b_n - b_{n+1} = \dfrac{2n}{4n^2+1} - \dfrac{2n+2}{4n^2+8n+5} = \dfrac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)} > 0$ for $n \geq 1$], and

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{2/n}{4+1/n^2} = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{2n}{4n^2+1}$ converges by the Alternating Series Test.

9. $b_n = \dfrac{1}{4n^2+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

10. $a_n = (-1)^{n-1} \dfrac{2n^2}{4n^2+1}$, so $|a_n| = \dfrac{2n^2}{4n^2+1} \to \dfrac{1}{2}$ as $n \to \infty$. Therefore, $\lim\limits_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist) and the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{2n^2}{4n^2+1}$ diverges by the Test for Divergence.

11. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{\sqrt{n}}{n+4}$. $b_n = \dfrac{\sqrt{n}}{n+4} > 0$ for all n. Let $f(x) = \dfrac{\sqrt{x}}{x+4}$. Then $f'(x) = \dfrac{4-x}{2\sqrt{x}\,(x+4)^2} < 0$ if $x > 4$, so $\{b_n\}$ is decreasing after $n = 4$. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+4} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}+4/\sqrt{n}} = 0$. So the series converges by the Alternating Series Test.

12. $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{n}{2^n}$. $b_n = \dfrac{n}{2^n} > 0$ and $b_n \geq b_{n+1} \Leftrightarrow \dfrac{n}{2^n} \geq \dfrac{n+1}{2^{n+1}} \Leftrightarrow 2n \geq n+1 \Leftrightarrow n \geq 1$ which is certainly true. $\lim\limits_{n\to\infty} (n/2^n) = 0$ by l'Hospital's Rule, so the series converges by the Alternating Series Test.

13. $\sum\limits_{n=2}^{\infty} (-1)^n \dfrac{n}{\ln n}$. $\lim\limits_{n\to\infty} \dfrac{n}{\ln n} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \dfrac{1}{1/n} = \infty$, so the series diverges by the Test for Divergence.

14. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \left(\dfrac{\ln n}{n}\right) = 0 + \sum\limits_{n=2}^{\infty} (-1)^{n-1} \left(\dfrac{\ln n}{n}\right)$. $b_n = \dfrac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \dfrac{\ln x}{x}$ then $f'(x) = \dfrac{1-\ln x}{x^2} < 0$ if $x > e$, so $\{b_n\}$ is eventually decreasing. Also, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{\ln n}{n} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \dfrac{1/n}{1} = 0$, so the series converges by the Alternating Series Test.

15. $\sum\limits_{n=1}^{\infty} \dfrac{\cos n\pi}{n^{3/4}} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^{3/4}}$. $b_n = \dfrac{1}{n^{3/4}}$ is decreasing and positive and $\lim\limits_{n\to\infty} \dfrac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

17. $\sum\limits_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{n}$. $b_n = \sin\frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin\frac{\pi}{n} \geq \sin\frac{\pi}{n+1}$, and $\lim\limits_{n\to\infty} \sin\frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

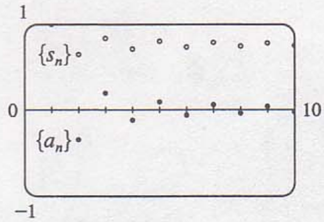
18. $\sum\limits_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim\limits_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim\limits_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \;\Rightarrow\; \lim\limits_{n\to\infty} \frac{n^n}{n!} = \infty \;\Rightarrow\; \lim\limits_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

20. $\frac{1}{\sqrt[3]{\ln n}}$ decreases and $\lim\limits_{n\to\infty} \frac{1}{\sqrt[3]{\ln n}} = 0$, so by the Alternating Series Test the series converges.

21.

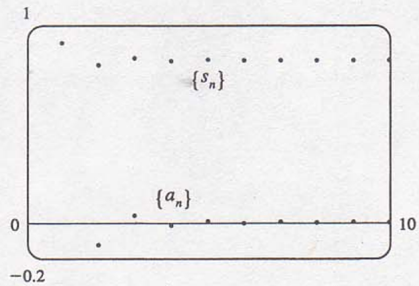
n	a_n	s_n
1	1	1
2	−0.35355	0.64645
3	0.19245	0.83890
4	−0.125	0.71390
5	0.08944	0.80334
6	−0.06804	0.73530
7	0.05399	0.78929
8	−0.04419	0.74510
9	0.03704	0.78214
10	−0.03162	0.75051



By the Alternating Series Estimation Theorem, the error in the approximation $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \approx 0.75051$ is $|s - s_{10}| \leq b_{11} = 1/(11)^{3/2} \approx 0.0275$ (to four decimal places, rounded up).

22.

n	a_n	s_n
1	1	1
2	−0.125	0.875
3	0.03704	0.91204
4	−0.01563	0.89641
5	0.008	0.90441
6	−0.00463	0.89978
7	0.00292	0.90270
8	−0.00195	0.90074
9	0.00137	0.90212
10	−0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the approximation $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \approx 0.90112$ is $|s - s_{10}| \leq b_{11} = 1/11^3 \approx 0.0007513$.

23. With $b_n = 1/n^2$, $b_{10} = 1/10^2 = 0.01$ and $b_{11} = 1/11^2 = 1/121 \approx 0.008 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 10$.

24. $b_5 = 1/5^4 = 0.0016 > 0.001$ and $b_6 = 1/6^4 \approx 0.00077 < 0.001$, so by the Alternating Series Estimation Theorem, $n = 5$.

25. $b_7 = 2^7/7! \approx 0.025 > 0.01$ and $b_8 = 2^8/8! \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. $b_5 = 5/4^5 \approx 0.0049 > 0.002$ and $b_6 = 6/4^6 \approx 0.0015 < 0.002$, so by the Alternating Series Estimation Theorem, $n = 5$.

27. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(2n-1)!}$. $b_5 = \dfrac{1}{(2 \cdot 5 - 1)!} = \dfrac{1}{362{,}880} < 0.00001$, so $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(2n-1)!} \approx \sum\limits_{n=1}^{4} \dfrac{(-1)^{n-1}}{(2n-1)!} \approx 0.8415$.

28. $b_4 = \dfrac{1}{(2 \cdot 4)!} = \dfrac{1}{40{,}320} \approx 0.000025$ and $s_3 = 1 - \dfrac{1}{2} + \dfrac{1}{24} - \dfrac{1}{720} \approx 0.54028$, so, correct to four decimal places, $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} \approx 0.5403$.

29. $b_6 = \dfrac{1}{2^6 6!} = \dfrac{1}{46{,}080} \approx 0.000022 < 0.00001$, so $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{2^n n!} \approx \sum\limits_{n=0}^{5} \dfrac{(-1)^n}{2^n n!} \approx 0.6065$.

30. $b_8 = 1/8^6 \approx 0.0000038 < 0.00001$ and $s_7 = 1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} + \frac{1}{15{,}625} - \frac{1}{46{,}656} + \frac{1}{117{,}649} \approx 0.9855537$, so correct to five decimal places, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^6} \approx 0.98555$.

31. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{1}{49} - \dfrac{1}{50} + \dfrac{1}{51} - \dfrac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} = s_{50} + \left(\dfrac{1}{51} - \dfrac{1}{52}\right) + \left(\dfrac{1}{53} - \dfrac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

32. If $p > 0$, $\dfrac{1}{(n+1)^p} \leq \dfrac{1}{n^p}$ and $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \leq 0$, $\lim\limits_{n\to\infty} \dfrac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

33. Clearly $b_n = \dfrac{1}{n+p}$ is decreasing and eventually positive and $\lim\limits_{n\to\infty} b_n = 0$ for any p. So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n + p \neq 0$ for $n \geq 1$, or p is not a negative integer.

34. Let $f(x) = \dfrac{(\ln x)^p}{x}$. Then $f'(x) = \dfrac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p. Clearly $\lim\limits_{n\to\infty} \dfrac{(\ln n)^p}{n} = 0$ if $p \leq 0$, and if $p > 0$ we can apply l'Hospital's Rule $[\![p+1]\!]$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).

35. $\sum b_{2n} = \sum 1/(2n)^2$ clearly converges (by comparison with the p-series for $p = 2$). So suppose that $\sum (-1)^{n-1} b_n$ converges. Then by Theorem 12.2.8(ii) [ET 11.2.8(ii)], so does $\sum \left[(-1)^{n-1} b_n + b_n\right] = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = 2\sum \dfrac{1}{2n-1}$. But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\sum (-1)^{n-1} b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.

36. (a) We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$. $P(1)$ is true by an easy calculation. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.

$$h_{2n+2} - h_{n+1} = \left(h_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(h_n + \frac{1}{n+1}\right) = (h_{2n} - h_n) + \frac{1}{2n+1} - \frac{1}{2n+2}$$
$$= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} = s_{2n+2}$$

which is $P(n+1)$, and proves that $s_{2n} = h_{2n} - h_n$ for all n.

(b) We know that $h_{2n} - \ln 2n \to \gamma$ and $h_n - \ln n \to \gamma$ as $n \to \infty$.
So $s_{2n} = h_{2n} - h_n = (h_{2n} - \ln 2n) - (h_n - \ln n) + (\ln 2n - \ln n)$, and
$\lim\limits_{n\to\infty} s_{2n} = \gamma - \gamma + \lim\limits_{n\to\infty} [\ln 2n - \ln n] = \lim\limits_{n\to\infty} (\ln 2 + \ln n - \ln n) = \ln 2$.

12.6 Absolute Convergence and the Ratio and Root Tests — ET 11.6

1. (a) Since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 8 > 1$, part (ii) of the Ratio Test tells us that $\sum a_n$ is divergent.

(b) Since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (i) of the Ratio Test tells us that $\sum a_n$ is convergent.

(c) Since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and $\sum a_n$ might converge or it might diverge.

2. The series $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}$ has positive terms and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^2}{2^{n+1}} \cdot \dfrac{2^n}{n^2}\right] = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^2 \cdot \dfrac{1}{2} = \dfrac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

3. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n\sqrt{n}} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$), so the given series is absolutely convergent.

4. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^{1/2}}$ converges by the Alternating Series Test, but $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1/2}}$ is a divergent p-series ($p = \frac{1}{2} < 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^{1/2}}$ converges conditionally.

5. Using the Ratio Test, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-3)^{n+1}/(n+1)^3}{(-3)^n/n^3}\right| = 3 \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^3 = 3 > 1$, so the series diverges.

6. Using the Ratio Test, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-3)^{n+1}/(n+1)!}{(-3)^n/n!}\right| = 3 \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1$, so the series is absolutely convergent.

7. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{5+n}$ converges by the Alternating Series Test, but $\sum\limits_{n=1}^{\infty} \dfrac{1}{5+n}$ diverges by the Limit Comparison Test with the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, so the given series is conditionally convergent.

8. $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \dfrac{n!}{(n+1)!} = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n!}$ is absolutely convergent by the Ratio Test.

9. $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \dfrac{n}{5+n} = \lim\limits_{n\to\infty} \dfrac{1}{5/n+1} = 1$, so $\lim\limits_{n\to\infty} a_n \neq 0$. Thus, the given series is divergent by the Test for Divergence.

10. $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:

$\lim\limits_{n\to\infty} \dfrac{n/\left(n^2+1\right)}{1/n} = \lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1} = 1$. But $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n}{n^2+1}$ converges by the Alternating Series Test:

$\left\{ \dfrac{n}{n^2+1} \right\}$ has positive terms, is decreasing since $\left(\dfrac{x}{x^2+1} \right)' = \dfrac{1-x^2}{\left(x^2+1\right)^2} \leq 0$ for $x \geq 1$, and

$\lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = 0$. Thus, $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n}{n^2+1}$ is conditionally convergent.

11. $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \dfrac{1/(2n+2)!}{1/(2n)!} = \lim\limits_{n\to\infty} \dfrac{(2n)!}{(2n+2)!} = \lim\limits_{n\to\infty} \dfrac{(2n)!}{(2n+2)(2n+1)(2n)!} =$

$\lim\limits_{n\to\infty} \dfrac{1}{(2n+2)(2n+1)} = 0 < 1$, so the series is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.

12. $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{(n+1)!/e^{n+1}}{n!/e^n} \right| = \dfrac{1}{e} \lim\limits_{n\to\infty} (n+1) = \infty$, so the series diverges by the Ratio Test.

13. $\left| \dfrac{\sin 2n}{n^2} \right| \leq \dfrac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges (p-series, $p = 2 > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{\sin 2n}{n^2}$ converges absolutely by the Comparison Test.

14. $\dfrac{\arctan n}{n^3} < \dfrac{\pi/2}{n^3}$ and $\sum\limits_{n=1}^{\infty} \dfrac{\pi/2}{n^3}$ converges ($p = 3 > 1$), so $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\arctan n}{n^3}$ converges absolutely by the Comparison Test.

15. $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)\,3^{n+1}}{4^n} \cdot \dfrac{4^{n-1}}{n \cdot 3^n} \right] = \lim\limits_{n\to\infty} \left(\dfrac{3}{4} \cdot \dfrac{n+1}{n} \right) = \dfrac{3}{4} < 1$, so the series is absolutely convergent by the Ratio Test.

16. $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^2\, 2^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^2 2^n} \right] = \lim\limits_{n\to\infty} \left[\left(1 + \dfrac{1}{n}\right)^2 \cdot \dfrac{2}{n+1} \right] = 0$, so the series

$\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

17. $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left[\dfrac{10^{n+1}}{(n+2)\,4^{2n+3}} \cdot \dfrac{(n+1)\,4^{2n+1}}{10^n} \right] = \lim\limits_{n\to\infty} \left(\dfrac{10}{4^2} \cdot \dfrac{n+1}{n+2} \right) = \dfrac{5}{8} < 1$, so the series is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

18. $\left|\cos\dfrac{n\pi}{6}\right| \le 1$, so since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ converges ($p = \frac{3}{2} > 1$), the given series converges absolutely by the Comparison Test.

19. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/10^{n+1}}{n!/10^n} = \lim_{n\to\infty}\frac{n+1}{10} = \infty$, so the series diverges by the Ratio Test.

20. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series converges absolutely by the Ratio Test.

21. $\dfrac{|\cos(n\pi/3)|}{n!} \le \dfrac{1}{n!}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ converges (Exercise 12.4.29 [ET 11.4.29]), so the given series converges absolutely by the Comparison Test.

22. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{1}{\ln n} = 0 < 1$ so the series converges absolutely by the Root Test.

23. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{n^n}{3^{1+3n}}\right)^{1/n} = \lim_{n\to\infty}\frac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$, so the series is divergent by the Root Test.

Or: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{3^{4+3n}}\cdot\frac{3^{1+3n}}{n^n}\right] = \lim_{n\to\infty}\left[\frac{1}{3^3}\cdot\left(\frac{n+1}{n}\right)^n(n+1)\right]$

$$= \frac{1}{27}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}(n+1) = \tfrac{1}{27}e\lim_{n\to\infty}(n+1) = \infty$$

so the series is divergent by the Ratio Test.

24. Since $\left\{\dfrac{1}{n\ln n}\right\}$ is decreasing and $\displaystyle\lim_{n\to\infty}\frac{1}{n\ln n} = 0$, the series converges by the Alternating Series Test, but since $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the Integral Test (Exercise 12.3.19 [ET 11.3.19]), the given series converges only conditionally.

25. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \lim_{n\to\infty}\frac{1+1/n^2}{2+1/n^2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Root Test.

26. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{1}{\arctan n} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1$ so the series converges absolutely by the Root Test.

27. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/[1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)]}{n!/[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]} = \lim_{n\to\infty}\frac{n+1}{2n+1} = \frac{1}{2} < 1$, so the series converges absolutely by the Ratio Test.

28. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)(3n+1)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)(2n+3)}}{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}\right| = \lim_{n\to\infty}\frac{3n+1}{2n+3} = \frac{3}{2} > 1$, so the series diverges by the Ratio Test.

29. $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!} = \sum_{n=1}^{\infty}\frac{2^n n!}{n!} = \sum_{n=1}^{\infty}2^n$ which diverges by the Test for Divergence since $\displaystyle\lim_{n\to\infty}2^n = \infty$.

30. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+5)}}{\dfrac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}}\right| = \lim_{n\to\infty}\frac{2(n+1)}{3n+5} = \frac{2}{3} < 1$ so the series converges absolutely by the Ratio Test.

31. By the recursive definition, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{5n+1}{4n+3}\right| = \dfrac{5}{4} > 1$, so the series diverges by the Ratio Test.

32. By the recursive definition, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

33. (a) $\lim\limits_{n\to\infty}\left|\dfrac{1/(n+1)^3}{1/n^3}\right| = \lim\limits_{n\to\infty}\dfrac{n^3}{(n+1)^3} = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^3} = 1$. Inconclusive.

(b) $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)}{2^{n+1}}\cdot\dfrac{2^n}{n}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{2n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{2}+\dfrac{1}{2n}\right) = \dfrac{1}{2}$. Conclusive (convergent).

(c) $\lim\limits_{n\to\infty}\left|\dfrac{(-3)^n}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{n}{n+1}} = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{1}{1+1/n}} = 3$. Conclusive (divergent).

(d) $\lim\limits_{n\to\infty}\left|\dfrac{\sqrt{n+1}}{1+(n+1)^2}\cdot\dfrac{1+n^2}{\sqrt{n}}\right| = \lim\limits_{n\to\infty}\left[\sqrt{1+\dfrac{1}{n}}\cdot\dfrac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive.

34. We use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{[(n+1)!]^2/[k(n+1)!]}{(n!)^2/(kn)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)][k(n+1)-1]\cdots[kn+1]}\right|\end{aligned}$$

Now if $k=1$, then this is equal to $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k=2$, the limit is $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)^2}{(2n+2)(2n+1)}\right| = \dfrac{1}{4} < 1$, so the series converges, and if $k>2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

35. (a) $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0$, so by the Ratio Test the series converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim\limits_{n\to\infty}\dfrac{x^n}{n!} = 0$ by Theorem 12.2.6 [ET 11.2.6].

36. (a)
$$\begin{aligned}R_n &= a_{n+1}+a_{n+2}+a_{n+3}+a_{n+4}+\cdots = a_{n+1}\left(1+\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+3}}{a_{n+1}}+\frac{a_{n+4}}{a_{n+1}}+\cdots\right) \\ &= a_{n+1}\left(1+\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+4}}{a_{n+3}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}}+\cdots\right) \\ &= a_{n+1}\left(1+r_{n+1}+r_{n+2}r_{n+1}+r_{n+3}r_{n+2}r_{n+1}+\cdots\right)\quad(\bigstar) \\ &\leq a_{n+1}\left(1+r_{n+1}+r_{n+1}^2+r_{n+1}^3+\cdots\right)\quad[\text{since } \{r_n\} \text{ is decreasing}] = \frac{a_{n+1}}{1-r_{n+1}}\end{aligned}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n \to L$ as $n\to\infty$, we have $r_n < L$ for all n. So, starting with equation $\bigstar$,

$$R_n = a_{n+1}\left(1+r_{n+1}+r_{n+2}r_{n+1}+r_{n+3}r_{n+2}r_{n+1}+\cdots\right) \leq a_{n+1}\left(1+L+L^2+L^3+\cdots\right) = \frac{a_{n+1}}{1-L}$$

37. (a) $s_5 = \sum\limits_{n=1}^{5} \dfrac{1}{n2^n} = \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{24} + \dfrac{1}{64} + \dfrac{1}{160} = \dfrac{661}{960} \approx 0.68854$. Now the ratios

$r_n = \dfrac{a_{n+1}}{a_n} = \dfrac{n2^n}{(n+1)\,2^{n+1}} = \dfrac{n}{2\,(n+1)}$ form an increasing sequence, since

$r_{n+1} - r_n = \dfrac{n+1}{2\,(n+2)} - \dfrac{n}{2\,(n+1)} = \dfrac{(n+1)^2 - n\,(n+2)}{2\,(n+1)\,(n+2)} = \dfrac{1}{2\,(n+1)\,(n+2)} > 0$. So by

Exercise 36(b), the error is less than $\dfrac{a_6}{1 - \lim\limits_{n\to\infty} r_n} = \dfrac{1/\left(6 \cdot 2^6\right)}{1 - 1/2} = \dfrac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \dfrac{a_{n+1}}{1 - \frac{1}{2}} = \dfrac{2}{(n+1)\,2^{n+1}}$. We want

$R_n < 0.00005 \quad \Leftrightarrow \quad \dfrac{1}{(n+1)\,2^n} < 0.00005 \quad \Leftrightarrow \quad (n+1)\,2^n > 20{,}000$. To find such an n we can use trial and error or a graph. We calculate $(11+1)\,2^{11} = 24{,}576$, so $s_{11} = \sum\limits_{n=1}^{11} \dfrac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

38. $\sum\limits_{n=1}^{10} \dfrac{n}{2^n} = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \cdots + \dfrac{10}{1024} \approx 1.988$. The ratios $r_n = \dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{2n} = \dfrac{1}{2}\left(1 + \dfrac{1}{n}\right)$ form a decreasing sequence, so by Exercise 36(a), using $a_{11} = \dfrac{11}{2048}$ and $r_{11} = \dfrac{12}{22} = \dfrac{6}{11}$, the error in the above approximation is less than $\dfrac{a_{11}}{1 - r_{11}} \approx 0.0118$.

39. Summing the inequalities $-|a_i| \le a_i \le |a_i|$ for $i = 1, 2, \ldots, n$, we get $-\sum_{i=1}^n |a_i| \le \sum_{i=1}^n a_i \le \sum_{i=1}^n |a_i|$ $\Rightarrow \quad -\lim\limits_{n\to\infty} \sum_{i=1}^n |a_i| \le \lim\limits_{n\to\infty} \sum_{i=1}^n a_i \le \lim\limits_{n\to\infty} \sum_{i=1}^n |a_i| \quad \Rightarrow \quad -\sum_{n=1}^\infty |a_n| \le \sum_{n=1}^\infty a_n \le \sum_{n=1}^\infty |a_n| \quad \Rightarrow$ $\left|\sum_{n=1}^\infty a_n\right| \le \sum_{n=1}^\infty |a_n|$.

40. (a) Following the hint, we get that $|a_n| < r^n$ for $n \ge N$, and so since the geometric series $\sum_{n=1}^\infty r^n$ converges $(0 < r < 1)$, the series $\sum_{n=N}^\infty |a_n|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^\infty |a_n|$, so $\sum_{n=1}^\infty a_n$ is absolutely convergent.

(b) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \ge N$, so $|a_n| > 1$ for $n \ge N$. Thus, $\lim\limits_{n\to\infty} a_n \ne 0$, so $\sum_{n=1}^\infty a_n$ diverges by the Test for Divergence.

41. (a) Since $\sum a_n$ is absolutely convergent, and since $\left|a_n^+\right| \le |a_n|$ and $\left|a_n^-\right| \le |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent. (Or use Theorem 12.2.8 [ET 11.2.8].)

(b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum \left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem 12.2.8 [ET 11.2.8]. But $\sum \left(a_n^+ - \frac{1}{2}a_n\right) = \sum \left[\frac{1}{2}\left(a_n + |a_n|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.

42. Let $\sum b_n$ be the rearranged series constructed in the hint. [This series can be constructed by virtue of the result of Exercise 41(b).] This series will have partial sums s_n that oscillate in value back and forth across r. Since $\lim\limits_{n\to\infty} a_n = 0$ (by Theorem 12.2.6 [ET 11.2.6]), and since the size of the oscillations $|s_n - r|$ is always less than $|a_n|$ because of the way $\sum b_n$ was constructed, we have that $\sum b_n = \lim\limits_{n\to\infty} s_n = r$.

12.7 Strategy for Testing Series ET 11.7

1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^2-1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series diverges by the Test for Divergence.

2. If $a_n = \dfrac{n-1}{n^2+n}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^2-n}{n^2+n} = \lim\limits_{n\to\infty} \dfrac{1-1/n}{1+1/n} = 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

3. $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n}$ converges by the Comparison Test with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, a p-series that converges because $p = 2 > 1$.

4. Let $b_n = \dfrac{n-1}{n^2+n}$. Then $b_1 = 0$, and $b_2 = b_3 = \frac{1}{6}$, but $b_n > b_{n+1}$ for $n \geq 3$ since $\left(\dfrac{x-1}{x^2+x}\right)' = \dfrac{(x^2+x)-(x-1)(2x+1)}{(x^2+x)^2} = \dfrac{-x^2+2x+1}{(x^2+x)^2} < 0$ for $x \geq 3$. (This can be confirmed with a graph.) Thus, $\{b_n \mid n \geq 3\}$ is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=3}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ converges by the Alternating Series Test. Hence, the full series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ also converges.

5. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left(\dfrac{3^{n+2}}{2^{3n+3}} \cdot \dfrac{2^{3n}}{3^{n+1}}\right) = \lim\limits_{n\to\infty} \dfrac{3}{2^3} = \dfrac{3}{8} < 1$, so the series is absolutely convergent by the Ratio Test.

6. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{3n}{1+8n} = \lim\limits_{n\to\infty} \dfrac{3}{1/n+8} = \dfrac{3}{8} < 1$, so $\sum\limits_{n=1}^{\infty} \left(\dfrac{3n}{1+8n}\right)^n$ converges by the Root Test.

7. $\sum\limits_{k=1}^{\infty} k^{-1.7} = \sum\limits_{k=1}^{\infty} \dfrac{1}{k^{1.7}}$ is a convergent p-series ($p = 1.7 > 1$).

8. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{10^{n+1}/(n+1)!}{10^n/n!} = \lim\limits_{n\to\infty} \dfrac{10}{n+1} = 0 < 1$, so the series converges by the Ratio Test.

9. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)/e^{n+1}}{n/e^n} = \dfrac{1}{e} \lim\limits_{n\to\infty} \dfrac{n+1}{n} = \dfrac{1}{e} < 1$, so the series converges by the Ratio Test.

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1, \infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1, \infty)$ as well, and we can apply the Integral Test. $\int_1^\infty x^2 e^{-x^3}\,dx = \lim\limits_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the series converges.

11. $b_n = \dfrac{1}{n\ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n\ln n}$ converges by the Alternating Series Test. The series is conditionally convergent since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n}$ diverges by Exercise 12.3.19 [ET 11.3.19].

12. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim\limits_{n\to\infty} \sin n$ does not exist.

13. Let $f(x) = \dfrac{2}{x(\ln x)^3}$. $f(x)$ is clearly positive and decreasing for $x \geq 2$, so we apply the Integral Test.

$\displaystyle\int_2^\infty \frac{2}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\left[\frac{-1}{(\ln x)^2}\right]_2^t = 0 - \frac{-1}{(\ln 2)^2}$, which is finite, so $\displaystyle\sum_{n=2}^\infty \frac{2}{n(\ln n)^3}$ converges.

14. Using the Limit Comparison Test with $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, we have

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3+n}{n^3+1} = \lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^\infty b_n$ is the divergent harmonic series, $\sum_{n=1}^\infty a_n$ is also divergent.

15. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}(n+1)^2/(n+1)!}{3^n n^2/n!} = 3\lim_{n\to\infty}\frac{n+1}{n^2} = 0$, so the series converges by the Ratio Test.

16. $b_n = \dfrac{1}{\sqrt{n}-1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\displaystyle\lim_{n\to\infty} b_n = 0$, so $\displaystyle\sum_{n=2}^\infty \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

17. $\dfrac{3^n}{5^n+n} \leq \dfrac{3^n}{5^n} = \left(\dfrac{3}{5}\right)^n$. Since $\displaystyle\sum_{n=1}^\infty \left(\frac{3}{5}\right)^n$ is a convergent geometric series ($|r| = \frac{3}{5} < 1$), $\displaystyle\sum_{n=1}^\infty \frac{3^n}{5^n+n}$ converges by the Comparison Test.

18. $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+6)/5^{k+1}}{(k+5)/5^k} = \frac{1}{5}\lim_{k\to\infty}\frac{k+6}{k+5} = \frac{1}{5} < 1$, so the series converges by the Ratio Test.

19. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{(n+1)!}{2\cdot5\cdot8\cdot\cdots\cdot(3n+5)}}{\dfrac{n!}{2\cdot5\cdot8\cdot\cdots\cdot(3n+2)}} = \lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3} < 1$, so the series converges by the Ratio Test.

20. $\displaystyle\lim_{n\to\infty}\frac{n}{(n+1)(n+2)} = \lim_{n\to\infty}\frac{1}{n+3+2/n} = 0$, so since $\{b_n\}$ is a positive, decreasing sequence, $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n n}{(n+1)(n+2)}$ converges by the Alternating Series Test.

21. Use the Limit Comparison Test with $a_i = \dfrac{1}{\sqrt{i(i+1)}}$ and $b_i = \dfrac{1}{i}$.

$\displaystyle\lim_{i\to\infty}\frac{a_i}{b_i} = \lim_{i\to\infty}\frac{i}{\sqrt{i(i+1)}} = \lim_{i\to\infty}\frac{1}{\sqrt{1+1/i}} = 1$. Since $\displaystyle\sum_{i=1}^\infty b_i$ diverges (harmonic series) so does $\displaystyle\sum_{i=1}^\infty \frac{1}{\sqrt{i(i+1)}}$.

22. $\dfrac{\sqrt{n^2-1}}{n^3+2n^2+5} < \dfrac{n}{n^3+2n^2+5} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for $n \geq 1$, so $\displaystyle\sum_{n=1}^\infty \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p-series $\sum_{n=1}^\infty 1/n^2$.

23. $\displaystyle\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\displaystyle\lim_{n\to\infty}(-1)^n 2^{1/n}$ does not exist and the series diverges by the Test for Divergence.

24. $\dfrac{|\cos(n/2)|}{n^2+4n} < \dfrac{1}{n^2}$ and since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ converges ($p = 2 > 1$), $\displaystyle\sum_{n=1}^\infty \frac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.

25. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/(2\sqrt{n})} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$, so the series converges by the Alternating Series Test.

26. Let $a_n = \dfrac{\tan(1/n)}{n}$ and $b_n = \dfrac{1}{n^2}$. Then

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} n \cdot \tan(1/n) = \lim\limits_{n\to\infty} \dfrac{\tan(1/n)}{1/n} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \dfrac{(-1/n^2)\sec^2(1/n)}{-1/n^2} = \sec^2(0) = 1 > 0$, so since $\sum\limits_{n=1}^{\infty} b_n$ converges ($p = 2 > 1$), $\sum\limits_{n=1}^{\infty} \dfrac{\tan(1/n)}{n}$ converges also, by the Limit Comparison Test.

27. $\sum\limits_{n=1}^{\infty} \dfrac{(-2)^{2n}}{n^n} = \sum\limits_{n=1}^{\infty} \left(\dfrac{4}{n}\right)^n$. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{4}{n} = 0$, so the series converges by the Root Test.

28. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left(\dfrac{n^2+2n+2}{5^{n+1}} \cdot \dfrac{5^n}{n^2+1}\right) = \lim\limits_{n\to\infty} \left(\dfrac{1+2/n+2/n^2}{1+1/n^2} \cdot \dfrac{1}{5}\right) = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^2+1}{5^n}$ converges by the Ratio Test.

29. $\displaystyle\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (using integration by parts) $\overset{\text{H}}{=} 1$. So $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ converges by the Integral Test, and since $\dfrac{k\ln k}{(k+1)^3} < \dfrac{k\ln k}{k^3} = \dfrac{\ln k}{k^2}$, the given series converges by the Comparison Test.

30. Since $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \le e^{1/1} = e$ for all $n \ge 1$, and $\sum\limits_{n=1}^{\infty} \dfrac{e}{n^2}$ converges ($p = 2 > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

31. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{2^{n+1}/(2n+3)!}{2^n/(2n+1)!} = 2\lim\limits_{n\to\infty} \dfrac{1}{(2n+3)(2n+2)} = 0$, so the series converges by the Ratio Test.

32. Let $f(x) = \dfrac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{5-x}{2\sqrt{x}(x+5)^2} < 0$ for $x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+5} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/2} + 5n^{-1/2}} = 0$, so the series converges.

33. $0 < \dfrac{\tan^{-1} n}{n^{3/2}} < \dfrac{\pi/2}{n^{3/2}}$. $\sum\limits_{n=1}^{\infty} \dfrac{\pi/2}{n^{3/2}} = \dfrac{\pi}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ which is a convergent p-series ($p = \frac{3}{2} > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{\tan^{-1} n}{n^{3/2}}$ converges by the Comparison Test.

34. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{2n}{n^2} = \lim\limits_{n\to\infty} \dfrac{2}{n} = 0$ so the series converges by the Root Test.

35. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{n^2/n} = \lim\limits_{n\to\infty} \dfrac{1}{[(n+1)/n]^n} = \dfrac{1}{\lim\limits_{n\to\infty}(1+1/n)^n} = \dfrac{1}{e} < 1$ (see Equation 7.4.9 [ET 3.8.6]), so the series converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \to \infty$ as $n \to \infty$, so $\ln \ln n > 2$ for sufficiently large n. For these n we have $(\ln n)^{\ln n} > n^2$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges ($p = 2 > 1$), so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(2^{1/n} - 1\right) = 1 - 1 = 0$, so the series converges by the Root Test.

38. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2^{1/n} - 1}{1/n} \overset{\text{H}}{=} \ln 2 > 0$.

So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$.

Alternate Solution:

$\sqrt[n]{2} - 1 = \dfrac{1}{2^{(n-1)/n} + 2^{(n-2)/n} + 2^{(n-3)/n} + \cdots + 2^{1/n} + 1}$ [rationalize the numerator] $\geq \dfrac{1}{2n}$, and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series), so does $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ by the Comparison Test.

12.8 Power Series

ET 11.8

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$ is called a power series in $(x - a)$ or a power series centered at a or a power series about a.

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{x}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{\sqrt{n+1}/\sqrt{n}}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{\sqrt{1 + 1/n}} = |x|$. By the Ratio Test, the series converges when $|x| < 1$, so the radius of convergence $R = 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges because it is a p-series with $p = \frac{1}{2} \leq 1$, but when $x = -1$, it converges by the Alternating Series Test. So the interval of convergence is $I = [-1, 1)$.

4. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{n+2} \cdot \dfrac{n+1}{x^n}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{1 + 1/(n+1)} = |x|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

5. If $a_n = nx^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)\,x^{n+1}}{nx^n}\right| = |x|\lim\limits_{n\to\infty}\dfrac{n+1}{n} = |x| < 1$ for convergence (by the Ratio Test). So $R = 1$. When $x = 1$ or -1, $\lim\limits_{n\to\infty} nx^n$ does not exist, so $\sum_{n=0}^{\infty} nx^n$ diverges for $x = \pm 1$. So $I = (-1, 1)$.

6. If $a_n = \dfrac{x^n}{n^2}$ then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x|\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2 = |x| < 1$ for convergence (by the Ratio Test), so $R = 1$. If $x = \pm 1$, $\sum\limits_{n=1}^{\infty}|a_n| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$, which converges ($p = 2 > 1$), so $I = [-1, 1]$.

7. If $a_n = \dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}/\,(n+1)!}{x^n/n!}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0 < 1$ for all x. So, by the Ratio Test, $R = \infty$, and $I = (-\infty, \infty)$.

8. Here the Root Test is easier. If $a_n = n^n x^n$ then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} n\,|x| = \infty$ if $x \neq 0$, so $R = 0$ and $I = \{0\}$.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)\,4^{n+1}\,|x|^{n+1}}{n4^n\,|x|^n} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)4\,|x| = 4\,|x|$. Now $4\,|x| < 1 \;\Leftrightarrow\; |x| < \frac{1}{4}$, so by the Ratio Test, $R = \frac{1}{4}$. When $x = \frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty}(-1)^n\,n$, and when $x = -\frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty} n$. Thus, $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.

10. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)\,3^{n+1}}\cdot\dfrac{n3^n}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{3\,(1+1/n)} = \dfrac{|x|}{3}$, so by the Ratio Test, the series converges when $\dfrac{|x|}{3} < 1 \;\Leftrightarrow\; |x| < 3$, so $R = 3$. When $x = -3$, the series is an alternating harmonic series, which converges by the Alternating Series Test. When $x = 3$, it is the harmonic series, which diverges. Thus, the interval of convergence is $[-3, 3)$.

11. If $a_n = \dfrac{3^n x^n}{(n+1)^2}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{3^{n+1}x^{n+1}}{(n+2)^2}\cdot\dfrac{(n+1)^2}{3^n x^n}\right| = 3\,|x|\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n+2}\right)^2 = 3\,|x| < 1$ for convergence, so $|x| < \frac{1}{3}$ and $R = \frac{1}{3}$. When $x = \frac{1}{3}$, $\sum\limits_{n=0}^{\infty}\dfrac{3^n x^n}{(n+1)^2} = \sum\limits_{n=0}^{\infty}\dfrac{1}{(n+1)^2} = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$). When $x = -\frac{1}{3}$, $\sum\limits_{n=0}^{\infty}\dfrac{3^n x^n}{(n+1)^2} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{(n+1)^2}$ which converges by the Alternating Series Test, so $I = \left[-\frac{1}{3}, \frac{1}{3}\right]$.

12. If $a_n = \dfrac{n^2 x^n}{10^n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x|}{10}\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^2 = \dfrac{|x|}{10} < 1$ for convergence (by the Ratio Test), so $R = 10$. If $x = \pm 10$, $|a_n| = n^2 \to \infty$ as $n \to \infty$, so $\sum_{n=0}^{\infty} a_n$ diverges (Test for Divergence) and $I = (-10, 10)$.

13. If $a_n = \dfrac{x^n}{\ln n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{\ln(n+1)}\cdot\dfrac{\ln n}{x^n}\right| = |x|\lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)} \overset{\text{H}}{=} |x|$, so $R = 1$. When $x = 1$, $\sum\limits_{n=2}^{\infty}\dfrac{x^n}{\ln n} = \sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ which diverges because $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ is the divergent harmonic series. When $x = -1$, $\sum\limits_{n=2}^{\infty}\dfrac{x^n}{\ln n} = \sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ which converges by the Alternating Series Test. So $I = [-1, 1)$.

14. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^3|x-5| = |x-5|$, so by the Ratio Test, the series converges when $|x-5|<1 \Leftrightarrow 4<x<6$. When $x=4$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n^3$, which diverges by the Test for Divergence. When $x=6$, the series becomes $\sum_{n=0}^{\infty} n^3$, which also diverges. Thus, $R=1$ and $I=(4,6)$.

15. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{\sqrt{n+1}\,|x-1|^{n+1}}{\sqrt{n}\,|x-1|^n} = \lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x-1| = |x-1|$, so by the Ratio Test, the series converges when $|x-1|<1 \Leftrightarrow -1<x-1<1 \Leftrightarrow 0<x<2$. $R=1$. When $x=2$, the series becomes $\sum_{n=0}^{\infty}\sqrt{n}$, which diverges by the Test for Divergence. When $x=0$, the series becomes $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I=(0,2)$.

16. If $a_n=\dfrac{(-1)^n x^{2n-1}}{(2n-1)!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{x^2}{(2n+1)\,2n} = 0 < 1$ for all x. By the Ratio Test the series converges for all x, so $R=\infty$ and $I=(-\infty,\infty)$.

17. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{|x+2|^{n+1}}{(n+1)\,2^{n+1}}\cdot\dfrac{n2^n}{|x+2|^n}\right] = \lim\limits_{n\to\infty}\dfrac{n}{n+1}\cdot\dfrac{|x+2|}{2} = \dfrac{|x+2|}{2}$, so by the Ratio Test, the series converges when $\dfrac{|x+2|}{2}<1 \Leftrightarrow |x+2|<2 \Leftrightarrow -2<x+2<2 \Leftrightarrow -4<x<0$. $R=2$. When $x=-4$, the series becomes $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{(-2)^n}{n2^n} = \sum\limits_{n=1}^{\infty}\dfrac{2^n}{n2^n} = \sum\limits_{n=1}^{\infty}\dfrac{1}{n}$, which is the divergent harmonic series. When $x=0$, the series is $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I=(-4,0]$.

18. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim\limits_{n\to\infty}\dfrac{2\,|x+3|}{\sqrt{1+1/n}} = 2\,|x+3| < 1 \Leftrightarrow$ $|x+3|<\frac{1}{2} \Leftrightarrow -\frac{7}{2}<x<-\frac{5}{2}$. Thus, $R=\frac{1}{2}$. When $x=-\frac{5}{2}$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. When $x=-\frac{7}{2}$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p=\frac{1}{2}\le 1$. Thus, $I=\left(-\frac{7}{2},-\frac{5}{2}\right]$.

19. If $a_n=\dfrac{(x-2)^n}{n^n}$, then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\dfrac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R=\infty$ and $I=(-\infty,\infty)$.

20. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(3x-2)^{n+1}}{(n+1)\,3^{n+1}}\cdot\dfrac{n3^n}{(3x-2)^n}\right| = \lim\limits_{n\to\infty}\left(\dfrac{|3x-2|}{3}\cdot\dfrac{1}{1+1/n}\right) = \dfrac{|3x-2|}{3} = \left|x-\frac{2}{3}\right|$, so by the Ratio Test, the series converges when $\left|x-\frac{2}{3}\right|<1 \Leftrightarrow -\frac{1}{3}<x<\frac{5}{3}$. $R=1$. When $x=-\frac{1}{3}$, the series is $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, the convergent alternating harmonic series. When $x=\frac{5}{3}$, the series becomes the divergent harmonic series. Thus, $I=\left[-\frac{1}{3},\frac{5}{3}\right)$.

21. If $a_n = \dfrac{2^n (x-3)^n}{n+3}$, then

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-3)^{n+1}}{n+4}\cdot\dfrac{n+3}{2^n(x-3)^n}\right| = 2\,|x-3|\lim\limits_{n\to\infty}\dfrac{n+3}{n+4} = 2\,|x-3| < 1$ for convergence, or $|x-3| < \frac{1}{2} \;\Leftrightarrow\; \frac{5}{2} < x < \frac{7}{2}$, and $R = \frac{1}{2}$. When $x = \frac{5}{2}$, $\sum\limits_{n=0}^{\infty}\dfrac{2^n(x-3)^n}{n+3} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{n+3}$ which converges by the Alternating Series Test. When $x = \frac{7}{2}$, $\sum\limits_{n=0}^{\infty}\dfrac{2^n(x-3)^n}{n+3} = \sum\limits_{n=0}^{\infty}\dfrac{1}{n+3}$, similar to the harmonic series, which diverges. So $I = \left[\frac{5}{2},\frac{7}{2}\right)$.

22. If $a_n = \dfrac{(x+1)^n}{n(n+1)}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x+1|\lim\limits_{n\to\infty}\dfrac{n}{n+2} = |x+1| < 1$ for convergence, or $-2 < x < 0$ and $R = 1$. If $x = -2$ or 0, then $|a_n| = \dfrac{1}{n^2+n} < \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty}|a_n|$ converges since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ does ($p = 2 > 1$), and $I = [-2, 0]$.

23. If $a_n = n!\,(2x-1)^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n}\right| = \lim\limits_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n\to\infty$ for all $x \neq \frac{1}{2}$. Since the series diverges for all $x \neq \frac{1}{2}$, $R = 0$ and $I = \left\{\frac{1}{2}\right\}$.

24. If $a_n = \dfrac{nx^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$ then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x|\lim\limits_{n\to\infty}\dfrac{n+1}{n(2n+1)} = 0$ for all x. So the series converges for all $x \;\Rightarrow\; R = \infty$ and $I = (-\infty, \infty)$.

25. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{|4x+1|^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{|4x+1|^n}\right] = \lim\limits_{n\to\infty}\dfrac{|4x+1|}{(1+1/n)^2} = |4x+1|$, so by the Ratio Test, the series converges when $|4x+1| < 1 \;\Leftrightarrow\; -1 < 4x+1 < 1 \;\Leftrightarrow\; -2 < 4x < 0 \;\Leftrightarrow\; -\frac{1}{2} < x < 0$, so $R = \frac{1}{4}$. When $x = -\frac{1}{2}$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x = 0$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$, a convergent p-series ($p = 2 > 1$). $I = \left[-\frac{1}{2}, 0\right]$.

26. If $a_n = \dfrac{(-1)^n(2x+3)^n}{n\ln n}$ then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |2x+3|\lim\limits_{n\to\infty}\dfrac{n\ln n}{(n+1)\ln(n+1)} = |2x+3| < 1$ for convergence, so $-2 < x < -1$ and $R = \frac{1}{2}$. When $x = -2$, $\sum\limits_{n=2}^{\infty}a_n = \sum\limits_{n=2}^{\infty}\dfrac{1}{n\ln n}$ which diverges (Integral Test), and when $x = -1$, $\sum\limits_{n=2}^{\infty}a_n = \sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{n\ln n}$ which converges (Alternating Series Test), so $I = (-2, -1]$.

27. If $a_n = \dfrac{x^n}{(\ln n)^n}$ then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\dfrac{|x|}{\ln n} = 0 < 1$ for all x, so $R = \infty$ and $I = (-\infty, \infty)$ by the Root Test.

28. If $a_n = \dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)\,x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$ then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}|x|\left(\dfrac{2n+2}{2n+1}\right) = |x| < 1$ for convergence, so $R = 1$. If $x = \pm 1$, $|a_n| = \dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} > 1$ for all n since each integer in the numerator is larger than the corresponding one in the denominator, so $\sum a_n$ diverges in both cases by the Test for Divergence, and $I = (-1, 1)$.

29. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \leq 4$. In particular, it converges when $x = -2$, that is, $\sum_{n=0}^{\infty} c_n (-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n (-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n / (n4^n)$.]

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \leq x < 4$ and diverges for at least $x \geq 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$, that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$, that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$, that is, $\sum c_n (-3)^n$ is convergent.

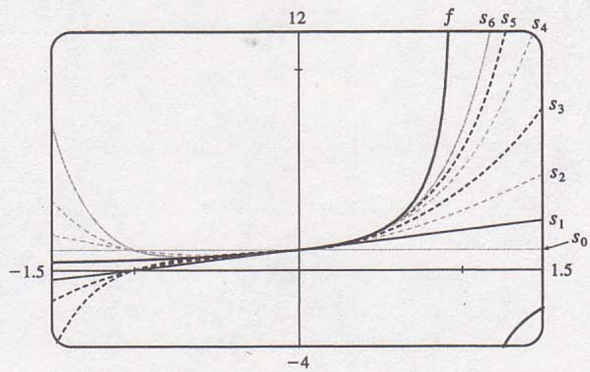
(d) It diverges when $x = -9$, that is, $\sum c_n (-9)^n = \sum (-1)^n c_n 9^n$ is divergent.

31. If $a_n = \dfrac{(n!)^k}{(kn)!} x^n$, then

$$\begin{aligned}
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \frac{[(n+1)!]^k (kn)!}{(n!)^k [k(n+1)]!} |x| = \lim_{n \to \infty} \frac{(n+1)^k}{(kn+k)(kn+k-1) \cdots (kn+2)(kn+1)} |x| \\
&= \lim_{n \to \infty} \left[\frac{(n+1)}{(kn+1)} \frac{(n+1)}{(kn+2)} \cdots \frac{(n+1)}{(kn+k)} \right] |x| \\
&= \lim_{n \to \infty} \left[\frac{n+1}{kn+1} \right] \lim_{n \to \infty} \left[\frac{n+1}{kn+2} \right] \cdots \lim_{n \to \infty} \left[\frac{n+1}{kn+k} \right] |x| = \left(\frac{1}{k} \right)^k |x| < 1 \quad \Leftrightarrow
\end{aligned}$$

$|x| < k^k$ for convergence, and the radius of convergence is $R = k^k$.

32. The partial sums definitely do not converge to $f(x)$ for $x \geq 1$, since f is undefined at $x = 1$ and negative on $(1, \infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x \leq -1$, since $0 < f(x) < 1$ on this interval, while the partial sums are either larger than 1 or less than 0. The partial sums seem to converge to f on $(-1, 1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x| < 1$, divergence for $|x| \geq 1$ (see Example 12.2.5 [ET 11.2.5]).



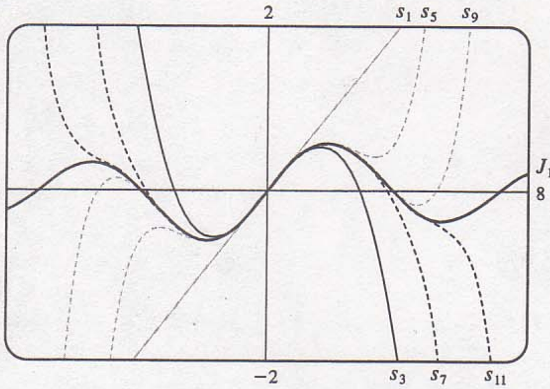
33. (a) If $a_n = \dfrac{(-1)^n x^{2n+1}}{n!\,(n+1)!2^{2n+1}}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \left(\dfrac{x}{2}\right)^2 \lim\limits_{n\to\infty}\dfrac{1}{(n+1)(n+2)} = 0$ for all x. So $J_1(x)$ converges for all x; the domain is $(-\infty,\infty)$.

(b), (c) The initial terms of $J_1(x)$ up to $n=5$ are

$a_0 = \dfrac{x}{2}$, $a_1 = -\dfrac{x^3}{16}$, $a_2 = \dfrac{x^5}{384}$,

$a_3 = -\dfrac{x^7}{18{,}432}$, $a_4 = \dfrac{x^9}{1{,}474{,}560}$, and

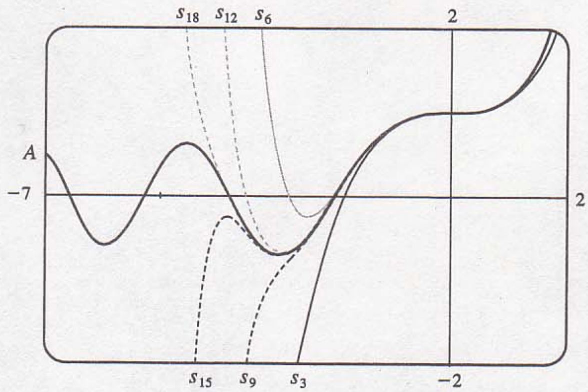
$a_5 = -\dfrac{x^{11}}{176{,}947{,}200}$. The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



34. (a) $A(x) = 1 + \sum\limits_{n=1}^{\infty} a_n$, where $a_n = \dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\cdots\cdot(3n-1)(3n)}$, so

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x|^3 \lim\limits_{n\to\infty}\dfrac{1}{(3n+2)(3n+3)} = 0$ for all x, so the domain is $\mathbb{R}$.

(b)



$s_0 = 1$ has been omitted from the graph. The partial sums seem to approximate $A(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.

To plot A, we must first define $A(x)$ for the CAS. Note that for $n \geq 1$, the denominator of a_n is

$2\cdot 3\cdot 5\cdot 6\cdot\cdots\cdot(3n-1)\cdot 3n = \dfrac{(3n)!}{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)} = \dfrac{(3n)!}{\prod_{k=1}^{n}(3k-2)}$, so

$a_n = 1 + \dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$ and thus $A(x) = 1 + \sum\limits_{n=1}^{\infty}\dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$. Both Maple and Mathematica are able to plot A if we define it this way, and Derive is able to produce a similar graph using a suitable partial sum of $A(x)$.

Derive, Maple and Mathematica all have two initially known Airy functions, called `AI_SERIES(z,m)` and `BI_SERIES(z,m)` from `BESSEL.MTH` in Derive and `AiryAi` and `AiryBi` in Maple and Mathematica (just `Ai` and `Bi` in older versions of Maple). However, it is very difficult to solve for A in terms of the CAS's Airy functions, although in fact $A(x) = \dfrac{\sqrt{3}\texttt{AiryAi}(x) + \texttt{AiryBi}(x)}{\sqrt{3}\texttt{AiryAi}(0) + \texttt{AiryBi}(0)}$.

35. $s_{2n-1} = 1 + 2x + x^2 + 2x^3 + \cdots + x^{2n-2} + 2x^{2n-1} = (1+2x)\left(1 + x^2 + x^4 + \cdots + x^{2n-2}\right)$

$= (1+2x)\dfrac{1-x^{2n}}{1-x^2}$ [by (12.2.3 [ET 11.2.3]) with $r = x^2$]$\rightarrow \dfrac{1+2x}{1-x^2}$ as $n \to \infty$ [by (12.2.4 [ET 11.2.4])],

when $|x| < 1$. Also $s_{2n} = s_{2n-1} + x^{2n} \to \dfrac{1+2x}{1-x^2}$ since $x^{2n} \to 0$ for $|x| < 1$. Therefore, $s_n \to \dfrac{1+2x}{1-x^2}$

since s_{2n} and s_{2n-1} both approach $\dfrac{1+2x}{1-x^2}$ as $n \to \infty$. Thus, the interval of convergence is $(-1,1)$ and

$f(x) = \dfrac{1+2x}{1-x^2}$.

36. $s_{4n-1} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_0x^4 + c_1x^5 + c_2x^6 + c_3x^7 + \cdots + c_3x^{4n-1}$

$= \left(c_0 + c_1x + c_2x^2 + c_3x^3\right)\left(1 + x^4 + x^8 + \cdots + x^{4n-4}\right) \to \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1-x^4}$ as $n \to \infty$

[by (12.2.4 [ET 11.2.4]) with $r = x^4$] for $|x| < 1$. Also s_{4n}, s_{4n+1}, s_{4n+2} have the same limits (for example, $s_{4n} = s_{4n-1} + c_0x^{4n}$ and $x^{4n} \to 0$ for $|x| < 1$.) So if at least one of c_0, c_1, c_2, and c_3 is nonzero, then the interval of convergence is $(-1,1)$ and $f(x) = \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1-x^4}$.

37. We use the Root Test on the series $\sum c_nx^n$. $\lim\limits_{n\to\infty} \sqrt[n]{|c_nx^n|} = |x| \lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = c\,|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

38. Since $\sum c_nx^n$ converges whenever $|x| < R$, $\sum c_nx^{2n} = \sum c_n\left(x^2\right)^n$ converges whenever $\left|x^2\right| < R \iff |x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

39. For $2 < x < 3$, $\sum c_nx^n$ diverges and $\sum d_nx^n$ converges. By Exercise 12.2.61 [ET 11.2.61], $\sum (c_n + d_n)x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n)x^n$ is 2.

12.9 Representations of Functions as Power Series — ET 11.9

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_nx^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} nc_nx^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum\limits_{n=0}^{\infty} b_nx^n$ converges on $(-2,2)$, then $\int f(x)\,dx = C + \sum\limits_{n=0}^{\infty} \dfrac{b_n}{n+1}x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence — it may happen that the integrated series converges at an endpoint (or both endpoints).

3. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ with $|-x| < 1 \iff |x| < 1$, so $R = 1$ and $I = (-1,1)$.

4. $f(x) = \dfrac{x}{1-x} = x\left(\dfrac{1}{1-x}\right) = x\sum\limits_{n=0}^{\infty} x^n = \sum\limits_{n=0}^{\infty} x^{n+1} = \sum\limits_{n=1}^{\infty} x^n$ with $R = 1$ and $I = (-1,1)$.

5. Replacing x with x^3 in (1) gives $f(x) = \dfrac{1}{1-x^3} = \sum\limits_{n=0}^{\infty} \left(x^3\right)^n = \sum\limits_{n=0}^{\infty} x^{3n}$. The series converges when $\left|x^3\right| < 1$; that is, when $|x| < 1$, so $I = (-1,1)$.

6. $f(x) = \dfrac{1}{1+9x^2} = \dfrac{1}{1-(-9x^2)} = \sum\limits_{n=0}^{\infty} (-9x^2)^n = \sum\limits_{n=0}^{\infty} (-1)^n 3^{2n} x^{2n}$. The series converges when $|-9x^2| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

7. $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\left(\dfrac{1}{1+x^2/4}\right) = \dfrac{1}{4}\sum\limits_{n=0}^{\infty} (-1)^n \left(\dfrac{x^2}{4}\right)^n$ (using Exercise 3) $= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{4^{n+1}}$, with $\left|\dfrac{x^2}{4}\right| < 1 \;\Leftrightarrow\; x^2 < 4 \;\Leftrightarrow\; |x| < 2$, so $R = 2$ and $I = (-2, 2)$.

8. $f(x) = \dfrac{1+x^2}{1-x^2} = 1 + \dfrac{2x^2}{1-x^2} = 1 + 2x^2 \sum\limits_{n=0}^{\infty} (x^2)^n = 1 + \sum\limits_{n=0}^{\infty} 2x^{2n+2} = 1 + \sum\limits_{n=1}^{\infty} 2x^{2n}$, with $|x^2| < 1 \;\Leftrightarrow\; |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

9. $f(x) = \dfrac{1}{x-5} = -\dfrac{1}{5}\left(\dfrac{1}{1-x/5}\right) = -\dfrac{1}{5}\sum\limits_{n=0}^{\infty} \left(\dfrac{x}{5}\right)^n$. The series converges when $\left|\dfrac{x}{5}\right| < 1$; that is, when $|x| < 5$, so $I = (-5, 5)$.

10. $f(x) = \dfrac{x}{4x+1} = x \cdot \dfrac{1}{1-(-4x)} = x\sum\limits_{n=0}^{\infty} (-4x)^n = \sum\limits_{n=0}^{\infty} (-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x| < 1$; that is, when $|x| < \frac{1}{4}$, so $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.

11. $f(x) = \dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1} \;\Rightarrow\; 3 = A(x-1) + B(x+2)$. Taking $x = -2$, we get $A = -1$. Taking $x = 1$, we get $B = 1$. Thus,

$$\begin{aligned}\frac{3}{x^2+x-2} &= \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x/2} = -\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\left[-1 - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n\right]x^n = \sum_{n=0}^{\infty}\left[-1 + \left(-\tfrac{1}{2}\right)^{n+1}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]x^n\end{aligned}$$

We represented the given function as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

12. $f(x) = \dfrac{7x-1}{3x^2+2x-1} = \dfrac{7x-1}{(3x-1)(x+1)} = \dfrac{A}{3x-1} + \dfrac{B}{x+1} = \dfrac{1}{3x-1} + \dfrac{2}{x+1} = 2 \cdot \dfrac{1}{1-(-x)} - \dfrac{1}{1-3x}$
$= 2\sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} [2(-1)^n - 3^n] x^n$

The series $\sum (-x)^n$ converges for $x \in (-1, 1)$ and the series $\sum (3x)^n$ converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, so their sum converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right) = I$.

13. $f(x) = \dfrac{1}{(1+x)^2} = -\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n x^n\right]$ (from Exercise 3)
$= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$ [from Theorem 2(a)] $= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$ with $R = 1$.

14. $f(x) = \dfrac{1}{1+x} = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ (geometric series with $R = 1$), so

$$\begin{aligned}f(x) = \ln(1+x) &= \int \frac{dx}{1+x} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad [C = 0 \text{ since } f(0) = 0], \text{ with } R = 1\end{aligned}$$

15. $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ (from Exercise 13)

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R=1$.

16. $f(x) = x\ln(1+x) = x\left[\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}\right]$ (by Exercise 14) $= \displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\dfrac{(-1)^n x^n}{n-1}$ with $R=1$.

17. $f(x) = \ln(5-x) = -\displaystyle\int\dfrac{dx}{5-x} = -\dfrac{1}{5}\int\dfrac{dx}{1-x/5}$

$= -\dfrac{1}{5}\displaystyle\int\left[\sum_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n\right]dx = C - \dfrac{1}{5}\sum_{n=0}^{\infty}\dfrac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\dfrac{x^n}{n5^n}$

Putting $x=0$, we get $C=\ln 5$. The series converges for $|x/5|<1 \Leftrightarrow |x|<5$, so $R=5$.

18. We know that $\dfrac{1}{1-2x} = \displaystyle\sum_{n=0}^{\infty}(2x)^n$. Differentiating, we get $\dfrac{2}{(1-2x)^2} = \displaystyle\sum_{n=1}^{\infty}2^n nx^{n-1} = \sum_{n=0}^{\infty}2^{n+1}(n+1)x^n$,

so $f(x) = \dfrac{x^2}{(1-2x)^2} = \dfrac{x^2}{2}\cdot\dfrac{2}{(1-2x)^2} = \dfrac{x^2}{2}\displaystyle\sum_{n=0}^{\infty}2^{n+1}(n+1)x^n = \sum_{n=0}^{\infty}2^n(n+1)x^{n+2}$ or

$\displaystyle\sum_{n=2}^{\infty}2^{n-2}(n-1)x^n$, with $R=\frac{1}{2}$.

19. $\dfrac{1}{2-x} = \dfrac{1}{2(1-x/2)} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n$ for $\left|\dfrac{x}{2}\right|<1 \Leftrightarrow |x|<2$. Now

$\dfrac{1}{(x-2)^2} = \left(\dfrac{1}{2-x}\right)' = \left(\displaystyle\sum_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n\right)' = \sum_{n=0}^{\infty}\left(\dfrac{1}{2^{n+1}}x^n\right)' = \sum_{n=1}^{\infty}\dfrac{n}{2^{n+1}}x^{n-1} = \sum_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^n$. So

$f(x) = \dfrac{x^3}{(x-2)^2} = x^3\displaystyle\sum_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^n = \sum_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^{n+3}$ or $\displaystyle\sum_{n=3}^{\infty}\dfrac{n-2}{2^{n-1}}x^n$ for $|x|<2$. Thus, $R=2$ and $I=(-2,2)$.

20. From Example 7, $g(x) = \arctan x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$. Thus,

$f(x) = \arctan(x/3) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\dfrac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\dfrac{1}{3^{2n+1}(2n+1)}x^{2n+1}$ for $\left|\dfrac{x}{3}\right|<1 \Leftrightarrow |x|<3$,

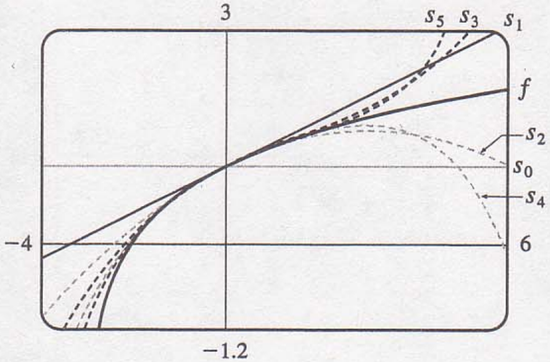
so $R=3$.

21. $f(x) = \ln(3+x) = \int \frac{dx}{3+x} = \frac{1}{3}\int \frac{dx}{1+x/3} = \frac{1}{3}\int \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{3}\right)^n dx$ (from Exercise 3)

$= C + \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1/3)^n}{n+1}x^{n+1} = \ln 3 + \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1/3)^{n-1}}{n}x^n \quad [C = f(0) = \ln 3]$

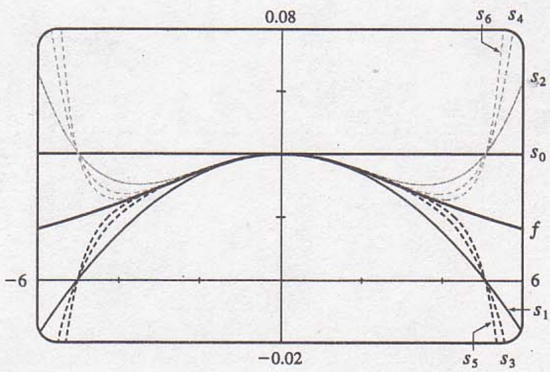
$= \ln 3 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n3^n}x^n$ with $R = 3$.

The terms of the series are $a_0 = \ln 3, a_1 = \frac{x}{3}, a_2 = -\frac{x^2}{18}, a_3 = \frac{x^3}{81}, a_4 = -\frac{x^4}{324}, a_5 = \frac{x^5}{1215}, \ldots$.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3, 3)$.

22. $f(x) = \frac{1}{x^2+25} = \frac{1/25}{1+(x/5)^2} = \frac{1}{25}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{5}\right)^{2n}$ (by Exercise 3) with $R = 5$. The terms of the series are $a_0 = \frac{1}{25}, a_1 = -\frac{x^2}{625}, a_2 = \frac{x^4}{15{,}625}, \ldots$.

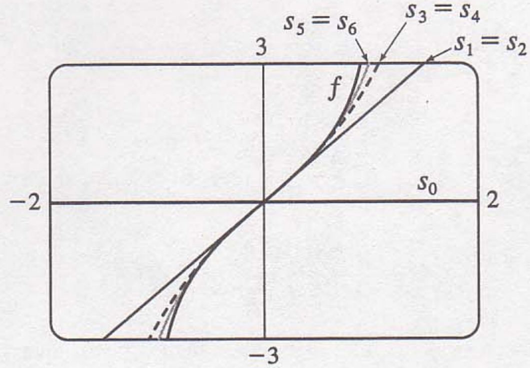


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5, 5)$.

23. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \displaystyle\int \frac{dx}{1+x} + \int \frac{dx}{1-x}$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} + C$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then $f(x) = \pm 2 \displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \dfrac{1}{n}$.

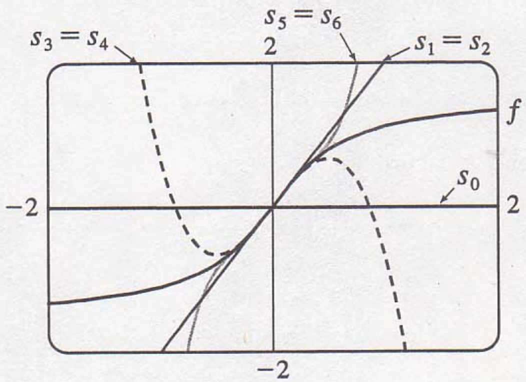


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

24. $f(x) = \tan^{-1} 2x = 2\displaystyle\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n (4x^2)^n\, dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$

$$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \text{ for } |4x^2| < 1 \quad\Leftrightarrow\quad |x| < \tfrac{1}{2}, \text{ so } R = \tfrac{1}{2}.$$

If $x = \pm\frac{1}{2}$, then $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

25. $\displaystyle\int \frac{dx}{1+x^4} = \int \sum_{n=0}^{\infty} (-1)^n x^{4n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$ with $R = 1$.

26. $\dfrac{1}{1+x^5} = \sum\limits_{n=0}^{\infty} (-1)^n x^{5n} \quad\Rightarrow\quad \dfrac{x}{1+x^5} = \sum\limits_{n=0}^{\infty} (-1)^n x^{5n+1} \quad\Rightarrow\quad \displaystyle\int \frac{x}{1+x^5}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+2}}{5n+2}$ with $R = 1$.

27. By Example 7, $\arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$, so

$$\int \frac{\arctan x}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2} \text{ with } R = 1.$$

28. By Example 7, $\displaystyle\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$, with $R = 1$.

29. We use the representation $\displaystyle\int \frac{dx}{1+x^4} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$ from Exercise 25 with $C = 0$. So

$$\int_0^{0.2} \frac{dx}{1+x^4} = \left[x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots\right]_0^{0.2} = 0.2 - \frac{0.2^5}{5} + \frac{0.2^9}{9} - \frac{0.2^{13}}{13} + \cdots$$

Since the series is alternating, the error in the nth-order approximation is less than the first neglected term, by The Alternating Series Estimation Theorem. If we use only the first two terms of the series, then the error is at most $0.2^9/9 \approx 5.7 \times 10^{-8}$. So, to six decimal places, $\displaystyle\int_0^{0.2} \frac{dx}{1+x^4} \approx 0.2 - \frac{0.2^5}{5} = 0.199936$.

30. We use the representation $\int \tan^{-1}(x^2)\,dx = C + \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+3}}{(2n+1)(4n+3)}$ from Exercise 28 with $C = 0$:

$$\int_0^{1/2} \tan^{-1}(x^2)\,dx = \left[\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} + \frac{x^{19}}{171} - \cdots\right]_0^{1/2} = \frac{0.5^3}{3} - \frac{0.5^7}{21} + \frac{0.5^{11}}{55} - \frac{0.5^{15}}{105} + \frac{0.5^{19}}{171} - \cdots$$

The series is alternating, so if we use only the first four terms of the series, then the error is at most $0.5^{19}/171 \approx 1.1 \times 10^{-8}$. So, to six decimal places,

$$\textstyle\int_0^{1/2} \tan^{-1}(x^2)\,dx \approx \frac{1}{3}(0.5)^3 - \frac{1}{21}(0.5)^7 + \frac{1}{55}(0.5)^{11} - \frac{1}{105}(0.5)^{15} \approx 0.041303$$

31. We substitute x^4 for x in Example 7, and find that

$$\begin{aligned}\int x^2 \tan^{-1}(x^4)\,dx &= \int x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{2n+1}\,dx \\ &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+6}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+7}}{(2n+1)(8n+7)}\end{aligned}$$

So $\int_0^{1/3} x^2 \tan^{-1}(x^4)\,dx = \left[\dfrac{x^7}{7} - \dfrac{x^{15}}{45} + \cdots\right]_0^{1/3} = \dfrac{1}{7 \cdot 3^7} - \dfrac{1}{45 \cdot 3^{15}} + \cdots$. The series is alternating, so if we use only one term, the error is at most $1/\left(45 \cdot 3^{15}\right) \approx 1.5 \times 10^{-9}$. So $\int_0^{1/3} x^2 \tan^{-1}(x^4)\,dx \approx 1/\left(7 \cdot 3^7\right) \approx 0.000065$ to six decimal places.

32. $\displaystyle\int_0^{0.5} \frac{dx}{1+x^6} = \int_0^{0.5} \sum_{n=0}^{\infty} (-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{6n+1}}{6n+1}\right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)\,2^{6n+1}}$

$$= \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} - \frac{1}{19\cdot 2^{19}} + \cdots$$

The series is alternating, so if we use only three terms, the error is at most $\dfrac{1}{19\cdot 2^{19}} \approx 1.0\times 10^{-7}$. So, to six decimal places, $\displaystyle\int_0^{0.5} \frac{dx}{1+x^6} \approx \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} \approx 0.498893$.

33. Using the result of Example 6, $\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n$, with $x=-0.1$, we have $\ln 1.1 = \ln[1-(-0.1)] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5} = 0.000002$. So $\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531$.

34. $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad\Rightarrow\quad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n x^{2n-1}}{(2n)!}$ (the first term disappears), so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)\,x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad \text{(substituting } n+1 \text{ for } n\text{)}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \quad\Rightarrow\quad f''(x) + f(x) = 0.$$

35. (a) $\displaystyle J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2},\ J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n x^{2n-1}}{2^{2n}(n!)^2}$, and $\displaystyle J_0''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n(2n-1)\,x^{2n-2}}{2^{2n}(n!)^2}$, so

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n(2n-1)\,x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n(2n-1)\,x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2}[(n-1)!]^2}$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1) + 2n - 2^2n^2}{2^{2n}(n!)^2}\right] x^{2n} = \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 - 2n + 2n - 4n^2}{2^{2n}(n!)^2}\right] x^{2n} = 0$$

(b) $\displaystyle\int_0^1 J_0(x)\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx$

$$= \left[x - \frac{x^3}{3\cdot 4} + \frac{x^5}{5\cdot 64} - \frac{x^7}{7\cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

36. (a) $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, $J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)\,x^{2n}}{n!\,(n+1)!\,2^{2n+1}}$, and

$$J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)\,x^{2n-1}}{n!\,(n+1)!\,2^{2n+1}}.$$

$$\begin{aligned}
&x^2 J_1''(x) + xJ_1'(x) + (x^2-1)\,J_1(x) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)\,x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)\,x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&\qquad + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)\,x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)\,x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&\qquad - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(n-1)!\,n!\,2^{2n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \qquad \left(\begin{array}{c}\text{Replace } n \text{ with } n-1 \\ \text{in the third term}\end{array}\right) \\
&= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) - (n)(n+1)\,2^2 - 1}{n!\,(n+1)!\,2^{2n+1}}\right] x^{2n+1} = 0
\end{aligned}$$

(b) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \quad \Rightarrow$

$$\begin{aligned}
J_0'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n)\,x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,2\,(n+1)\,x^{2n+1}}{2^{2n+2}\,[(n+1)!]^2} \quad \text{(Replace } n \text{ with } n+1\text{)} \\
&= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!\,n!} \quad \text{(cancel 2 and } n+1\text{; take } -1 \text{ outside sum)} \quad = -J_1(x)
\end{aligned}$$

37. (a) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 10.4.2 [ET 9.4.2], the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.
Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.

38. $\frac{|\sin nx|}{n^2} \le \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges by the Comparison Test. $\frac{d}{dx}\left(\frac{\sin nx}{n^2}\right) = \frac{\cos nx}{n}$, so when $x = 2k\pi$ (k an integer), $\sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos(2kn\pi)}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series). $f_n''(x) = -\sin nx$, so $\sum_{n=1}^{\infty} f_n''(x) = -\sum_{n=1}^{\infty} \sin nx$, which converges only if $\sin nx = 0$, or $x = k\pi$ (k an integer).

39. If $a_n = \dfrac{x^n}{n^2}$, then by the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x| \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R = 1$. When $x = \pm 1$, $\sum\limits_{n=1}^{\infty}\left|\dfrac{x^n}{n^2}\right| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$), so the interval of convergence for f is $[-1, 1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f(x) = \sum\limits_{n=1}^{\infty}\dfrac{x^n}{n^2} \Rightarrow f'(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n^2} = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n+1}$, and this series diverges for $x = 1$ (harmonic series) and converges for $x = -1$ (Alternating Series Test), so the interval of convergence is $[-1, 1)$. $f''(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1 \neq 0$, so its interval of convergence is $(-1, 1)$.

40. (a) $\sum\limits_{n=1}^{\infty} nx^{n-1} = \sum\limits_{n=0}^{\infty}\dfrac{d}{dx}x^n = \dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty} x_n\right] = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = -\dfrac{1}{(1-x)^2}(-1) = \dfrac{1}{(1-x)^2}$, $|x| < 1$.

(b) (i) $\sum\limits_{n=1}^{\infty} nx^n = x\sum\limits_{n=1}^{\infty} nx^{n-1} = x\left[\dfrac{1}{(1-x)^2}\right]$ [from (a)] $= \dfrac{x}{(1-x)^2}$ for $|x| < 1$.

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = \sum\limits_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \dfrac{1/2}{(1-1/2)^2} = 2$.

(c) (i) $\sum\limits_{n=2}^{\infty} n(n-1)x^n = x^2\sum\limits_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\dfrac{d}{dx}\left[\sum\limits_{n=1}^{\infty} nx^{n-1}\right] = x^2\dfrac{d}{dx}\dfrac{1}{(1-x)^2}$

$$= x^2\frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3} \text{ for } |x| < 1.$$

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=2}^{\infty}\dfrac{n^2-n}{2^n} = \sum\limits_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \dfrac{2(1/2)^2}{(1-1/2)^3} = 4$.

(iii) From (b)(ii) and (c)(ii), we have $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n} = \sum\limits_{n=1}^{\infty}\dfrac{n^2-n}{2^n} + \sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = 4 + 2 = 6$.

12.10 Taylor and Maclaurin Series ET 11.10

1. Using Theorem 5 with $\sum\limits_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $0.4 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\cdots$	$\cdots$	$\cdots$

$$\cos x = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = x^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x. \text{ So}$$

$R = \infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n\, 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^2\,|x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x, \text{ so } R = \infty$$

(Ratio Test).

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
$\cdots$	$\cdots$	$\cdots$

$$(1+x)^{-3} = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 - 3x + \tfrac{12}{2}x^2 - \tfrac{60}{6}x^3 + \tfrac{360}{24}x^4 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!\, x^n}{2\,(n!)} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)\, x^n}{2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+3)(n+2)\,|x|^{n+1}}{(n+2)(n+1)\,|x|} = |x| < 1 \text{ for}$$

convergence, so $R = 1$.

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\cdots$	$\cdots$	$\cdots$

$$\ln(1+x) = = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$
$$= x - \tfrac{1}{2}x^2 + \tfrac{2}{6}x^3 - \tfrac{6}{24}x^4 + \tfrac{24}{120}x^5 - \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{1+1/n} = |x| < 1 \text{ for}$$

convergence, so $R = 1$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\cdots$	$\cdots$	$\cdots$

So $f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$ and $\sinh x = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$. If $a_n = \dfrac{x^{2n+1}}{(2n+1)!}$ then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = x^2 \lim\limits_{n\to\infty} \dfrac{1}{(2n+3)(2n+2)} = 0 < 1$ for all x, so $R = \infty$.

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ so $\cosh x = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$ with $R = \infty$, by the Ratio Test.

9.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1 + x + x^2$	7
1	$1 + 2x$	5
2	2	2
3	0	0
4	0	0
$\cdots$	$\cdots$	$\cdots$

$$\begin{aligned} f(x) &= 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-2)^n \\ &= 7 + 5(x-2) + (x-2)^2 \end{aligned}$$

Since $a_n = 0$ for large n, $R = \infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
$\cdots$	$\cdots$	$\cdots$

$$\begin{aligned} f(x) &= -1 + 3(x+1) - \tfrac{6}{2}(x+1)^2 + \tfrac{6}{6}(x+1)^3 \\ &= -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3 \end{aligned}$$

Since $a_n = 0$ for large n, $R = \infty$.

11. Clearly, $f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{e^3}{n!}(x-3)^n$. If $a_n = \dfrac{e^3}{n!}(x-3)^n$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-3|}{n+1} = 0 \text{ for all } x, \text{ so } R = \infty.$$

12.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3 \cdot 2x^{-4}$	$-\frac{3 \cdot 2}{16}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(2) = \dfrac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so $\ln x = \ln 2 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}(x-2)^n}{n \cdot 2^n}$.

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x-2|}{2} \lim\limits_{n\to\infty} \dfrac{n}{n+1} = \dfrac{|x-2|}{2} < 1$ for convergence, so $|x-2| < 2 \quad \Rightarrow \quad R = 2$.

13.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-3 \cdot 2x^{-4}$	$-3 \cdot 2$
4	$4 \cdot 3 \cdot 2x^{-5}$	$4 \cdot 3 \cdot 2$
$\cdots$	$\cdots$	$\cdots$

So $f^{(n)}(1) = (-1)^n n!$, and $\dfrac{1}{x} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n n!}{n!}(x-1)^n = \sum\limits_{n=0}^{\infty}(-1)^n (x-1)^n$. If $a_n = (-1)^n (x-1)^n$ then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = |x-1| < 1$ for convergence, so $0 < x < 2$ and $R = 1$.

14.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2}$	2^{-2}
2	$-\frac{1}{4}x^{-3/2}$	-2^{-5}
3	$\frac{3}{8}x^{-5/2}$	$3 \cdot 2^{-8}$
4	$-\frac{15}{16}x^{-7/2}$	$-15 \cdot 2^{-11}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(4) = \dfrac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^{3n-1}}$ for $n \geq 2$, so

$\sqrt{x} = 2 + \dfrac{x-4}{4} + \sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^{3n-1} n!}(x-4)^n$.

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x-4|}{8} \lim\limits_{n\to\infty} \left(\dfrac{2n-1}{n+1}\right) = \dfrac{|x-4|}{4} < 1$ for convergence, so $|x-4| < 4 \quad \Rightarrow \quad R = 4$.

15.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\sin x$	$\sqrt{2}/2$
1	$\cos x$	$\sqrt{2}/2$
2	$-\sin x$	$-\sqrt{2}/2$
3	$-\cos x$	$-\sqrt{2}/2$
4	$\sin x$	$\sqrt{2}/2$
$\cdots$	$\cdots$	$\cdots$

$$\begin{aligned}\sin x &= f\left(\tfrac{\pi}{4}\right) + f'\left(\tfrac{\pi}{4}\right)\left(x-\tfrac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\tfrac{\pi}{4}\right)^2 \\ &\quad + \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x-\tfrac{\pi}{4}\right)^4 + \cdots \\ &= \tfrac{\sqrt{2}}{2}\left[1 + \left(x-\tfrac{\pi}{4}\right) - \tfrac{1}{2!}\left(x-\tfrac{\pi}{4}\right)^2 \right. \\ &\qquad \left. - \tfrac{1}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \tfrac{1}{4!}\left(x-\tfrac{\pi}{4}\right)^4 + \cdots\right] \\ &= \tfrac{\sqrt{2}}{2}\left[1 - \tfrac{1}{2!}\left(x-\tfrac{\pi}{4}\right)^2 + \tfrac{1}{4!}\left(x-\tfrac{\pi}{4}\right)^4 - \cdots\right] \\ &\quad + \tfrac{\sqrt{2}}{2}\left[\left(x-\tfrac{\pi}{4}\right) - \tfrac{1}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \cdots\right] \\ &= \tfrac{\sqrt{2}}{2}\textstyle\sum_{n=0}^{\infty}(-1)^n\left[\tfrac{1}{(2n)!}\left(x-\tfrac{\pi}{4}\right)^{2n} + \tfrac{1}{(2n+1)!}\left(x-\tfrac{\pi}{4}\right)^{2n+1}\right]\end{aligned}$$

The series can also be written in the more elegant form $\sin x = \dfrac{\sqrt{2}}{2}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2}\left(x-\frac{\pi}{4}\right)^n}{n!}$. If $a_n = \dfrac{(-1)^{n(n-1)/2}\left(x-\frac{\pi}{4}\right)^n}{n!}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|x-\frac{\pi}{4}\right|}{n+1} = 0 < 1$ for all x, so $R = \infty$.

16.

n	$f^{(n)}(x)$	$f^{(n)}\left(-\frac{\pi}{4}\right)$
0	$\cos x$	$\frac{\sqrt{2}}{2}$
1	$-\sin x$	$\frac{\sqrt{2}}{2}$
2	$-\cos x$	$-\frac{\sqrt{2}}{2}$
3	$\sin x$	$-\frac{\sqrt{2}}{2}$
4	$\cos x$	$\frac{\sqrt{2}}{2}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}\left(-\frac{\pi}{4}\right) = (-1)^{n(n-1)/2}\,\frac{\sqrt{2}}{2}$, so

$$\cos x = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(-\frac{\pi}{4}\right)}{n!}\left(x+\tfrac{\pi}{4}\right)^n = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2}\left(x+\frac{\pi}{4}\right)^n}{n!}$$

with $R = \infty$ by the Ratio Test (as in Exercise 15).

17. If $f(x) = \cos x$, then by Formula 9 with $a = 0$, $|R_n(x)| \le \dfrac{\left|f^{(n+1)}(x)\right|}{(n+1)!}|x|^{n+1}$. But $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so $|R_n(x)| \le \dfrac{1}{(n+1)!}|x|^{n+1} \to 0$ as $n \to \infty$ by Equation 10. So $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 3 represents $\cos x$ for all x.

18. If $f(x) = \sin x$, then $|R_n(x)| \le \dfrac{\left|f^{(n+1)}(x)\right|}{(n+1)!}\left|x-\frac{\pi}{4}\right|^{n+1}$. But $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so $|R_n(x)| \le \dfrac{1}{(n+1)!}\left|x-\frac{\pi}{4}\right|^{n+1} \to 0$ as $n \to \infty$ by Equation 10. So by Theorem 8, the series in Exercise 15 represents $\sin x$ for all x.

19. If $f(x) = \sinh x$, then $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$, where $0 < |z| < |x|$. But for all n, $\left|f^{(n+1)}(z)\right| \le \cosh z \le \cosh x$ (since all derivatives are either sinh or cosh, $|\sinh z| < |\cosh z|$ for all z, and $|z| < |x| \Rightarrow \cosh z < \cosh x$), so $|R_n(z)| \le \dfrac{\cosh x}{(n+1)!}x^{n+1} \to 0$ as $n \to \infty$ (by Equation 10). So by Theorem 8, the series represents $\sinh x$ for all x.

20. If $f(x) = \cosh x$, then $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$, where $0 < |z| < |x|$. But for all n,

$\left|f^{(n+1)}(z)\right| \leq \cosh z \leq \cosh x$ (since all derivatives are either sinh or cosh, $|\sinh z| < |\cosh z|$ for all z, and

$|z| < |x| \Rightarrow \cosh z < \cosh x$), so $|R_n(z)| \leq \dfrac{\cosh x}{(n+1)!}x^{n+1} \to 0$ as $n \to \infty$ (by Equation 10). So by

Theorem 8, the series represents $\cosh x$ for all x.

21. $\cos x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(\pi x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}$, $R = \infty$

22. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \Rightarrow f(x) = e^{-x/2} = \sum\limits_{n=0}^{\infty} \dfrac{(-x/2)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{2^n\, n!}x^n$, $R = \infty$

23. $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} \Rightarrow f(x) = x\tan^{-1} x = x\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+2}}{2n+1}$, $R = 1$

24. $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} \Rightarrow f(x) = \sin(x^4) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!}x^{8n+4}$, $R = \infty$

25. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \Rightarrow f(x) = x^2 e^{-x} = x^2 \sum\limits_{n=0}^{\infty} \dfrac{(-x)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{n+2}}{n!}$, $R = \infty$

26. $\cos x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \Rightarrow \cos 2x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(2x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n)!}x^{2n} \Rightarrow$

$f(x) = x\cos 2x = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n)!}x^{2n+1}$, $R = \infty$

27. $\sin^2 x = \frac{1}{2}[1 - \cos 2x] = \dfrac{1}{2}\left[1 - \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!}\right] = 2^{-1}\left[1 - 1 - \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!}\right]$

$= \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$, $R = \infty$

28. $\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \dfrac{1}{2}\left[1 + \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!}\right] = \dfrac{1}{2}\left[1 + 1 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 2^{2n} x^{2n}}{(2n)!}\right]$

$= 1 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$, $R = \infty$

Another Method: Use $\cos^2 x = 1 - \sin^2 x$ and Exercise 27.

29. $\dfrac{\sin x}{x} = \dfrac{1}{x}\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n+1)!}$ and this series also gives the required value at $x = 0$ (namely 1), so $R = \infty$.

30. $\dfrac{1 - \cos x}{x^2} = x^{-2}\left[1 - \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}\right] = x^{-2}\left[-\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}\right] = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^{2n-2}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n+2)!}$,

since the series is equal to $\frac{1}{2}$ when $x = 0$; $R = \infty$.

31.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$
...	...	...

So $f^{(n)}(0) = \dfrac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n}$ for $n \geq 2$,

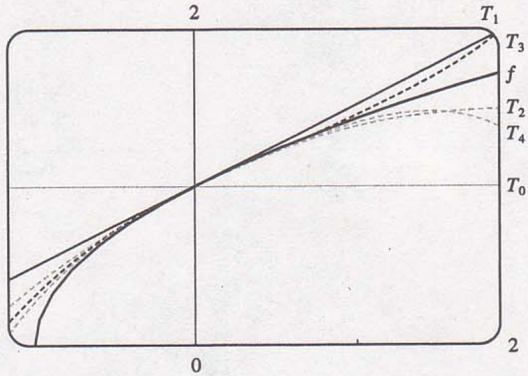
and

$$\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n.$$

If $a_n = \dfrac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n$, then

$\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{2} \lim_{n\to\infty} \frac{2n-1}{n+1} = |x| < 1$ for

convergence, so $R = 1$.



32.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+2x)^{-1/2}$	1
1	$-\frac{1}{2}(1+2x)^{-3/2}(2)$	-1
2	$\frac{3}{2}(1+2x)^{-5/2}(2)$	3
3	$-3 \cdot \frac{5}{2}(1+2x)^{-7/2}(2)$	$-3 \cdot 5$
...	...	...

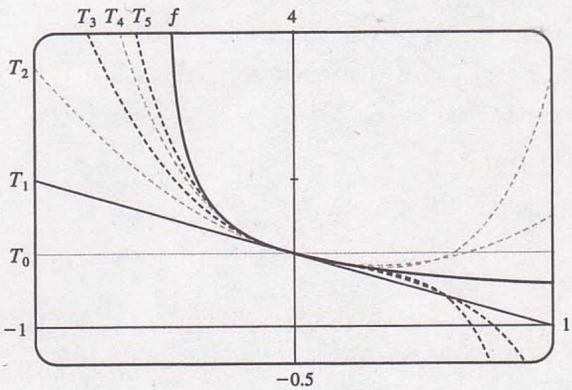
$f^{(n)}(0) = (-1)^n\, 1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)$, so

$$\begin{aligned}(1+2x)^{-1/2} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!} x^n\end{aligned}$$

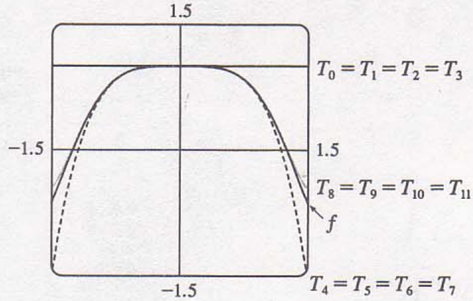
$\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2n+1}{n+1} |x| = 2|x| < 1$ for

convergence, so $R = \frac{1}{2}$.

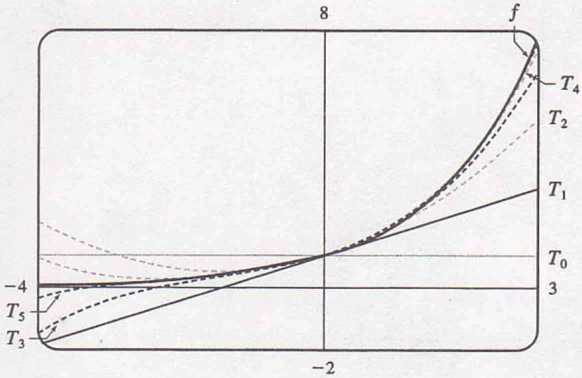
Another method: Use Exercise 31 and differentiate.



33. $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n}}{(2n)!}$, $R = \infty$



34. $2^x = \left(e^{\ln 2}\right)^x = e^{x \ln 2} = \sum_{n=0}^{\infty} \dfrac{(x \ln 2)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{(\ln 2)^n x^n}{n!}$, $R = \infty$.



35. $\ln(1+x) = \displaystyle\int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ with $C = 0$ and $R = 1$, so $\ln(1.1) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (0.1)^n}{n}$. This is an alternating series with $b_5 = \dfrac{(0.1)^5}{5} = 0.000002$, so to five decimal places, $\ln(1.1) \approx \displaystyle\sum_{n=1}^{4} \frac{(-1)^{n-1} (0.1)^n}{n} \approx 0.09531$.

36. $3^\circ = \frac{\pi}{60}$ radians and $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, so

$\sin \frac{\pi}{60} = \frac{\pi}{60} - \dfrac{\left(\frac{\pi}{60}\right)^3}{3!} + \dfrac{\left(\frac{\pi}{60}\right)^5}{5!} - \cdots = \dfrac{\pi}{60} - \dfrac{\pi^3}{1{,}296{,}000} + \dfrac{\pi^5}{93{,}312{,}000{,}000} - \cdots$. But $\frac{\pi^5}{93{,}312{,}000{,}000} < 10^{-8}$, so by the Alternating Series Estimation Theorem, $\sin \frac{\pi}{60} \approx \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} \approx 0.05234$.

37. $\displaystyle\int \sin(x^2)\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$

38. $\dfrac{\sin x}{x} = \dfrac{1}{x} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$, so

$\displaystyle\int \frac{\sin x}{x} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$, with $R = \infty$.

39. Using the series from Exercise 31 and substituting x^3 for x, we get

$$\int \sqrt{x^3+1}\,dx = \int \left[1+\frac{x^3}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n n!}x^{3n}\right]dx$$
$$= C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n n!\,(3n+1)}x^{3n+1}$$

40. $\displaystyle\int e^{x^3}\,dx = \int \sum_{n=0}^{\infty}\frac{(x^3)^n}{n!}\,dx = C + \sum_{n=0}^{\infty}\frac{x^{3n+1}}{(3n+1)\,n!}$, with $R=\infty$.

41. Using our series from Exercise 37, we get

$\displaystyle\int_0^1 \sin(x^2)\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n\,x^{4n+3}}{(4n+3)\,(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)\,(2n+1)!}$ and $|c_3| = \dfrac{1}{75{,}600} < 0.000014$, so by the Alternating Series Estimation Theorem, we have $\displaystyle\sum_{n=0}^{2}\frac{(-1)^n}{(4n+3)\,(2n+1)!} = \frac{1}{3}-\frac{1}{42}+\frac{1}{1320} \approx 0.310$ (correct to three decimal places).

42. $\displaystyle\cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n\,(x^2)^{2n}}{(2n)!}$, so

$\displaystyle\int_0^{0.5}\cos(x^2)\,dx = \int_0^{0.5}\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{4n}}{(2n)!}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n\,x^{4n+1}}{(4n+1)\,(2n)!}\right]_0^{0.5} = 0.5 - \frac{(0.5)^5}{5\cdot 2!} + \frac{(0.5)^9}{9\cdot 4!} - \cdots$, but $\dfrac{(0.5)^9}{9\cdot 4!} \approx 0.000009$, so by the Alternating Series Estimation Theorem, $\int_0^{0.5}\cos(x^2)\,dx \approx 0.5 - \dfrac{(0.5)^5}{5\cdot 2!} \approx 0.497$ (correct to three decimal places).

43. We first find a series representation for $f(x) = (1+x)^{-1/2}$, and then substitute.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
$\cdots$	$\cdots$	$\cdots$

$\displaystyle\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{4}\left(\frac{x^2}{2!}\right) - \frac{15}{8}\left(\frac{x^3}{3!}\right) + \cdots \;\Rightarrow\; \frac{1}{\sqrt{1+x^3}} = 1 - \tfrac{1}{2}x^3 + \tfrac{3}{8}x^6 - \tfrac{5}{16}x^9 + \cdots \;\Rightarrow$

$\displaystyle\int_0^{0.1}\frac{dx}{\sqrt{1+x^3}} = \left[x - \frac{1}{8}x^4 + \frac{3}{56}x^7 - \frac{1}{32}x^{10} + \cdots\right]_0^{0.1} \approx (0.1) - \frac{1}{8}(0.1)^4$, by the Alternating Series Estimation Theorem, since $\frac{3}{56}(0.1)^7 \approx 0.0000000054 < 10^{-8}$, which is the maximum desired error. Therefore, $\displaystyle\int_0^{0.1}\frac{dx}{\sqrt{1+x^3}} \approx 0.09998750$.

44. $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+3}}{n!\,(2n+3)}\right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+3)\,2^{2n+3}}$ and since $c_2 = \dfrac{1}{1792} < 0.001$ we use $\displaystyle\sum_{n=0}^{1} \frac{(-1)^n}{n!\,(2n+3)\,2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354.$

45. $$\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x\to 0} \frac{x - \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3}$$
$$= \lim_{x\to 0} \left(\tfrac{1}{3} - \tfrac{1}{5}x^2 + \tfrac{1}{7}x^4 - \cdots\right) = \tfrac{1}{3}$$
since power series are continuous functions.

46. $$\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)}{1 + x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right)}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots}{-\frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 - \cdots}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^4 - \cdots}{-\frac{1}{2!} - \frac{1}{3!}x - \frac{1}{4!}x^2 - \frac{1}{5!}x^3 - \frac{1}{6!}x^4 - \cdots} = \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1$$
since power series are continuous functions.

47. $$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x + \frac{1}{6}x^3}{x^5}$$
$$= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots\right) = \frac{1}{5!} = \frac{1}{120}$$
since power series are continuous functions.

48. $$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\right) - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \lim_{x\to 0} \left(\tfrac{1}{3} + \tfrac{2}{15}x^2 + \cdots\right) = \tfrac{1}{3}$$
since power series are continuous functions.

49. As in Example 8(a), we have $e^{-x^2} = 1 - \dfrac{x^2}{1!} + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \cdots$ and we know that $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x = \left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right)$. Writing only the terms with degree ≤ 4, we get $e^{-x^2}\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^4 + \cdots = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$.

50.

$$\begin{array}{r|l} & 1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots \\ \hline 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots & 1 \\ & 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots \\ \hline & \frac{1}{2}x^2-\frac{1}{24}x^4+\cdots \\ & \frac{1}{2}x^2-\frac{1}{4}x^4+\cdots \\ \hline & \frac{5}{24}x^4+\cdots \\ & \frac{5}{24}x^4+\cdots \\ \hline & \cdots \end{array}$$

$\sec x = \dfrac{1}{\cos x} = \dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots}$.

From the long division above,

$\sec x = 1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$.

51.

$$\begin{array}{r|l} & -x+\frac{1}{2}x^2-\frac{1}{3}x^3+\cdots \\ \hline 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots & -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots \\ & -x-x^2-\frac{1}{2}x^3-\cdots \\ \hline & \frac{1}{2}x^2+\frac{1}{6}x^3-\cdots \\ & \frac{1}{2}x^2+\frac{1}{2}x^3+\cdots \\ \hline & -\frac{1}{3}x^3+\cdots \\ & -\frac{1}{3}x^3+\cdots \\ \hline & \cdots \end{array}$$

From Example 6 in Section 12.9 [ET 11.9], we have $\ln(1-x) = -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$. Therefore,

$y = \dfrac{\ln(1-x)}{e^x} = \dfrac{-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots}{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots}$. So by the long division above,

$\dfrac{\ln(1-x)}{e^x} = -x+\dfrac{x^2}{2}-\dfrac{x^3}{3}+\cdots$, $|x|<1$.

52. From Example 6 in Section 12.9 [ET 11.9], we have $\ln(1-x) = -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$. Therefore,

$$\begin{aligned} e^x\ln(1-x) &= \left(1+x+\tfrac{1}{2}x^2+\cdots\right)\left(-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3+\cdots\right) \\ &= -x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-x^2-\tfrac{1}{2}x^3-\tfrac{1}{2}x^3-\cdots \\ &= -x-\tfrac{3}{2}x^2-\tfrac{4}{3}x^3-\cdots,\ |x|<1 \end{aligned}$$

53. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!} = \sum_{n=0}^{\infty}\frac{(-x^4)^n}{n!} = e^{-x^4}$, by (11).

54. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{2n!} = \cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2}$, by (16).

55. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}$, by (15).

56. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{5^n\,n!} = \sum_{n=0}^{\infty}\frac{(3/5)^n}{n!} = e^{3/5}$, by (11).

57. $\displaystyle 3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots = \frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots = \sum_{n=1}^{\infty}\frac{3^n}{n!} = \sum_{n=0}^{\infty}\frac{3^n}{n!}-1 = e^3-1$, by (11).

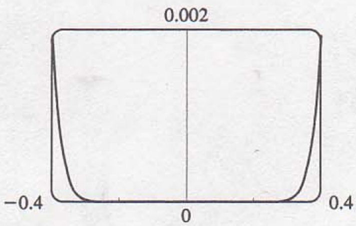
58. $\displaystyle 1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 3)^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{(-\ln 2)^n}{n!} = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \tfrac{1}{2}$, by (11).

59. Assume that $|f'''(x)| \leq M$, so $f'''(x) \leq M$ for $a \leq x \leq d$. Now $\int_a^x f'''(t)\,dt \leq \int_a^x M\,dt$
$\Rightarrow \quad f''(x) - f''(a) \leq M(x-a) \quad \Rightarrow \quad f''(x) \leq f''(a) + M(x-a)$. Thus,
$\int_a^x f''(t)\,dt \leq \int_a^x [f''(a) + M(t-a)]\,dt \quad \Rightarrow \quad f'(x) - f'(a) \leq f''(a)(x-a) + \frac{1}{2}M(x-a)^2$
$\Rightarrow \quad f'(x) \leq f'(a) + f''(a)(x-a) + \frac{1}{2}M(x-a)^2 \quad \Rightarrow$
$\int_a^x f'(t)\,dt \leq \int_a^x \left[f'(a) + f''(a)(t-a) + \frac{1}{2}M(t-a)^2\right]dt$
$\Rightarrow \quad f(x) - f(a) \leq f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}M(x-a)^3$. So
$f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2 \leq \frac{1}{6}M(x-a)^3$. But
$R_2(x) = f(x) - T_2(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2$, so $R_2(x) \leq \frac{1}{6}M(x-a)^3$. A similar argument using $f'''(x) \geq -M$ shows that $R_2(x) \geq -\frac{1}{6}M(x-a)^3$. So $|R_2(x_2)| \leq \frac{1}{6}M|x-a|^3$. Although we have assumed that $x > a$, a similar calculation shows that this inequality is also true if $x < a$.

60. (a) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ so

$f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{e^{-1/x^2}}{x} = \lim\limits_{x\to 0} \dfrac{1/x}{e^{1/x^2}} = \lim\limits_{x\to 0} \dfrac{x}{2e^{1/x^2}} = 0$ (using l'Hospital's Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule to show that $f''(0) = 0$, $f^{(3)}(0) = 0, \ldots, f^{(n)}(0) = 0$, so that the Maclaurin series for f consists entirely of zero terms. But since $f(x) \neq 0$ except for $x = 0$, we see that f cannot equal its Maclaurin series except at $x = 0$.

(b)



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be "infinitely flat" at $x = 0$, since all of its derivatives are 0 there.

12.11 The Binomial Series ET 11.11

1. The general binomial series in (2) is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots.$$

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1\cdot 3\cdot x^3}{2^3\cdot 3!} - \frac{1\cdot 3\cdot 5\cdot x^4}{2^4\cdot 4!} + \cdots \\ &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)\,x^n}{2^n\cdot n!},\ R = 1\end{aligned}$$

2. $\dfrac{1}{(1+x)^4} = (1+x)^{-4} = \sum\limits_{n=0}^{\infty} \dbinom{-4}{n} x^n$. The binomial coefficient is

$$\binom{-4}{n} = \frac{(-4)(-5)(-6)\cdots(-4-n+1)}{n!} = \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots \cdot (n+1)(n+2)(n+3)}{2 \cdot 3 \cdot n!}$$
$$= \frac{(-1)^n (n+1)(n+2)(n+3)}{6}$$

so $\dfrac{1}{(1+x)^4} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n$ for $|x| < 1$, so $R = 1$.

3. $\dfrac{1}{(2+x)^3} = \dfrac{1}{[2(1+x/2)]^3} = \dfrac{1}{8}\left(1+\dfrac{x}{2}\right)^{-3} = \dfrac{1}{8} \sum\limits_{n=0}^{\infty} \dbinom{-3}{n} \left(\dfrac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n} = \frac{(-3)(-4)(-5)\cdot \cdots \cdot(-3-n+1)}{n!} = \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (n+1)(n+2)}{2 \cdot n!}$$
$$= \frac{(-1)^n (n+1)(n+2)}{2}$$

so $\dfrac{1}{(2+x)^3} = \dfrac{1}{8} \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)}{2} \dfrac{x^n}{2^n} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)\, x^n}{2^{n+4}}$ for $\left|\dfrac{x}{2}\right| < 1 \;\Leftrightarrow\; |x| < 2$,
so $R = 2$.

4. $(1+x^2)^{1/3} = \sum\limits_{n=0}^{\infty} \dbinom{\frac{1}{3}}{n} x^{2n} = 1 + \dfrac{x^2}{3} + \dfrac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} x^4 + \dfrac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} x^6 + \cdots$

$$= 1 + \frac{x^2}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-4)\, x^{2n}}{3^n\, n!}, \text{ with } R = 1.$$

5. $\sqrt[4]{1-8x} = (1-8x)^{1/4} = \sum\limits_{n=0}^{\infty} \dbinom{\frac{1}{4}}{n} (-8x)^n$

$$= 1 + \tfrac{1}{4}(-8x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-8x)^2 + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-8x)^3 + \cdots$$
$$= 1 - 2x + \sum_{n=2}^{\infty} \frac{(-1)^n (-1)^{n-1} \cdot 3 \cdot 7 \cdot \cdots \cdot (4n-5)\, 8^n}{4^n \cdot n!} x^n$$
$$= 1 - 2x - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)\, 2^n}{n!} x^n$$

and $|-8x| < 1 \;\Leftrightarrow\; |x| < \frac{1}{8}$, so $R = \frac{1}{8}$.

6. $\dfrac{1}{\sqrt[5]{32-x}} = \dfrac{1}{2\sqrt[5]{1-x/32}} = \dfrac{1}{2}\left(1-\dfrac{x}{32}\right)^{-1/5} = \dfrac{1}{2} \sum\limits_{n=0}^{\infty} \dbinom{-\frac{1}{5}}{n} \left(-\dfrac{x}{32}\right)^n = \dfrac{1}{2} \sum\limits_{n=0}^{\infty} \dbinom{-\frac{1}{5}}{n} \dfrac{(-1)^n x^n}{2^{5n}}$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{5}\right)\left(-\frac{x}{2^5}\right) + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{2!}\frac{x^2}{2^{10}} + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)}{3!}\left(-\frac{x^3}{2^{15}}\right) + \cdots\right]$$
$$= \frac{1}{2} + \frac{1}{5 \cdot 2^6} x + \frac{1 \cdot 6}{5^2 \cdot 2! \cdot 2^{11}} x^2 + \frac{1 \cdot 6 \cdot 11}{5^3 \cdot 3! \cdot 2^{16}} x^3 + \cdots = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 6 \cdot \cdots \cdot (5n-4)}{5^n 2^{5n+1}\, n!} x^n$$

The radius of convergence is 32.

7. $\dfrac{x}{\sqrt{4+x^2}} = \dfrac{x}{2\sqrt{1+x^2/4}} = \dfrac{x}{2}\left(1+\dfrac{x^2}{4}\right)^{-1/2} = \dfrac{x}{2}\sum\limits_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\dfrac{x^2}{4}\right)^n$

$$= \frac{x}{2}\left[1+\left(-\tfrac{1}{2}\right)\frac{x^2}{4}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3+\cdots\right]$$

$$= \frac{x}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4}<1 \;\Leftrightarrow\; \frac{|x|}{2}<1 \;\Leftrightarrow\; |x|<2, \text{ so } R=2.$$

8. $\dfrac{x^2}{\sqrt{2+x}} = \dfrac{x^2}{\sqrt{2\,(1+x/2)}} = \dfrac{x^2}{\sqrt{2}}\left(1+\dfrac{x}{2}\right)^{-1/2} = \dfrac{x^2}{\sqrt{2}}\sum\limits_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\dfrac{x}{2}\right)^n$

$$= \frac{x^2}{\sqrt{2}}\left[1+\left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$= \frac{x^2}{\sqrt{2}}+\frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n}}x^n$$

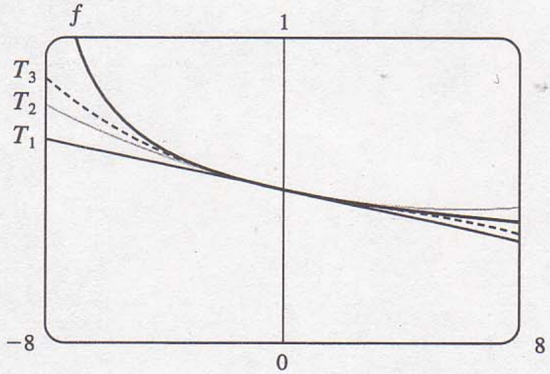
$$= \frac{x^2}{\sqrt{2}}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right|<1 \;\Leftrightarrow\; |x|<2, \text{ so } R=2.$$

9. $\dfrac{1}{\sqrt[3]{8+x}} = (8+x)^{-1/3} = 8^{-1/3}\left(1+\dfrac{x}{8}\right)^{-1/3} = \tfrac{1}{2}\left(1+\dfrac{x}{8}\right)^{-1/3}$

$$= \tfrac{1}{2}\left[1+\left(-\tfrac{1}{3}\right)\left(\frac{x}{8}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\cdots\right]$$

$$= \frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{3^n\cdot n!\,8^n}x^n\right] \text{ and } \left|\frac{x}{8}\right|<1 \;\Leftrightarrow\; |x|<8, \text{ so } R=8.$$

The three Taylor polynomials are $T_1(x)=\frac{1}{2}-\frac{1}{48}x$, $T_2(x)=\frac{1}{2}-\frac{1}{48}x+\frac{1}{576}x^2$, and $T_3(x)=\frac{1}{2}-\frac{1}{48}x+\frac{1}{576}x^2-\frac{4\cdot 7}{2\cdot 27\cdot 6\cdot 512}x^3=\frac{1}{2}-\frac{1}{48}x+\frac{1}{576}x^2-\frac{7}{41{,}472}x^3$.

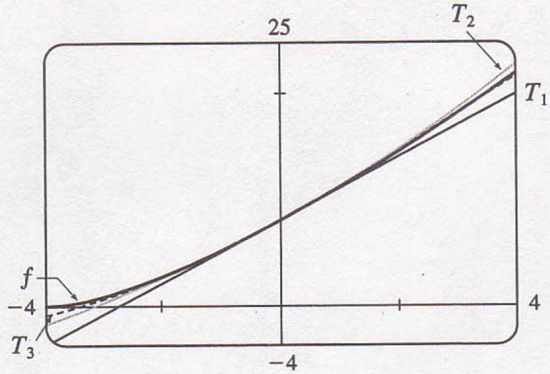


10. $(4+x)^{3/2} = 8\left(1+\frac{x}{4}\right)^{3/2} = 8\sum_{n=0}^{\infty}\binom{\frac{3}{2}}{n}\left(\frac{x}{4}\right)^n$

$$= 8\left[1+\tfrac{3}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}\left(\frac{x}{4}\right)^3+\cdots\right]$$

$$= 8+3x+\sum_{n=2}^{\infty}\frac{(3)(1)(-1)\cdot\cdots\cdot(5-2n)\,x^n}{8^{n-1}\cdot n!} \text{ and } \left|\frac{x}{4}\right|<1 \quad\Leftrightarrow\quad |x|<4, \text{ so } R=4.$$

The three Taylor polynomials are $T_1(x) = 8+3x$, $T_2(x) = 8+3x+\frac{3}{16}x^2$, and $T_3(x) = 8+3x+\frac{3}{16}x^2-\frac{1}{128}x^3$.



11. (a) $\left[1+(-x^2)\right]^{-1/2} = 1+\left(-\frac{1}{2}\right)(-x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3+\cdots$

$$= 1+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^{2n}$$

(b) $\sin^{-1}x = \int\frac{1}{\sqrt{1-x^2}}\,dx = C+x+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{(2n+1)\,2^n\cdot n!}x^{2n+1}$

$$= x+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{(2n+1)\,2^n\cdot n!}x^{2n+1} \text{ since } 0=\sin^{-1}0=C.$$

12. (a) $\left(1+x^2\right)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}x^{2n} = 1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\,x^{2n}}{2^n\cdot n!}$

(b) $\sinh^{-1}x = \int\frac{dx}{\sqrt{1+x^2}} = C+x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\,x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, but $C=0$ since

$\sinh^{-1}0=0$, so $\sinh^{-1}x = x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\,x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, $R=1$.

13. (a) $(1+x)^{-1/2} = 1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3+\cdots$

$$= 1+\sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^n$$

(b) Take $x=0.1$ in the above series. $\frac{1\cdot3\cdot5\cdot7}{2^4 4!}(0.1)^4 < 0.00003$, so

$$\frac{1}{\sqrt{1.1}} \approx 1-\frac{0.1}{2}+\frac{1\cdot3}{2^2\cdot2!}(0.1)^2-\frac{1\cdot3\cdot5}{2^3\cdot3!}(0.1)^3 \approx 0.953.$$

14. (a) $(8+x)^{1/3} = 2\left(1+\frac{x}{8}\right)^{1/3} = 2\sum\limits_{n=0}^{\infty}\binom{\frac{1}{3}}{n}\left(\frac{x}{8}\right)^n$

$$= 2\left[1+\tfrac{1}{3}\left(\frac{x}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3+\cdots\right]$$

$$= 2\left[1+\frac{x}{24}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot 2\cdot 5\cdot\cdots\cdot(3n-4)\,x^n}{24^n\cdot n!}\right]$$

(b) $(8+0.2)^{1/3} = 2\left[1+\frac{0.2}{24}-\frac{(0.2)^2}{24^2}+\frac{2\cdot 5\,(0.2)^3}{24^3\cdot 3!}-\cdots\right] \approx 2\left[1+\frac{0.2}{24}-\frac{(0.2)^2}{24^2}\right]$ since

$2\cdot\frac{2\cdot 5\,(0.2)^3}{24^3\cdot 3!} \approx 0.000002$, so $\sqrt[3]{8.2} \approx 2.0165$.

15. (a) $[1+(-x)]^{-2} = 1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots$

$$= 1+2x+3x^2+4x^3+\cdots = \sum_{n=0}^{\infty}(n+1)\,x^n,$$

so $\frac{x}{(1-x)^2} = \sum\limits_{n=0}^{\infty}(n+1)\,x^{n+1} = \sum\limits_{n=1}^{\infty}nx^n$.

(b) With $x=\frac{1}{2}$ in part (a), we have $\sum\limits_{n=1}^{\infty}\frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$.

16. (a) $[1+(-x)]^{-3} = \sum\limits_{n=0}^{\infty}\binom{-3}{n}(-x)^n$

$$= 1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^2+\frac{(-3)(-4)(-5)}{3!}(-x)^3+\cdots$$

$$= 1+\sum_{n=1}^{\infty}\frac{3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{n!}x^n = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n \quad\Rightarrow$$

$$(x+x^2)\,[1+(-x)]^{-3} = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+1}+\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+2}$$

$$= x+\sum_{n=2}^{\infty}\left[\frac{n(n+1)}{2}+\frac{(n-1)\,n}{2}\right]x^n = x+\sum_{n=2}^{\infty}n^2x^n = \sum_{n=1}^{\infty}n^2x^n,\ -1<x<1$$

(b) Setting $x=\frac{1}{2}$ in the last series above gives the required series, so $\sum\limits_{n=1}^{\infty}\frac{n^2}{2^n} = \frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} = 6$.

17. (a) $\left(1+x^2\right)^{1/2} = 1+\left(\frac{1}{2}\right)x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^2\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x^2\right)^3+\cdots$

$$= 1+\frac{x^2}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^{2n}$$

(b) The coefficient of x^{10} (corresponding to $n=5$) in the above Maclaurin series is $\frac{f^{(10)}(0)}{10!}$, so

$\frac{f^{(10)}(0)}{10!} = \frac{(-1)^4\cdot 1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!} \quad\Rightarrow\quad f^{(10)}(0) = 10!\left(\frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\right) = 99{,}225$.

18. (a) $$\left(1+x^3\right)^{-1/2} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(x^3\right)^n$$
$$= 1 + \left(-\tfrac{1}{2}\right)\left(x^3\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(x^3\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(x^3\right)^3 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^{3n}}{2^n\cdot n!}$$

(b) The coefficient of x^9 in the preceding series is $\dfrac{f^{(9)}(0)}{9!}$, so $\dfrac{f^{(9)}(0)}{9!} = \dfrac{(-1)^3\,1\cdot 3\cdot 5}{2^3\cdot 3!}$ $\Rightarrow$

$f^{(9)}(0) = -\dfrac{9!\cdot 5}{8\cdot 2} = -113{,}400.$

19. (a) $g(x) = \sum_{n=0}^{\infty}\binom{k}{n}x^n \;\Rightarrow\; g'(x) = \sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1}$, so

$$\begin{aligned}(1+x)\,g'(x) &= (1+x)\sum_{n=1}^{\infty}\tbinom{k}{n}nx^{n-1} = \sum_{n=1}^{\infty}\tbinom{k}{n}nx^{n-1} + \sum_{n=1}^{\infty}\tbinom{k}{n}nx^n\\
&= \sum_{n=0}^{\infty}\tbinom{k}{n+1}(n+1)\,x^n + \sum_{n=0}^{\infty}\tbinom{k}{n}nx^n \qquad \left[\begin{array}{c}\text{Replace } n \text{ with } n+1\\ \text{in the first series}\end{array}\right]\\
&= \sum_{n=0}^{\infty}(n+1)\,\frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!}x^n\\
&\qquad + \sum_{n=0}^{\infty}\left[(n)\,\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\right]x^n\\
&= \sum_{n=0}^{\infty}\frac{(n+1)\,k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!}\left[(k-n)+n\right]x^n\\
&= k\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n = k\sum_{n=0}^{\infty}\tbinom{k}{n}x^n = kg(x)\end{aligned}$$

Thus, $g'(x) = \dfrac{kg(x)}{1+x}$.

(b) $h(x) = (1+x)^{-k}\,g(x) \;\Rightarrow$

$$\begin{aligned}h'(x) &= -k(1+x)^{-k-1}g(x) + (1+x)^{-k}g'(x) \quad \text{[Product Rule]}\\
&= -k(1+x)^{-k-1}g(x) + (1+x)^{-k}\frac{kg(x)}{1+x} \quad \text{[from part (a)]}\\
&= -k(1+x)^{-k-1}g(x) + k(1+x)^{-k-1}g(x) = 0\end{aligned}$$

(c) From part (b) we see that $h(x)$ must be constant for $x\in(-1,1)$, so $h(x) = h(0) = 1$ for $x\in(-1,1)$. Thus, $h(x) = 1 = (1+x)^{-k}g(x) \;\Leftrightarrow\; g(x) = (1+x)^k$ for $x\in(-1,1)$.

20. (a)

$$4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(-k^2\sin^2 x\right)\right]^{-1/2}dx$$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1-\frac{1}{2}\left(-k^2\sin^2 x\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(-k^2\sin^2 x\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(-k^2\sin^2 x\right)^3+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(\frac{1}{2}\right)k^2\sin^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x+\cdots\right]dx$$

(split up the integral and use the result from Exercise 8.1.40 [ET 7.1.40])

$$=4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4\right.$$
$$\left.+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6+\cdots\right]$$

$$=2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

$$T=2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8+\cdots\right]$$
$$\leq 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{1}{4}k^4+\frac{1}{4}k^6+\frac{1}{4}k^8+\cdots\right]$$

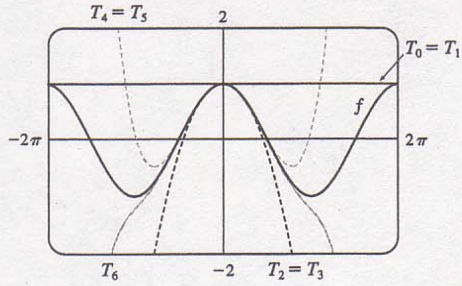
The terms in brackets (after the first) form a geometric series with $a=\frac{1}{4}k^2$ and $r=k^2=\sin^2\left(\frac{1}{2}\theta_0\right)<1$. So $T\leq 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{k^2/4}{1-k^2}\right]=2\pi\sqrt{\frac{L}{g}}\frac{4-3k^2}{4-4k^2}$.

(c) We substitute $L=1$, $g=9.8$, and $k=\sin\left(10^\circ/2\right)\approx 0.08716$, and the inequality from part (b) becomes $2.01090\leq T\leq 2.01093$, so $T\approx 2.0109$. The estimate $T\approx 2\pi\sqrt{L/g}\approx 2.0071$ differs by about 0.2%. If $\theta_0=42^\circ$, then $k\approx 0.35837$ and the inequality becomes $2.07153\leq T\leq 2.08103$, so $T\approx 2.0763$. The one-term estimate is the same, and the discrepancy between the two estimates increases to about 3.7%.

12.12 Applications of Taylor Polynomials ET 11.12

1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



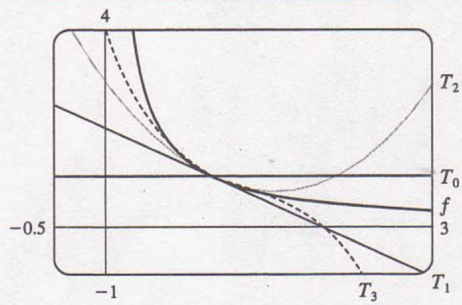
(b)

x	f	$T_0=T_1$	$T_2=T_3$	$T_4=T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$T_n(x)$
0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



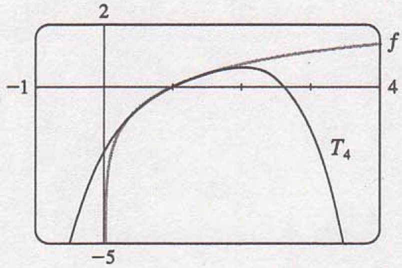
(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

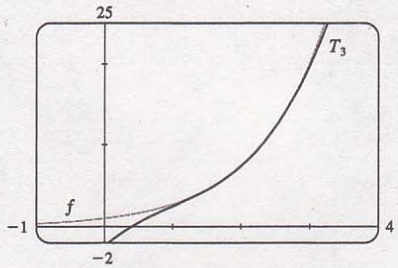
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = 0 + (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4$$

4.

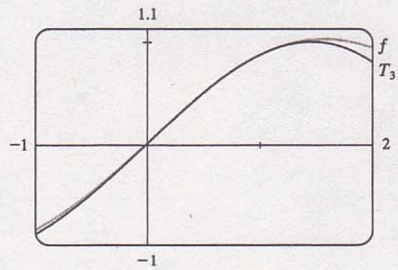
n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3$$

5.

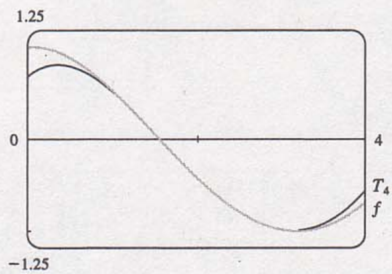
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\tfrac{\pi}{6}\right)^n = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(x-\tfrac{\pi}{6}\right) - \tfrac{1}{4}\left(x-\tfrac{\pi}{6}\right)^2 - \tfrac{\sqrt{3}}{12}\left(x-\tfrac{\pi}{6}\right)^3$$

6.

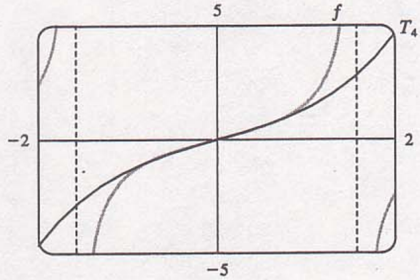
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{2\pi}{3}\right)$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}\left(\frac{2\pi}{3}\right)}{n!}\left(x-\tfrac{2\pi}{3}\right)^n = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\left(x-\tfrac{2\pi}{3}\right) + \tfrac{1}{4}\left(x-\tfrac{2\pi}{3}\right)^2 + \tfrac{\sqrt{3}}{12}\left(x-\tfrac{2\pi}{3}\right)^3 - \tfrac{1}{48}\left(x-\tfrac{2\pi}{3}\right)^4$$

7.

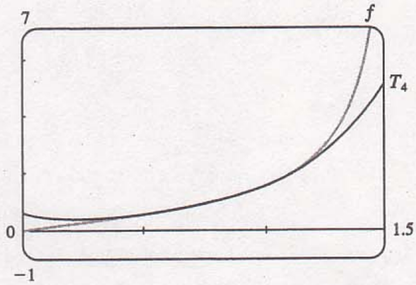
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	0



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} x^n = x + \frac{2x^3}{3!} = x + \frac{x^3}{3}$$

8.

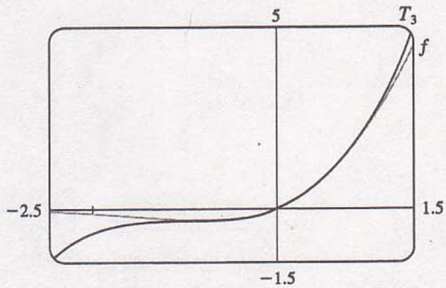
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\tan x$	1
1	$\sec^2 x$	2
2	$2\sec^2 x \tan x$	4
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	16
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	80



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!} \left(x - \tfrac{\pi}{4}\right)^n = 1 + 2\left(x - \tfrac{\pi}{4}\right) + 2\left(x - \tfrac{\pi}{4}\right)^2 + \tfrac{8}{3}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{10}{3}\left(x - \tfrac{\pi}{4}\right)^4$$

9.

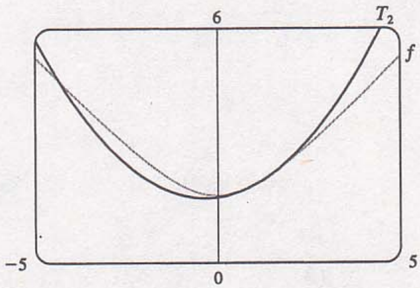
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^x \sin x$	0
1	$e^x(\sin x + \cos x)$	1
2	$2e^x \cos x$	2
3	$2e^x(\cos x - \sin x)$	2



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = x + x^2 + \tfrac{1}{3}x^3$$

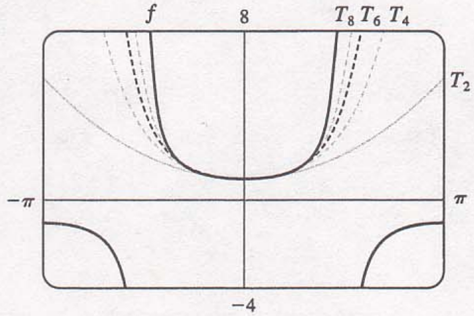
10.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(3 + x^2)^{1/2}$	2
1	$x(3 + x^2)^{-1/2}$	$\frac{1}{2}$
2	$3(3 + x^2)^{-3/2}$	$\frac{3}{8}$

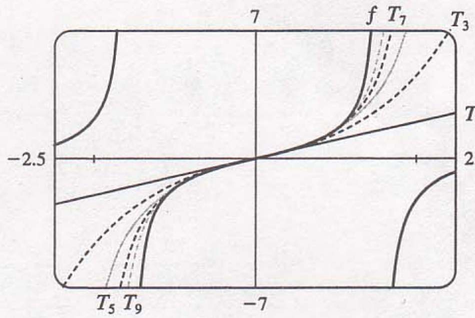


$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(1)}{n!} (x-1)^n = 2 + \tfrac{1}{2}(x-1) + \tfrac{3/8}{2}(x-1)^2 = 2 + \tfrac{1}{2}(x-1) + \tfrac{3}{16}(x-1)^2$$

11. In Maple, we can find the Taylor polynomials by the following method: first define `f:=sec(x);` and then set `T2:=convert(taylor(f,x=0,3),polynom);`, `T4:=convert(taylor(f,x=0,5),polynom);`, etc. (The third argument in the `taylor` function is one more than the degree of the desired polynomial). We must `convert` to the type `polynom` because the output of the `taylor` function contains an error term which we do not want. In Mathematica, we use `Tn:=Normal[Series[f,{x,0,n}]]`, with `n=2, 4,` etc. Note that in Mathematica, the "degree" argument is the same as the degree of the desired polynomial. In Derive, author sec x, then enter `Calculus, Taylor, 8, 0;` and then simplify the expression. The eighth Taylor polynomial is $T_8(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8$.



12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for $\tan x$ is $T_9(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9$.



13.

$$\begin{aligned} f(x) &= \sqrt{x} & f(4) &= 2 \\ f'(x) &= \tfrac{1}{2}x^{-1/2} & f'(4) &= \tfrac{1}{4} \\ f''(x) &= -\tfrac{1}{4}x^{-3/2} & f''(4) &= -\tfrac{1}{32} \\ f'''(x) &= \tfrac{3}{8}x^{-5/2} \end{aligned}$$

(a) $\sqrt{x} \approx T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2$

$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-4|^3$, where $|f'''(x)| \le M$. Now

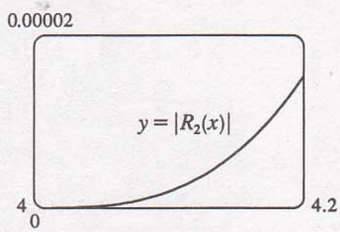
$4 \le x \le 4.2 \;\Rightarrow\; |x-4| \le 0.2 \;\Rightarrow\; |x-4|^3 \le 0.008.$

Since $f'''(x)$ is decreasing on $[4, 4.2]$, we can take

$M = |f'''(4)| = \frac{3}{8}4^{-5/2} = \frac{3}{256}$, so

$|R_2(x)| \le \frac{3/256}{6}(0.008) = \frac{0.008}{512} = 0.000015625.$

(c)



From the graph of $|R_2(x)| = |\sqrt{x} - T_2(x)|$, it seems that the error is less than 1.52×10^{-5} on $[4, 4.2]$.

14.

$$\begin{array}{ll} f(x) = x^{-2} & f(1) = 1 \\ f'(x) = -2x^{-3} & f'(1) = -2 \\ f''(x) = 6x^{-4} & f''(4) = 6 \\ f'''(x) = -24x^{-5} & \end{array}$$

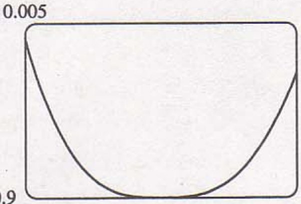
(a) $x^{-2} \approx T_2(x) = 1 - 2(x-1) + \frac{6}{2!}(x-1)^2$
$= 1 - 2(x-1) + 3(x-1)^2$

(b) $|R_2(x)| \leq \dfrac{M}{3!}|x-1|^3$, where $|f'''(x)| \leq M$.
Now $0.9 \leq x \leq 1.1 \Rightarrow |x-1| \leq 0.1 \Rightarrow$ $|x-1|^3 \leq 0.001$. Since $f'''(x)$ is decreasing on $[0.9, 1.1]$, we can take $M = |f'''(0.9)| = \frac{24}{(0.9)^5}$, so

$|R_2(x)| \leq \frac{24/(0.9)^5}{6}(0.001) = \frac{0.004}{0.59049}$
≈ 0.00677404

(c)

0.005



0.9 0 1.1

From the graph of $|R_2(x)| = |x^{-2} - T_2(x)|$, it seems that the error is less than 0.0046 on $[0.9, 1.1]$.

15.

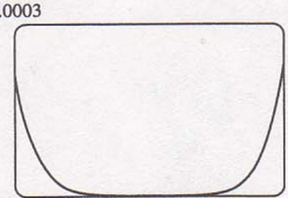
$$\begin{array}{llll} f(x) = \sin x & f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} & f^{(4)}(x) = \sin x & f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\ f'(x) = \cos x & f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} & f^{(5)}(x) = \cos x & f^{(5)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\ f''(x) = -\sin x & f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} & f^{(6)}(x) = -\sin x & \\ f'''(x) = -\cos x & f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} & & \end{array}$$

(a) $\sin x \approx T_5(x)$
$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$
$+ \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{4}\right)^4 + \frac{\sqrt{2}}{240}\left(x - \frac{\pi}{4}\right)^5$

(b) $|R_5(x)| \leq \dfrac{M}{6!}\left|x - \frac{\pi}{4}\right|^6$, where $\left|f^{(6)}(x)\right| \leq M$. Now $0 \leq x \leq \frac{\pi}{2} \Rightarrow \left(x - \frac{\pi}{4}\right)^6 \leq \left(\frac{\pi}{4}\right)^6$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so
$|R_5(x)| \leq \frac{1}{6!}\left(\frac{\pi}{4}\right)^6 = \frac{1}{720}\left(\frac{\pi}{4}\right)^6 \approx 0.00033.$

(c)

0.0003



0 $\frac{\pi}{2}$

From the graph of $|R_5(x)| = |\sin x - T_5(x)|$, it seems that the error is less than 0.00026 on $\left[0, \frac{\pi}{2}\right]$.

16.

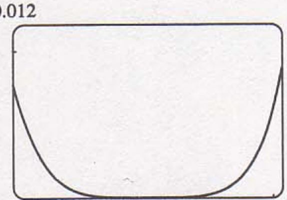
$$\begin{array}{llll} f(x) = \cos x & f\left(\frac{\pi}{3}\right) = \frac{1}{2} & f'''(x) = \sin x & f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \\ f'(x) = -\sin x & f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} & f^{(4)}(x) = \cos x & f^{(4)}\left(\frac{\pi}{3}\right) = \frac{1}{2} \\ f''(x) = -\cos x & f''\left(\frac{\pi}{3}\right) = -\frac{1}{2} & f^{(5)}(x) = -\sin x & \end{array}$$

(a) $\cos x \approx T_4(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2$
$+ \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4$

(b) $|R_4(x)| \leq \dfrac{M}{5!}\left|x - \frac{\pi}{3}\right|^5$, where $\left|f^{(5)}(x)\right| \leq M$.
Now $0 \leq x \leq \frac{2\pi}{3} \Rightarrow \left(x - \frac{\pi}{3}\right)^5 \leq \left(\frac{\pi}{3}\right)^5$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so
$|R_4(x)| \leq \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx 0.0105.$

(c)

0.012



0 $\frac{2\pi}{3}$

From the graph of $|R_4(x)| = |\cos x - T_4(x)|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17. $f(x) = \tan x$ $\quad f(0) = 0$ $\quad f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ $\quad f'''(0) = 2$

$f'(x) = \sec^2 x$ $\quad f'(0) = 1$ $\quad f^{(4)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''(0) = 0$

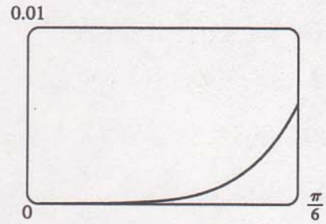
(a) $\tan x \approx T_3(x) = x + \frac{1}{3}x^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{6} \;\Rightarrow\; x^4 \le \left(\frac{\pi}{6}\right)^4$, and letting $x = \frac{\pi}{6}$ gives

$$|R_3(x)| \le \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \tfrac{4\sqrt{3}}{9}\left(\tfrac{\pi}{6}\right)^4 \approx 0.057859$$

(c)



From the graph, it seems that the error is less than 0.006 on $[0, \pi]$.

18. $f(x) = (1 + x^2)^{1/3}$ $\quad f(0) = 1$ $\quad f'''(x) = \dfrac{8x^3 - 72x}{27(1 + x^2)^{8/3}}$

$f'(x) = \frac{2}{3}x(1 + x^2)^{-2/3}$ $\quad f'(0) = 0$

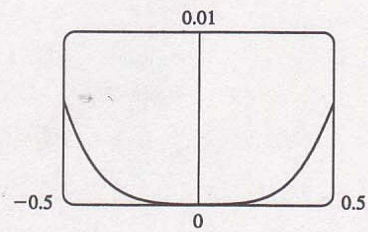
$f''(x) = \frac{2}{3}\left(1 - \frac{1}{3}x^2\right)(1 + x^2)^{-5/3}$ $\quad f''(0) = \frac{2}{3}$

(a) $\sqrt[3]{1 + x^2} \approx T_2(x) = 1 + \frac{1}{3}x^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $|f'''(x)| \le M$. By examining a graph of $|f'''(x)|$, we see that its maximum is approximately 0.71495314. Thus,

$$|R_2(x)| \le \frac{0.71495314}{3!}(0.5)^3 \approx 0.014895.$$

(c)



It seems that the error is less than 0.0061 on $[-0.5, 0.5]$.

19. $f(x) = e^{x^2}$ $\quad f(0) = 1$ $\quad f'''(x) = e^{x^2}(12x + 8x^3)$ $\quad f'''(0) = 0$

$f'(x) = e^{x^2}(2x)$ $\quad f'(0) = 0$ $\quad f^{(4)}(x) = e^{x^2}(12 + 48x^2 + 16x^4)$

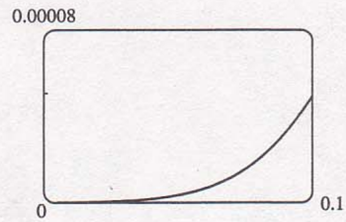
$f''(x) = e^{x^2}(2 + 4x^2)$ $\quad f''(0) = 2$

(a) $e^{x^2} \approx T_3(x) = 1 + \frac{2}{2!}x^2 = 1 + x^2$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0 \le x \le 0.1 \Rightarrow x^4 \le (0.1)^4$, and letting $x = 0.1$ gives

$$|R_3(x)| \le \frac{e^{0.01}(12 + 0.48 + 0.0016)}{24}(0.1)^4 \approx 0.00006.$$

(c)



From the graph of $|R_3(x)| = \left|e^{x^2} - (1 + x^2)\right|$, it appears that the error is less than 0.000051 on $[0, 0.1]$.

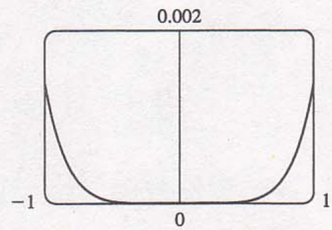
20. (a) Clearly $f^{(2n)}(0) = 1$ and $f^{(2n+1)}(0) = 0$, so

$$\cosh x \approx T_5(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}.$$

(b) $|R_5(x)| \le \dfrac{M}{6!}|x|^6$, where $\left|f^{(6)}(x)\right| \le M$. Since $f^{(6)}(x) = \cosh x$ and $\cosh x$ attains its maximum on $[-1, 1]$ at both endpoints, we let $x = 1$ and get

$$|R_5(x)| \le \frac{\cosh 1}{6!}(1)^6 \approx 0.002143.$$

(c)



It appears that the error is less than 0.0015 on $(-1, 1)$.

21. $f(x) = x^{3/4}$ $\quad f(16) = 8$ $\quad f'''(x) = \frac{15}{64}x^{-9/4}$ $\quad f'''(16) = \frac{15}{32{,}768}$

$f'(x) = \frac{3}{4}x^{-1/4}$ $\quad f'(16) = \frac{3}{8}$ $\quad f^{(4)}(x) = -\frac{135}{256}x^{-13/4}$

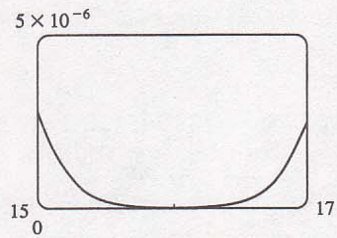
$f''(x) = -\frac{3}{16}x^{-5/4}$ $\quad f''(16) = -\frac{3}{512}$

(a) $x^{3/4} \approx T_3(x) = 8 + \frac{3}{8}(x - 16) - \frac{3}{1024}(x - 16)^2 + \frac{5}{65{,}536}(x - 16)^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x - 16|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $15 \le x \le 17 \Rightarrow |x - 16|^4 \le 1^4 = 1$, and letting $x = 15$ to minimize the denominator of $f^{(4)}(x)$ gives

$$|R_3(x)| \le \frac{135/\left[256(15)^{13/4}\right]}{4!}(1) \approx 0.0000033.$$

(c)



It appears that the error is less than 3×10^{-6} on $(15, 17)$.

22.
$f(x) = \ln x \qquad f(4) = \ln 4 \qquad f'''(x) = 2x^{-3} \qquad f'''(4) = \frac{1}{32}$

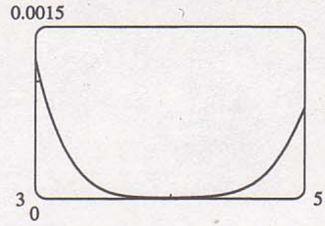
$f'(x) = x^{-1} \qquad f'(4) = \frac{1}{4} \qquad f^{(4)}(x) = -6x^{-4}$

$f''(x) = -x^{-2} \qquad f''(4) = -\frac{1}{16}$

(a) $\ln x \approx T_3(x) = \ln 4 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{1}{192}(x-4)^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x-4|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $3 \le x \le 5 \;\Rightarrow\; (x-4)^4 \le 1^4 = 1$, and letting $x = 3$ gives $M = 6/3^4$, so $|R_3(x)| \le \frac{6}{4!\,3^4} \cdot 1 = \frac{1}{324} \approx 0.0031$.

(c)



From the graph of $|R_3(x)| = |\ln x - T_3(x)|$, it appears that the error is less than 0.0013 on $[3, 5]$.

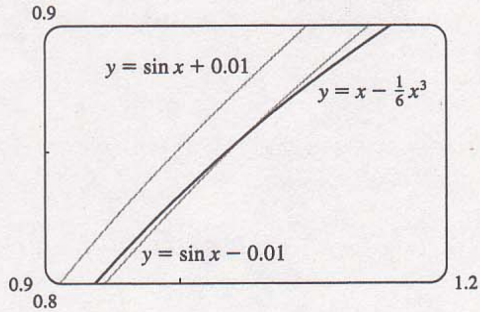
23. From Exercise 5, $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + R_3(x)$, where $R_3(x) \le \dfrac{M}{4!}\left|x - \frac{\pi}{6}\right|^4$ with $\left|f^{(4)}(x)\right| = |\sin x| \le M = 1$. Now $35^\circ = \left(\frac{\pi}{6} + \frac{\pi}{36}\right)$ radians, so the error is $\left|R_3\left(\frac{\pi}{36}\right)\right| \le \dfrac{\left(\frac{\pi}{36}\right)^4}{4!} < 0.000003$. Therefore, to five decimal places, $\sin 35^\circ \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \approx 0.57358$.

24. From Exercise 16, $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4 + R_4(x)$. Now since $x = 69^\circ = \left(\frac{\pi}{3} + \frac{\pi}{20}\right)$ radians, the error is $|R_4(x)| \le \dfrac{\left(\frac{\pi}{20}\right)^5}{5!} < 8 \times 10^{-7}$. Therefore, to five decimal places, $\cos 69^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right) - \frac{1}{4}\left(\frac{\pi}{20}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3 + \frac{1}{48}\left(\frac{\pi}{20}\right)^4 \approx 0.35837$.

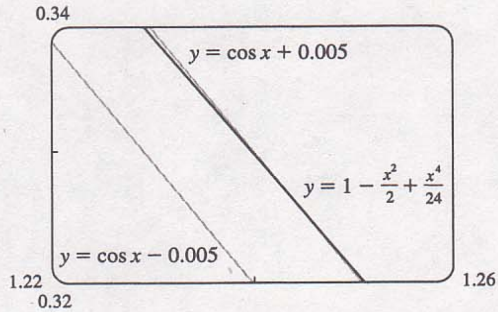
25. All derivatives of e^x are e^x, so $|R_n(x)| \le \dfrac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$, $R_n(0.1) \le \dfrac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$, and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus, by adding the three terms of the Maclaurin series for e^x corresponding to $n = 0, 1$, and 2, we can estimate $e^{0.1}$ to within 0.00001.

26. From Exercise 12.10.35 [ET 11.10.35], the Maclaurin series for $\ln(1+x)$ is $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} x^n$. So $\ln 1.4 = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}(0.4)^n$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (non-zero) terms of the Maclaurin series for the desired accuracy.

27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \Leftrightarrow \left|x^5\right| < 120\,(0.01) \Leftrightarrow$ $|x| < (1.2)^{1/5} \approx 1.037$. The curves intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate only for $x > 0$.



28. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \Leftrightarrow x^6 < 3.6 \Leftrightarrow$ $|x| < (3.6)^{1/6} \approx 1.238$. The curves intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$.



29. Let $s\,(t)$ be the position function of the car, and for convenience set $s\,(0) = 0$. The velocity of the car is $v\,(t) = s'\,(t)$ and the acceleration is $a\,(t) = s''\,(t)$, so the second degree Taylor polynomial is $T_2\,(t) = s\,(0) + v\,(0)\,t + \dfrac{a\,(0)}{2}t^2 = 20t + t^2$. We estimate the distance travelled during the next second to be $s\,(1) \approx T_2\,(1) = 20 + 1 = 21$ m. The function $T_2\,(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be 140 m/s ≈ 315 mi/h!)

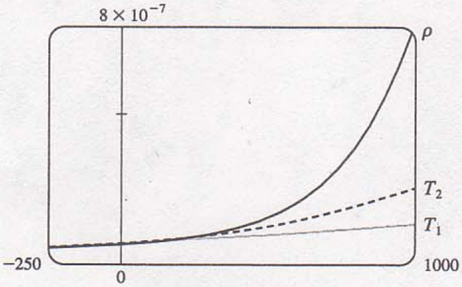
30. (a)

$$\rho(t) = \rho_{20} e^{\alpha(t-20)} \qquad \rho(20) = \rho_{20}$$
$$\rho'(t) = \alpha\rho_{20} e^{\alpha(t-20)} \qquad \rho'(20) = \alpha\rho_{20}$$
$$\rho''(t) = \alpha^2\rho_{20} e^{\alpha(t-20)} \qquad \rho''(20) = \alpha^2\rho_{20}$$

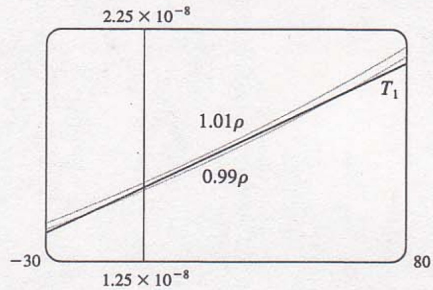
The linear approximation is $T_1(t) = \rho(20) + \rho'(20)(t-20) = \rho_{20}[1 + \alpha(t-20)]$. The quadratic approximation is

$$T_2(t) = \rho(20) + \rho'(20)(t-20) + \frac{\rho''(20)}{2}(t-20)^2 = \rho_{20}\left[1 + \alpha(t-20) + \tfrac{1}{2}\alpha^2(t-20)^2\right]$$

(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t) \le T_1(t) \le 1.01\rho(t)$, for $-14\ ^\circ\text{C} \le t \le 58\ ^\circ\text{C}$.

31. $E = \dfrac{q}{D^2} - \dfrac{q}{(D+d)^2} = \dfrac{q}{D^2} - \dfrac{q}{D^2(1+d/D)^2} = \dfrac{q}{D^2}\left[1 - \left(1 + \dfrac{d}{D}\right)^{-2}\right]$.

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$\begin{aligned} E &= \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 - \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right] \\ &= \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \cdots\right] \approx 2qd\cdot\frac{1}{D^3} \end{aligned}$$

when D is much larger than d, that is, when P is far away from the dipole.

32. (a) $\dfrac{n_1}{\ell_o} + \dfrac{n_2}{\ell_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{\ell_i} - \dfrac{n_1 s_o}{\ell_o}\right)$ (Equation 1) where

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi} \text{ and } \ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi}$$

Using $\cos\phi \approx 1$ gives

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)} = \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o - 2R^2} = \sqrt{s_o^2} = s_o$$

and similarly, $\ell_i = s_i$. Thus, Equation 1 becomes $\dfrac{n_1}{s_o} + \dfrac{n_2}{s_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{s_i} - \dfrac{n_1 s_o}{s_o}\right) \Rightarrow$

$\dfrac{n_1}{s_o} + \dfrac{n_2}{s_i} = \dfrac{n_2 - n_1}{R}$.

(b) Using $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ in (2) gives us

$$\begin{aligned}\ell_o &= \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\left(1 - \tfrac{1}{2}\phi^2\right)} \\ &= \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o + Rs_o\phi^2 - 2R^2 + R^2\phi^2} = \sqrt{s_o^2 + Rs_o\phi^2 + R^2\phi^2}\end{aligned}$$

Anticipating that we will use the binomial series expansion $(1+x)^k \approx 1 + kx$, we can write the last expression as $s_o\sqrt{1 + \phi^2\left(\dfrac{R}{s_o} + \dfrac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i = s_i\sqrt{1 - \phi^2\left(\dfrac{R}{s_i} - \dfrac{R^2}{s_i^2}\right)}$. Thus,

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = n_1\ell_o^{-1} + n_2\ell_i^{-1} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right) \quad \Leftrightarrow$$

$$\begin{aligned}&\frac{n_1}{s_o}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2} + \frac{n_2}{s_i}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2} \\ &\qquad = \frac{n_2}{R}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2} - \frac{n_1}{R}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2}\end{aligned}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_o^{-1} by the first two terms in their binomial series, we get

$$\begin{aligned}&\frac{n_1}{s_o}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] + \frac{n_2}{s_i}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right] \\ &\qquad = \frac{n_2}{R}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right] - \frac{n_1}{R}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] \quad \Leftrightarrow\end{aligned}$$

$$\begin{aligned}&\frac{n_1}{s_o} - \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_2}{s_i} + \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) \\ &\qquad = \frac{n_2}{R} + \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) - \frac{n_1}{R} + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) \quad \Leftrightarrow\end{aligned}$$

$$\begin{aligned}\frac{n_1}{s_o} + \frac{n_2}{s_i} &= \frac{n_2}{R} - \frac{n_1}{R} + \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) \\ &\qquad + \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) + \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) \\ &= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2}{2}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o} + \frac{1}{R}\right) + \frac{n_2\phi^2}{2}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right) \\ &= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2 R^2}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)\left(\frac{1}{R} + \frac{1}{s_o}\right) + \frac{n_2\phi^2 R^2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right) \\ &= \frac{n_2 - n_1}{R} + \phi^2 R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]\end{aligned}$$

From Figure 7, we see that $\sin\phi = h/R$. So if we approximate $\sin\phi$ with ϕ, we get $h = R\phi$ and $h^2 = \phi^2 R^2$ and hence, Equation 4, as desired.

33. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \to 1$ as $x \to \infty$. So we can approximate $\tanh(2\pi d/L) \approx 1$, and so $v^2 \approx gL/(2\pi) \quad \Leftrightarrow \quad v \approx \sqrt{gL/(2\pi)}$.

(b) From the calculations at right, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$, and so $v^2 \approx \dfrac{gL}{2\pi} \cdot \dfrac{2\pi d}{L} \quad \Leftrightarrow \quad v \approx \sqrt{gd}$.

$f(x) = \tanh x$	$f(0) = 0$
$f'(x) = \operatorname{sech}^2 x$	$f'(0) = 1$
$f''(x) = -2\operatorname{sech}^2 x \tanh x$	$f''(0) = 0$
$f'''(x) = 2\operatorname{sech}^2 x\,(3\tanh^2 x - 1)$	$f'''(0) = -2$

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$ is less than the first neglected term, which is $\dfrac{|f'''(0)|}{3!}\left(\dfrac{2\pi d}{L}\right)^3 = \dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3$. If $L > 10d$, then $\dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3 < \dfrac{1}{3}\left(2\pi \cdot \dfrac{1}{10}\right)^3 = \dfrac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\dfrac{gL}{2\pi} \cdot \dfrac{\pi^3}{375} \approx 0.0132gL$.

34. $T_n(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0 \le m \le n$. Then

$$T_n^{(m)}(x) = m!\frac{f^{(m)}(a)}{m!}(x-a)^0 + (m+1)(m)\cdots(2)\frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1 + \cdots$$
$$+ n(n-1)\cdots(n-m+1)\frac{f^{(n)}(z)}{n!}(x-a)^{n-m}$$

For $x = a$, all terms in this sum except the first one are 0, so $T_n^{(m)}(a) = \dfrac{m!\,f^{(m)}(a)}{m!} = f^{(m)}(a)$.

35. Using $f(x) = T_n(x) + R_n(x)$ with $n = 1$ and $x = r$, we have $f(r) = T_1(r) + R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a. Because $a = x_n$, $f(r) = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. But r is a root of f, so $f(r) = 0$ and we have $0 = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. Taking the first two terms to the left side and dividing by $f'(x_n)$, we have $f'(x_n)(x_n - r) - f(x_n) = R_1(r) \quad \Rightarrow \quad x_n - r - \dfrac{f(x_n)}{f'(x_n)} = \dfrac{R_1(r)}{f'(x_n)}$.

By the formula for Newton's method, the left side of the preceding equation is $x_{n+1} - r$, so $|x_{n+1} - r| = \left|\dfrac{R_1(r)}{f'(x_n)}\right|$. Taylor's Inequality gives us $|R_1(r)| \le \dfrac{|f''(r)|}{2!}|r - x_n|^2$. Combining this inequality with the facts $|f''(x)| \le M$ and $|f'(x)| \ge K$ gives us $|x_{n+1} - r| \le \dfrac{M}{2K}|x_n - r|^2$.

Applied Project □ Radiation from the Stars

1. If we write $f(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \dfrac{a\lambda^{-5}}{e^{b/(\lambda T)} - 1}$, then as $\lambda \to 0^+$, it is of the form ∞/∞, and as $\lambda \to \infty$ it is of the form $0/0$, so in either case we can use l'Hospital's Rule. First of all,

$$\lim_{\lambda\to\infty} f(\lambda) \overset{\text{H}}{=} \lim_{\lambda\to\infty} \frac{a\left(-5\lambda^{-6}\right)}{-\dfrac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 5\frac{aT}{b} \lim_{\lambda\to\infty} \frac{\lambda^2\lambda^{-6}}{e^{b/(\lambda T)}} = 5\frac{aT}{b} \lim_{\lambda\to\infty} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} = 0$$

Also,

$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} 5\frac{aT}{b} \lim_{\lambda\to 0^+} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} 5\frac{aT}{b} \lim_{\lambda\to 0^+} \frac{-4\lambda^{-5}}{-\dfrac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 20\frac{aT^2}{b^2} \lim_{\lambda\to 0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}}$$

This is still indeterminate, but note that each time we use l'Hospital's Rule, we gain a factor of λ in the numerator, as well as a constant factor, and the denominator is unchanged. So if we use l'Hospital's Rule three more times, the exponent of λ in the numerator will become 0. That is, for some $\{k_i\}$, all constant,

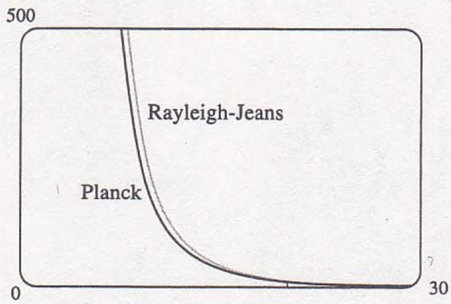
$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} k_1 \lim_{\lambda\to 0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_2 \lim_{\lambda\to 0^+} \frac{\lambda^{-2}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_3 \lim_{\lambda\to 0^+} \frac{\lambda^{-1}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_4 \lim_{\lambda\to 0^+} \frac{1}{e^{b/(\lambda T)}} = 0$$

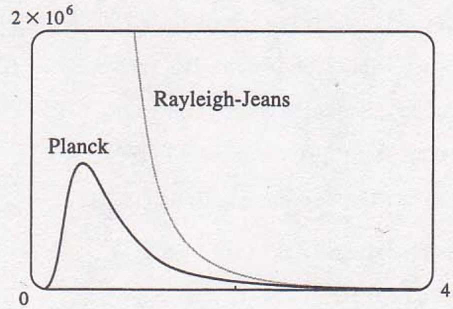
2. We expand the denominator of Planck's Law using the Taylor series $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ with $x = \dfrac{hc}{\lambda kT}$, and use the fact that if λ is large, then all subsequent terms in the Taylor expansion are very small compared to the first one, so we can approximate using the Taylor polynomial T_1:

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{8\pi hc\lambda^{-5}}{\left[1 + \dfrac{hc}{\lambda kT} + \dfrac{1}{2!}\left(\dfrac{hc}{\lambda kT}\right)^2 + \dfrac{1}{3!}\left(\dfrac{hc}{\lambda kT}\right)^3 + \cdots\right] - 1} \approx \frac{8\pi hc\lambda^{-5}}{\left(1 + \dfrac{hc}{\lambda kT}\right) - 1} = \frac{8\pi kT}{\lambda^4}$$

which is the Rayleigh-Jeans Law.

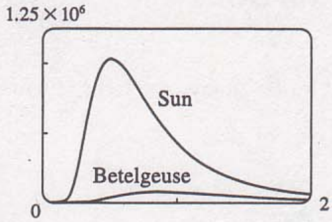
3. To convert to μm, we substitute $\lambda/10^6$ for λ in both laws. The first figure shows that the two laws are similar for large λ. The second figure shows that the two laws are very different for short wavelengths (Planck's Law gives a maximum at $\lambda \approx 0.51$ μm; the Rayleigh-Jeans Law gives no minimum or maximum.).

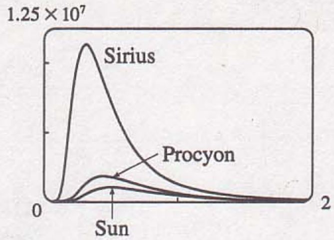




4. From the graph in Problem 3, $f(\lambda)$ has a maximum under Planck's Law at $\lambda \approx 0.51$ μm.

5.





As T gets larger, the total area under the curve increases, as we would expect: the hotter the star, the more energy it emits. Also, as T increases, the λ-value of the maximum decreases, so the higher the temperature, the shorter the peak wavelength (and consequently the average wavelength) of light emitted. This is why Sirius is a blue star and Betelgeuse is a red star: most of Sirius's light is of a fairly short wavelength, that is, a higher frequency, toward the blue end of the spectrum, whereas most of Betelgeuse's light is of a lower frequency, toward the red end of the spectrum.

12 Review — ET 11

CONCEPT CHECK

1. (a) See Definition 12.1.1 [ET 11.1.1].

(b) See Definition 12.2.2 [ET 11.2.2].

(c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) See Definition 12.1.9 [ET 11.1.9].

(b) A sequence is monotonic if it is either increasing or decreasing.

(c) By Theorem 12.1.10 [ET 11.1.10], every bounded, monotonic sequence is convergent.

3. (a) See (4) in Section 12.2 [ET 11.2].

(b) See (1) in Section 12.3 [ET 11.3].

4. If $\sum a_n = 3$, then $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} s_n = 3$.

5. (a) See the Test for Divergence (12.2.7 [ET 11.2.7]).

(b) See the Integral Test on page 749 [ET 715].

(c) See the Comparison Test on page 756 [ET 722].

(d) See the Limit Comparison Test on page 757 [ET 723].

(e) See the Alternating Series Test on page 761 [ET 727].

(f) See the Ratio Test on page 767 [ET 733].

(g) See the Root Test on page 769 [ET 735].

6. (a) See Definition 12.6.1 [ET 11.6.1].

(b) By (12.6.3 [ET 11.6.3]), it is convergent.

(c) See Definition 12.6.2 [ET 11.6.2].

7. (a) Use either (2) or (3) in Section 12.3 [ET 11.3].

(b) See Example 5 in Section 12.4 [ET 11.4].

(c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 763 [ET 729].

8. (a) $\sum_{n=0}^{\infty} c_n (x-a)^n$

(b) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a-R$ and $a+R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

9. (a), (b) See Theorem 12.9.2 [ET 11.9.2].

10. (a) $T_n(x) = \sum\limits_{i=0}^{n} \dfrac{f^{(i)}(a)}{i!}(x-a)^i$

(b) $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$ [$a = 0$ in part (b)]

(d) See Theorem 12.10.8 [ET 11.10.8].

(e) See Taylor's Inequality (12.10.9 [ET 11.10.9]).

11. (a) – (e) See the table on page 792 [ET 758].

12. See the Binomial Series (12.11.2 [ET 11.11.2]) for the expansion. The radius of convergence for the binomial series is 1.

TRUE-FALSE QUIZ

1. False. See Note 2 after Theorem 12.2.6 [ET 11.2.6].

2. True by Theorem 12.8.3 [ET 11.8.3].
Or: Use the Comparison Test to show that $\sum c_n (-2)^n$ converges absolutely.

3. False. For example, take $c_n = (-1)^n / (n6^n)$.

4. True by Theorem 12.8.3 [ET 11.8.3].

5. False, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n^3}{(n+1)^3}\right| = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^3} = 1.$

6. True, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n!}{(n+1)!}\right| = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1.$

7. False. See the note after Example 2 in Section 12.4 [ET 11.4].

8. True, since $\dfrac{1}{e} = e^{-1}$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-1} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n!}$.

9. True. See (7) in Section 12.1 [ET 11.1].

10. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 12.6.3 [ET 11.6.3].

11. True. By Theorem 12.10.5 [ET 11.10.5] the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \quad \Rightarrow \quad f'''(0) = 2$.

Or: Use Theorem 12.9.2 [ET 11.9.2] to differentiate f three times.

12. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.

13. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.

14. True. by Theorem 12.1.10 [ET 11.1.10] (the Monotonic Sequence Theorem), since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all $n \quad \Rightarrow \quad \{a_n\}$ is bounded.

15. True by Theorem 12.6.3 [ET 11.6.3]. [$\sum (-1)^n a_n$ is absolutely convergent and hence convergent.]

16. True. $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \quad \Rightarrow \quad \sum a_n$ converges (Ratio Test) $\quad \Rightarrow \quad \displaystyle\lim_{n\to\infty} a_n = 0$ (Theorem 12.2.6 [ET 11.2.6]).

EXERCISES

1. $\left\{\dfrac{2+n^3}{1+2n^3}\right\}$ converges since $\displaystyle\lim_{n\to\infty} \frac{2+n^3}{1+2n^3} = \lim_{n\to\infty} \frac{2/n^3+1}{1/n^3+2} = \frac{1}{2}$.

2. $a_n = \dfrac{9^{n+1}}{10^n} = 9 \cdot \left(\frac{9}{10}\right)^n$, so $\displaystyle\lim_{n\to\infty} a_n = 9 \lim_{n\to\infty} \left(\tfrac{9}{10}\right)^n = 9 \cdot 0 = 0$ by (12.1.7 [ET 11.1.7]).

3. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{1+n^2} = \lim_{n\to\infty} \frac{n}{1/n^2+1} = \infty$, so the sequence diverges.

4. $\left\{\dfrac{n}{\ln n}\right\}$ diverges, since $\displaystyle\lim_{x\to\infty} \frac{x}{\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$.

5. $\{\sin n\}$ is divergent since $\displaystyle\lim_{n\to\infty} \sin n$ does not exist.

6. $\left\{\dfrac{\sin n}{n}\right\}$ converges, since $-\dfrac{1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$ and $\pm\dfrac{1}{n} \to 0$ as $n \to \infty$, so $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = 0$ by the Squeeze Theorem.

7. $\left\{\left(1+\dfrac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y = \left(1+\dfrac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 4x \ln(1+3/x) = \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/(4x)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1+3/x}\left(-\dfrac{3}{x^2}\right)}{-1/(4x^2)} = \lim_{x\to\infty} \frac{12}{1+3/x} = 12$$

so $\displaystyle\lim_{x\to\infty} y = \lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{4n} = e^{12}$.

8. $\left\{\dfrac{(-10)^n}{n!}\right\}$ converges, since $\dfrac{10^n}{n!} = \dfrac{10 \cdot 10 \cdot 10 \cdot \cdots \cdot 10}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 10} \cdot \dfrac{10 \cdot 10 \cdot \cdots \cdot 10}{11 \cdot 12 \cdot \cdots \cdot n} \le 10^{10}\left(\dfrac{10}{11}\right)^{n-10} \to 0$ as $n \to \infty$,

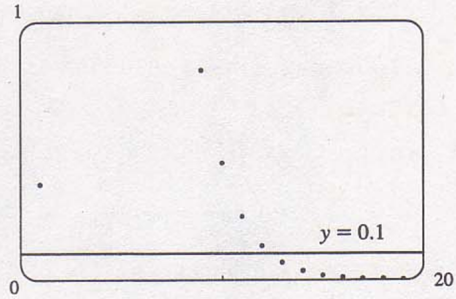
so $\displaystyle\lim_{n\to\infty} \frac{(-10)^n}{n!} = 0$ (Squeeze Theorem). *Or:* Use (12.10.10 [ET 11.10.10]).

9. We use induction, hypothesizing that $a_{n-1} < a_n < 2$. Note first that $1 < a_2 = \frac{1}{3}(1+5) = \frac{5}{3} < 2$, so the hypothesis holds for $n = 2$. Now assume that $a_{k-1} < a_k < 2$. Then $a_k = \frac{1}{3}(a_{k-1}+4) < \frac{1}{3}(a_k+4) < \frac{1}{3}(2+4) = 2$. So $a_k < a_{k+1} < 2$, and the induction is complete. To find the limit of the sequence, we note that $L = \displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} \quad \Rightarrow \quad L = \tfrac{1}{3}(L+4) \quad \Rightarrow \quad L = 2$.

10. $\lim\limits_{x\to\infty} \dfrac{x^4}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{4x^3}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{12x^2}{e^x}$

$\overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{24x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{24}{e^x} = 0$

Then we conclude from Theorem 12.1.2 [ET 11.1.2] that $\lim\limits_{n\to\infty} n^4 e^{-n} = 0$. From the graph, it seems that $12^4 e^{-12} > 0.1$, but $n^4 e^{-n} < 0.1$ whenever $n > 12$. So the smallest value of N corresponding to $\varepsilon = 0.1$ in the definition of the limit is $N = 12$.



11. $\dfrac{n}{n^3+1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^3+1}$ converges by the Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ $(p = 2 > 1)$.

12. Let $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, so $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3+n}{n^3+1} = \lim\limits_{n\to\infty} \dfrac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ also diverges by the Limit Comparison Test.

13. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^3}{5^{n+1}} \cdot \dfrac{5^n}{n^3}\right] = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^3 \cdot \dfrac{1}{5} = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n = \dfrac{1}{\sqrt{n+1}}$. Then b_n is positive for $n \geq 1$, the sequence $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

15. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3} < 1$, so the series converges by the Root Test.

16. $\lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3}$, so $\lim\limits_{n\to\infty} \ln\left(\dfrac{n}{3n+1}\right) = \ln\frac{1}{3} \neq 0$. Thus, $\sum\limits_{n=1}^{\infty} \ln\left(\dfrac{n}{3n+1}\right)$ diverges by the Test for Divergence.

17. $\left|\dfrac{\sin n}{1+n^2}\right| \leq \dfrac{1}{1+n^2} < \dfrac{1}{n^2}$ and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges (p-series with $p = 2 > 1$), so does $\sum\limits_{n=1}^{\infty} \left|\dfrac{\sin n}{1+n^2}\right|$ by the Comparison Test, and so does $\sum\limits_{n=1}^{\infty} \dfrac{\sin n}{1+n^2}$ by Theorem 12.6.3 [ET 11.6.3].

18. $f(x) = \dfrac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $(2, \infty)$, so we can use the Integral Test.

$\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t = \frac{1}{\ln 2}$, so the series converges.

19. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)}{5^{n+1}(n+1)!} \cdot \dfrac{5^n n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = \lim\limits_{n\to\infty} \dfrac{2n+1}{5(n+1)} = \dfrac{2}{5} < 1$, so the series converges by the Ratio Test.

20. $\sum\limits_{n=1}^{\infty} \dfrac{(-5)^{2n}}{n^2 9^n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}\left(\dfrac{25}{9}\right)^n$. Now $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{n^2}{(n+1)^2}\left(\dfrac{25}{9}\right)^{n+1}\left(\dfrac{9}{25}\right)^n = \dfrac{25}{9} > 1$, so the series diverges by the Ratio Test.

21. Let $b_n = \dfrac{\sqrt{n}}{n+1} > 0$. Then $0 \le \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+1} \le \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$, so $\lim\limits_{n\to\infty} b_n = 0$. If $f(x) = \dfrac{\sqrt{x}}{x+1}$ for $x > 0$, then $f'(x) = \dfrac{(x+1)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 1}{(x+1)^2} = \dfrac{(x+1)-2x}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$, so $f'(x) < 0$ for $x > 1$. It follows that $f(1) > f(2) > f(3) > \cdots$; that is, $b_n > b_{n+1}$ for all n. Thus, $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{\sqrt{n}}{n+1}$ converges by the Alternating Series Test.

22. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n+1}-\sqrt{n-1}}{n} = \dfrac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$ (rationalizing the numerator) and $b_n = \dfrac{1}{n^{3/2}}$. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} = 1$, so since $\sum_{n=1}^{\infty} b_n$ converges ($p = \frac{3}{2} > 1$), $\sum_{n=1}^{\infty} a_n$ converges also.

23. Consider the series of absolute values: $\sum_{n=1}^{\infty} n^{-1/3}$ is a p-series with $p = \frac{1}{3} < 1$ and is therefore divergent. But if we apply the Alternating Series Test we see that $a_{n+1} < a_n$ and $\lim\limits_{n\to\infty} n^{-1/3} = 0$. Therefore $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$ is conditionally convergent.

24. $\sum_{n=1}^{\infty} \left|(-1)^{n-1} n^{-3}\right| = \sum_{n=1}^{\infty} n^{-3}$ is a convergent p-series ($p = 3 > 1$.) Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$ is absolutely convergent.

25. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-1)^{n+1}(n+2)\,3^{n+1}}{2^{2n+3}} \cdot \dfrac{2^{2n+1}}{(-1)^n (n+1)\,3^n}\right| = \dfrac{n+2}{n+1}\cdot\dfrac{3}{4} = \dfrac{1+(2/n)}{1+(1/n)}\cdot\dfrac{3}{4} \to \dfrac{3}{4} < 1$ as $n \to \infty$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (n+1)\,3^n}{2^{2n+1}}$ is absolutely convergent.

26. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/(2\sqrt{x})}{1/x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{2} = \infty$. Therefore, $(-1)^{n+1}\dfrac{\sqrt{n}}{\ln n}$ does not approach 0, so the given series is divergent by the Test for Divergence.

27. Convergent geometric series. $\sum\limits_{n=1}^{\infty} \dfrac{2^{2n+1}}{5^n} = \sum\limits_{n=1}^{\infty} \dfrac{(2^2)^n \cdot 2^1}{5^n} = 2\sum\limits_{n=1}^{\infty} \dfrac{4^n}{5^n} = 2\left(\dfrac{\frac{4}{5}}{1-\frac{4}{5}}\right) = 8$.

28. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+3)} = \sum\limits_{n=1}^{\infty} \left[\dfrac{1}{3n} - \dfrac{1}{3(n+3)}\right]$ (partial fractions).

$s_n = \sum\limits_{i=1}^{n} \left[\dfrac{1}{3i} - \dfrac{1}{3(i+3)}\right] = \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} - \dfrac{1}{3(n+1)} - \dfrac{1}{3(n+2)} - \dfrac{1}{3(n+3)}$ (telescoping sum), so

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+3)} = \lim\limits_{n\to\infty} s_n = \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} = \dfrac{11}{18}$.

29.
$$\begin{aligned}\textstyle\sum_{n=1}^{\infty} \left[\tan^{-1}(n+1) - \tan^{-1} n\right] &= \lim_{n\to\infty} \left[\left(\tan^{-1} 2 - \tan^{-1} 1\right) + \left(\tan^{-1} 3 - \tan^{-1} 2\right) + \cdots \right. \\ &\qquad\qquad \left. + \left(\tan^{-1}(n+1) - \tan^{-1} n\right)\right] \\ &= \lim_{n\to\infty} \left[\tan^{-1}(n+1) - \tan^{-1} 1\right] = \tfrac{\pi}{2} - \tfrac{\pi}{4} = \tfrac{\pi}{4}\end{aligned}$$

30. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{2n} n!} = \sum_{n=0}^{\infty} \frac{(-x/4)^n}{n!} = e^{-x/4}$

31. $1.2 + 0.0\overline{345} = \frac{12}{10} + \frac{345/10{,}000}{1 - 1/1000} = \frac{12}{10} + \frac{345}{9990} = \frac{4111}{3330}$

32. This is a geometric series which converges whenever $|\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e$.

33. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16{,}807} - \frac{1}{32{,}768} + \cdots$. Since $\frac{1}{32{,}768} < 0.000031$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n^5} \approx 0.9721$.

34. (a) $s_5 = \sum_{n=1}^{5} \frac{1}{n^6} = 1 + \frac{1}{2^6} + \cdots + \frac{1}{5^6} \approx 1.017305$. The series $\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges by the Integral Test, so we estimate the remainder R_5 with (12.3.2 [ET 11.3.2]): $R_5 \le \int_5^{\infty} \frac{dx}{x^6} = \left[-\frac{x^{-5}}{5}\right]_5^{\infty} = \frac{5^{-5}}{5} = 0.000064$. So the error is at most 0.000064.

(b) In general, $R_n \le \int_n^{\infty} \frac{dx}{x^6} = \frac{1}{5n^5}$. If we take $n = 9$, then $s_9 \approx 1.01734$ and $R_9 \le \frac{1}{5 \cdot 9^5} \approx 3.4 \times 10^{-6}$. So to five decimal places, $\sum_{n=1}^{\infty} \frac{1}{n^5} \approx \sum_{n=1}^{9} \frac{1}{n^5} \approx 1.01734$.

Another Method: Use (12.3.3 [ET 11.3.3]) instead of (12.3.2 [ET 11.3.2]).

35. $\sum_{n=1}^{\infty} \frac{1}{2+5^n} \approx \sum_{n=1}^{8} \frac{1}{2+5^n} \approx 0.18976224$. To estimate the error, note that $\frac{1}{2+5^n} < \frac{1}{5^n}$, so the remainder term is $R_8 = \sum_{n=9}^{\infty} \frac{1}{2+5^n} < \sum_{n=9}^{\infty} \frac{1}{5^n} = \frac{1/5^9}{1 - 1/5} = 6.4 \times 10^{-7}$ (geometric series with $a = \frac{1}{5^9}$ and $r = \frac{1}{5}$).

36. (a)
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}(2n)!}{(2n+2)!\,n^n} = \lim_{n\to\infty} \frac{(n+1)^n (n+1)^1}{(2n+2)(2n+1)\,n^n}$$
$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \frac{1}{2(2n+1)} = e \cdot 0 = 0 < 1$$
so the series converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\lim_{n\to\infty} a_n = 0$ by Theorem 12.2.6 [ET 11.2.6].

37. Use the Limit Comparison Test. $\lim_{n\to\infty} \left|\frac{\left(\frac{n+1}{n}\right) a_n}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 > 0$. Since $\sum |a_n|$ is convergent, so is $\sum \left|\left(\frac{n+1}{n}\right) a_n\right|$, by the Limit Comparison Test.

38. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^2\,5^{n+1}}\cdot\dfrac{n^2 5^n}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^2}\dfrac{|x|}{5} = \dfrac{|x|}{5}$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{x^n}{n^2 5^n}$ converges when $|x|<5$. $R=5$. When $x=-5$, the series becomes the convergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ with $p=2>1$. When $x=5$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. Thus, $I=[-5,5]$.

39. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\dfrac{n4^n}{|x+2|^n}\right] = \lim\limits_{n\to\infty}\left[\dfrac{n}{n+1}\dfrac{|x+2|}{4}\right] = \dfrac{|x+2|}{4} < 1 \;\Leftrightarrow\; |x+2|<4$, so $R=4$. $|x+2|<4 \;\Leftrightarrow\; -4<x+2<4 \;\Leftrightarrow\; -6<x<2$. If $x=-6$, then the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-4)^n}{n4^n} = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x=2$, the series becomes the harmonic series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$, which diverges. Thus, $I=[-6,2)$.

40. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\dfrac{(n+2)!}{2^n(x-2)^n}\right| = \lim\limits_{n\to\infty}\dfrac{2}{n+3}|x-2| = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{2^n(x-2)^n}{(n+2)!}$ converges for all x. $R=\infty$ and $I=\mathbb{R}$.

41. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\dfrac{\sqrt{n+3}}{2^n(x-3)^n}\right| = 2|x-3|\lim\limits_{n\to\infty}\sqrt{\dfrac{n+3}{n+4}} = 2|x-3|<1 \;\Leftrightarrow\; |x-3|<\frac12$, so $R=\frac12$. For $x=\frac72$, the series becomes $\sum\limits_{n=0}^{\infty}\dfrac{1}{\sqrt{n+3}} = \sum\limits_{n=3}^{\infty}\dfrac{1}{n^{1/2}}$, which diverges ($p=\frac12\le 1$), but for $x=\frac52$, we get $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{\sqrt{n+3}}$, which is a convergent alternating series, so $I=\left[\frac52,\frac72\right)$.

42. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(2n+2)(2n+1)}{(n+1)(n+1)}|x| = 4|x|<1$ to converge, so $R=\frac14$.

43.

$f(x)=\sin x$	$f\left(\frac{\pi}{6}\right)=\frac12$	$f'''(x)=-\cos x$	$f'''\left(\frac{\pi}{6}\right)=-\frac{\sqrt3}{2}$
$f'(x)=\cos x$	$f'\left(\frac{\pi}{6}\right)=\frac{\sqrt3}{2}$	$f^{(4)}(x)=\sin x$	$f^{(4)}\left(\frac{\pi}{6}\right)=\frac12$
$f''(x)=-\sin x$	$f''\left(\frac{\pi}{6}\right)=-\frac12$	$\cdots$	$\cdots$

$f^{(2n)}\left(\frac{\pi}{6}\right)=(-1)^n\cdot\frac12$ and $f^{(2n+1)}\left(\frac{\pi}{6}\right)=(-1)^n\cdot\frac{\sqrt3}{2}$.

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\frac{\pi}{6}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2\,(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt3}{2\,(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}$$

44.

$f(x)=\cos x$	$f\left(\frac{\pi}{3}\right)=\frac12$	$f'''(x)=\sin x$	$f'''\left(\frac{\pi}{3}\right)=\frac{\sqrt3}{2}$
$f'(x)=-\sin x$	$f'\left(\frac{\pi}{3}\right)=-\frac{\sqrt3}{2}$	$f^{(4)}(x)=\cos x$	$f^{(4)}\left(\frac{\pi}{3}\right)=\frac12$
$f''(x)=-\cos x$	$f''\left(\frac{\pi}{3}\right)=-\frac12$		

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n\left(x-\frac{\pi}{3}\right)^{2n}}{2\,(2n)!} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\sqrt3\left(x-\frac{\pi}{3}\right)^{2n+1}}{2\,(2n+1)!}$$

45. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1 \;\Rightarrow\; \dfrac{x^2}{1+x} = \sum\limits_{n=0}^{\infty} (-1)^n x^{n+2}$ with $R = 1$.

46. $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1, 1]$, so

$\tan^{-1}(x^2) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^2)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+2}}{2n+1}$, which converges when $x^2 \in [-1, 1] \;\Leftrightarrow$ $x \in [-1, 1]$. Therefore, $R = 1$.

47. $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ for $|x| < 1 \;\Rightarrow\; \ln(1-x) = -\displaystyle\int \dfrac{dx}{1-x} = -\int \sum_{n=0}^{\infty} x^n\,dx = C - \sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1}$.

$\ln(1-0) = C - 0 \;\Rightarrow\; C = 0 \;\Rightarrow\; \ln(1-x) = -\sum\limits_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1} = \sum\limits_{n=1}^{\infty} \dfrac{-x^n}{n}$ with $R = 1$.

48. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \;\Rightarrow\; xe^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{x(2x)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{2^n x^{n+1}}{n!}$, $R = \infty$

49. $\sin x = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} \;\Rightarrow\; \sin(x^4) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{8n+4}}{(2n+1)!}$ for all x, so the radius of convergence is ∞.

50. $10^x = e^{x \ln 10} = \sum\limits_{n=0}^{\infty} \dfrac{(\ln 10)^n x^n}{n!}$, $R = \infty$

51. $f(x) = 1\,/\,\sqrt[4]{16-x} = (16-x)^{-1/4} = \frac{1}{2}\left(1 - \frac{1}{16}x\right)^{-1/4}$

$$= \tfrac{1}{2}\left[1 + \left(-\tfrac{1}{4}\right)\left(-\frac{x}{16}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\frac{x}{16}\right)^2 + \cdots\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1\cdot 5\cdot 9\cdot \cdots \cdot (4n-3)}{2\cdot 4^n\cdot n!\cdot 16^n} x^n = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1\cdot 5\cdot 9\cdot \cdots \cdot (4n-3)}{2^{6n+1}\, n!} x^n$$

for $\left|-\dfrac{x}{16}\right| < 1 \;\Rightarrow\; R = 16$.

52. $(1-3x)^{-5} = \sum\limits_{n=0}^{\infty} \dbinom{-5}{n} (-3x)^n = 1 + (-5)(-3x) + \dfrac{(-5)(-6)}{2!}(-3x)^2 + \dfrac{(-5)(-6)(-7)}{3!}(-3x)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{5\cdot 6\cdot \cdots \cdot (n+4)\cdot 3^n x^n}{n!},\ |-3x| < 1 \text{ so } R = \tfrac{1}{3}.$$

53. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ so $\dfrac{e^x}{x} = \dfrac{1}{x} + \sum\limits_{n=1}^{\infty} \dfrac{x^{n-1}}{n!}$ and $\displaystyle\int \dfrac{e^x}{x}\,dx = C + \ln|x| + \sum_{n=1}^{\infty} \dfrac{x^n}{n\cdot n!}$.

54. $(1+x^4)^{1/2} = \sum\limits_{n=0}^{\infty} \dbinom{\frac{1}{2}}{n} (x^4)^n = 1 + \left(\tfrac{1}{2}\right)x^4 + \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 + \cdots$

$$= 1 + \tfrac{1}{2}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{16}x^{12}$$

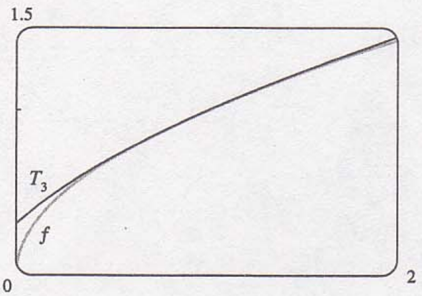
so $\int_0^1 (1+x^4)^{1/2}\,dx = \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots$. This is an alternating series, so by the Alternating Series Test, the error in the approximation $\int_0^1 (1+x^4)^{1/2}\,dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy. Thus, correct to two decimal places, $\int_0^1 (1+x^4)^{1/2}\,dx \approx 1.09$.

55. (a)

$f(x) = x^{1/2}$	$f(1) = 1$	$f'''(x) = \frac{3}{8}x^{-5/2}$	$f'''(1) = \frac{3}{8}$
$f'(x) = \frac{1}{2}x^{-1/2}$	$f'(1) = \frac{1}{2}$	$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$	
$f''(x) = -\frac{1}{4}x^{-3/2}$	$f''(1) = -\frac{1}{4}$		

$$\sqrt{x} \approx T_3(x) = 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$$
$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

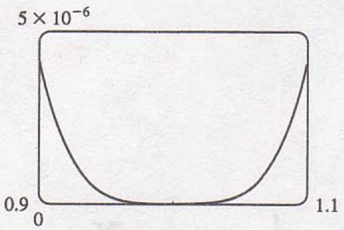
(b)



(c) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$ with $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \le x \le 1.1 \Rightarrow (x-1)^4 \le (0.1)^4$, and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so

$$|R_3(x)| \le \frac{15}{16(0.9)^{7/2}\,4!}(0.1)^4 \approx 0.000005648.$$

(d)

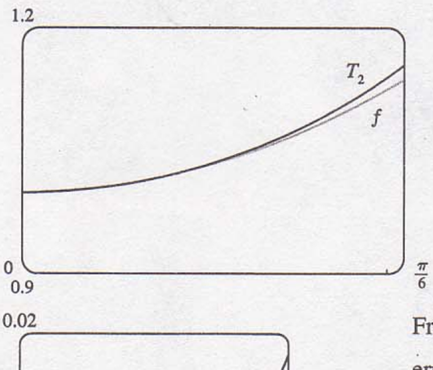


From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

56. (a)

$f(x) = \sec x$	$f(0) = 1$	$\sec x \approx T_2(x) = 1 + \frac{1}{2}x^2$
$f'(x) = \sec x \tan x$	$f'(0) = 0$	
$f''(x) = \sec x \tan^2 x + \sec^3 x$	$f''(0) = 1$	
$f'''(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$		

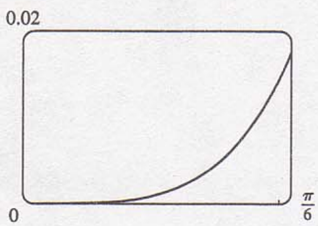
(b)



(c) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \le M$ with $f^{(3)}(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$. Now $0 \le x \le \frac{\pi}{6} \Rightarrow x^3 \le \left(\frac{\pi}{6}\right)^3$, and letting $x = \frac{\pi}{6}$ gives $M = \frac{14}{3}$, so

$$|R_2(x)| \le \tfrac{14}{3 \cdot 6}\left(\tfrac{\pi}{6}\right)^3 \approx 0.111648.$$

(d)



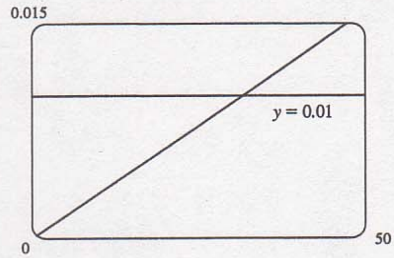
From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the error is less than 0.02 on $\left[0, \frac{\pi}{6}\right]$.

57. $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$, so $\sin x - x = -\dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ and $\dfrac{\sin x - x}{x^3} = -\dfrac{1}{3!} + \dfrac{x^2}{5!} - \dfrac{x^4}{7!} + \cdots$ and $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x^3} = \lim\limits_{x \to 0} \left(-\dfrac{1}{6} + \dfrac{x^2}{120} - \dfrac{x^4}{5040} + \cdots \right) = -\dfrac{1}{6}$.

58. (a) $F = \dfrac{mgR^2}{(R+h)^2} = \dfrac{mg}{(1+h/R)^2} = mg \sum\limits_{n=0}^{\infty} \binom{-2}{n} \left(\dfrac{h}{R} \right)^n$ (Binomial Series)

(b) We expand $F = mg\left[1 - 2(h/R) + 3(h/R)^2 - \cdots\right]$. This is an alternating series, so by the Alternating Series

Estimation Theorem, the error in the approximation $F = mg$ is less than $2mgh/R$, so for accuracy within 1% we want $\left| \dfrac{2mgh/R}{mgR^2/(R+h)^2} \right| < 0.01 \quad \Leftrightarrow \quad \dfrac{2h(R+h)^2}{R^3} < 0.01$. This inequality would be difficult to solve for h, so we substitute $R = 6{,}400$ km and plot both sides of the inequality. It appears that the approximation is accurate to within 1% for $h < 31$ km.



59. $f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad f(-x) = \sum_{n=0}^{\infty} c_n (-x)^n = \sum_{n=0}^{\infty} (-1)^n c_n x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n c_n x^n = \sum_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 12.10.5 [ET 11.10.5]), so $(-1)^n c_n = -c_n$. If n is even, then $(-1)^n = 1$, so $c_n = -c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all even coefficients are 0.

(b) If f is even, then $f(-x) = f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n c_n x^n = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad (-1)^n c_n = c_n$. If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all odd coefficients are 0.

60. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \quad \Rightarrow \quad e^{x^2} = \sum\limits_{n=0}^{\infty} \dfrac{(x^2)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{n!} = \sum\limits_{k=0}^{\infty} \dfrac{f^{(k)}(0)}{k!} x^k \quad \Rightarrow \quad \dfrac{f^{(2n)}(0)}{(2n)!} = \dfrac{1}{n!} \quad \Rightarrow$ $f^{(2n)}(0) = \dfrac{(2n)!}{n!}$.

Problems Plus

1. It would be far too much work to compute 15 derivatives of f. The key idea is to remember that $f^{(n)}(0)$ occurs in the coefficient of x^n in the Maclaurin series of f. We start with the Maclaurin series for sin: $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$. Then $\sin(x^3) = x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} - \cdots$ and so the coefficient of x^{15} is $\dfrac{f^{(15)}(0)}{15!} = \dfrac{1}{5!}$. Therefore, $f^{(15)}(0) = \dfrac{15!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 = 10{,}897{,}286{,}400$.

2. $|AP_2|^2 = 2$, $|AP_3|^2 = 2 + 2^2$, $|AP_4|^2 = 2 + 2^2 + (2^2)^2$, $|AP_5|^2 = 2 + 2^2 + (2^2)^2 + (2^3)^2, \ldots,$

$$\begin{aligned} |AP_n|^2 &= 2 + 2^2 + (2^2)^2 + \cdots + (2^{n-2})^2 \quad (\text{for } n \geq 3) \quad = 2 + (4 + 4^2 + 4^3 + \cdots + 4^{n-2}) \\ &= 2 + \frac{4(4^{n-2} - 1)}{4 - 1} \quad (\text{finite geometric sum with } a = 4,\ r = 4) \quad = \frac{6}{3} + \frac{4^{n-1} - 4}{3} = \frac{2}{3} + \frac{4^{n-1}}{3} \end{aligned}$$

So $\tan \angle P_nAP_{n+1} = \dfrac{|P_nP_{n+1}|}{|AP_n|} = \dfrac{2^{n-1}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \dfrac{\sqrt{4^{n-1}}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \dfrac{1}{\sqrt{\frac{2}{3 \cdot 4^{n-1}} + \frac{1}{3}}} \to \sqrt{3}$ as $n \to \infty$, so $\angle P_nAP_{n+1} \to \frac{\pi}{3}$ as $n \to \infty$.

3. (a) From Formula 14a in Appendix D, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, so $\cot 2\theta = \dfrac{1 - \tan^2\theta}{2\tan\theta} \Rightarrow$ $2\cot 2\theta = \dfrac{1 - \tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing θ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or $\tan\frac{1}{2}x = \cot\frac{1}{2}x - 2\cot x$.

(b) From part (a), $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the nth partial sum of $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}\tan\dfrac{x}{2^n}$ is

$$\begin{aligned} s_n &= \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \cdots + \frac{\tan(x/2^n)}{2^n} \\ &= \left[\frac{\cot(x/2)}{2} - \cot x\right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2}\right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4}\right] + \cdots \\ &\qquad + \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot(x/2^{n-1})}{2^{n-1}}\right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \quad (\text{telescoping sum}) \end{aligned}$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n \sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x} \cdot \dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x} \cdot 1 = \dfrac{1}{x}$ as $n \to \infty$ since $x/2^n \to 0$ for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq n\pi$, then $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}\tan\dfrac{x}{2^n} = \lim\limits_{n\to\infty}\left(-\cot x + \dfrac{1}{2^n}\cot\dfrac{x}{2^n}\right) = -\cot x + \dfrac{1}{x}$. If $x = 0$, then all terms in the series are 0, so the sum is 0.

4. We use the problem-solving strategy of taking cases:

Case (i): If $|x| < 1$, then $0 \le x^2 < 1$, so $\lim\limits_{n\to\infty} x^{2n} = 0$ (see Example 8 in Section 12.1 [ET 11.1])

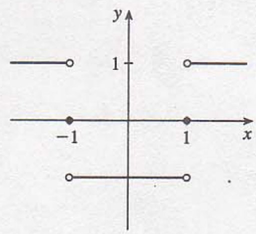
and $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1} = \dfrac{0-1}{0+1} = -1$.

Case (ii): If $|x| = 1$, that is, $x = \pm 1$, then $x^2 = 1$, so $f(x) = \lim\limits_{n\to\infty} \dfrac{1-1}{1+1} = 0$.

Case (iii): If $|x| > 1$, then $x^2 > 1$, so $\lim\limits_{n\to\infty} x^{2n} = \infty$ and

$$f(x) = \lim_{n\to\infty} \frac{x^{2n}-1}{x^{2n}+1} = \lim_{n\to\infty} \frac{1-(1/x^{2n})}{1+(1/x^{2n})} = \frac{1-0}{1+0} = 1.$$

Thus, $f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ -1 & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$



The graph shows that f is continuous everywhere except at $x = \pm 1$.

5. (a) At each stage, each side is replaced by four shorter sides, each of length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and ℓ_0 for the number of sides and the length of the side of the initial triangle, we generate the table at right. In general, we have $s_n = 3 \cdot 4^n$ and $\ell_n = \left(\frac{1}{3}\right)^n$, so the length of the perimeter at the nth stage of construction is $p_n = s_n \ell_n = 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n = 3 \cdot \left(\frac{4}{3}\right)^n$.

$s_0 = 3$	$\ell_0 = 1$
$s_1 = 3 \cdot 4$	$\ell_1 = 1/3$
$s_2 = 3 \cdot 4^2$	$\ell_2 = 1/3^2$
$s_3 = 3 \cdot 4^3$	$\ell_3 = 1/3^3$
$\cdots$	$\cdots$

(b) $p_n = \dfrac{4^n}{3^{n-1}} = 4\left(\dfrac{4}{3}\right)^{n-1}$. Since $\frac{4}{3} > 1$, $p_n \to \infty$ as $n \to \infty$.

(c) The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage. Let a be the area of the original triangle. Then the area a_n of each of the small triangles added at stage n is $a_n = a \cdot \dfrac{1}{9^n} = \dfrac{a}{9^n}$. Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at the nth stage is

$A_n = s_{n-1} \cdot a_n = 3 \cdot 4^{n-1} \cdot \dfrac{a}{9^n} = a \cdot \dfrac{4^{n-1}}{3^{2n-1}}$. Then the total area enclosed by the snowflake curve is

$A = a + A_1 + A_2 + A_3 + \cdots = a + a \cdot \dfrac{1}{3} + a \cdot \dfrac{4}{3^3} + a \cdot \dfrac{4^2}{3^5} + a \cdot \dfrac{4^3}{3^7} + \cdots$. After the first term, this is a geometric series with common ratio $\frac{4}{9}$, so $A = a + \dfrac{a/3}{1-\frac{4}{9}} = a + \dfrac{a}{3} \cdot \dfrac{9}{5} = \dfrac{8a}{5}$. But the area of the original equilateral triangle with side 1 is $a = \frac{1}{2} \cdot 1 \cdot \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}$. So the area enclosed by the snowflake curve is $\frac{8}{5} \cdot \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}$.

6. Let the series be S. Then every term in S is of the form $\dfrac{1}{2^m 3^n}$, $m, n \geq 0$, and furthermore each term occurs only once. So we can write

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2^m 3^n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2^m}\frac{1}{3^n} = \sum_{m=0}^{\infty}\frac{1}{2^m}\sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}} = 2\cdot\tfrac{3}{2} = 3$$

7. (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 14b in Appendix D,

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x)\tan(\arctan y)} = \frac{x-y}{1+xy} \quad \Rightarrow$$

$$\arctan x - \arctan y = a - b = \arctan\frac{x-y}{1+xy} \quad \text{since } -\tfrac{\pi}{2} < \arctan x - \arctan y < \tfrac{\pi}{2}$$

(b) From part (a) we have

$$\arctan\tfrac{120}{119} - \arctan\tfrac{1}{239} = \arctan\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}} = \arctan\frac{\frac{28{,}561}{28{,}441}}{\frac{28{,}561}{28{,}441}} = \arctan 1 = \tfrac{\pi}{4}$$

(c) Replacing y by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan\dfrac{x+y}{1-xy}$. So

$$4\arctan\tfrac{1}{5} = 2\left(\arctan\tfrac{1}{5} + \arctan\tfrac{1}{5}\right) = 2\arctan\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}} = 2\arctan\tfrac{5}{12} = \arctan\tfrac{5}{12} + \arctan\tfrac{5}{12}$$

$$= \arctan\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{5}{12}\cdot\frac{5}{12}} = \arctan\tfrac{120}{119}$$

Thus, from part (b), we have $4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4}$.

(d) From Example 7 in Section 12.9 [ET 11.9] we have $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$, so

$$\arctan\frac{1}{5} = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9} - \frac{1}{11\cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between s_5 and s_6, that is, $0.197395560 < \arctan\frac{1}{5} < 0.197395562$.

(e) From the series in part (d) we get $\arctan\dfrac{1}{239} = \dfrac{1}{239} - \dfrac{1}{3\cdot 239^3} + \dfrac{1}{5\cdot 239^5} - \cdots$. The third term is less than 2.6×10^{-13}, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan\frac{1}{239} \approx s_2 \approx 0.004184076$. Thus, $0.004184075 < \arctan\frac{1}{239} < 0.004184077$.

(f) From part (c) we have $\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$, so from parts (d) and (e) we have $16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \Rightarrow$ $3.141592652 < \pi < 3.141592692$. So, to 7 decimal places, $\pi \approx 3.1415927$.

8. (a) Let $a = \operatorname{arccot} x$ and $b = \operatorname{arccot} y$. Then

$$\cot(a-b) = \frac{1+\cot a \cot b}{\cot b - \cot a} = \frac{1+\cot(\operatorname{arccot} x)\cot(\operatorname{arccot} y)}{\cot(\operatorname{arccot} y) - \cot(\operatorname{arccot} x)} = \frac{1+xy}{y-x} \quad \Rightarrow$$

$$\operatorname{arccot} x - \operatorname{arccot} y = a - b = \operatorname{arccot} \frac{1+xy}{y-x}$$

(b) Applying the identity in part (a) with $x = n$ and $y = n+1$, we have

$$\operatorname{arccot}\left(n^2+n+1\right) = \operatorname{arccot}\left(1+n(n+1)\right) = \operatorname{arccot}\frac{1+n(n+1)}{(n+1)-n} = \operatorname{arccot} n - \operatorname{arccot}(n+1)$$

Thus, we have a telescoping series with nth partial sum

$$s_n = [\operatorname{arccot} 0 - \operatorname{arccot} 1] + [\operatorname{arccot} 1 - \operatorname{arccot} 2] + \cdots + [\operatorname{arccot} n - \operatorname{arccot}(n+1)] = \operatorname{arccot} 0 - \operatorname{arccot}(n+1)$$

Thus, $\sum_{n=0}^{\infty} \operatorname{arccot}\left(n^2+n+1\right) = \lim_{n\to\infty}\left[-\operatorname{arccot}(n+1)\right] = \frac{\pi}{2}$.

9. We start with the geometric series $\sum\limits_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2} \text{ for } |x| < 1. \text{ Differentiate again:}$$

$$\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{(1-x)^2 - x\cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^2x^n = \frac{x^2+x}{(1-x)^3} \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3x^{n-1} = \frac{d}{dx}\frac{x^2+x}{(1-x)^3} = \frac{(1-x)^3(2x+1) - \left(x^2+x\right)3(1-x)^2(-1)}{(1-x)^6} = \frac{x^2+4x+1}{(1-x)^4} \quad \Rightarrow$$

$\sum\limits_{n=1}^{\infty} n^3x^n = \dfrac{x^3+4x^2+x}{(1-x)^4}$, $|x| < 1$. The radius of convergence is 1 because that is the radius of convergence for the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3(\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

10. (a)
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\left(2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2} = 2\left(2\left(2\sin\frac{\theta}{8}\cos\frac{\theta}{8}\right)\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2}$$

$$= \cdots = 2\left(2\left(2\left(\cdots\left(2\left(2\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n}\right)\cos\frac{\theta}{2^{n-1}}\right)\cdots\right)\cos\frac{\theta}{8}\right)\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2}$$

$$= 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}$$

(b) $\sin\theta = 2^n\sin\dfrac{\theta}{2^n}\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}\cdots\cos\dfrac{\theta}{2^n} \quad \Leftrightarrow \quad \dfrac{\sin\theta}{\theta}\cdot\dfrac{\theta/2^n}{\sin(\theta/2^n)} = \cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}\cdots\cos\dfrac{\theta}{2^n}$.

Now we let $n\to\infty$, using $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ with $x = \dfrac{\theta}{2^n}$:

$$\lim_{n\to\infty}\left[\frac{\sin\theta}{\theta}\cdot\frac{\theta/2^n}{\sin(\theta/2^n)}\right] = \lim_{n\to\infty}\left[\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}\right] \quad \Leftrightarrow \quad \frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots.$$

(c) If we take $\theta = \frac{\pi}{2}$ in the result from part (b) and use the half-angle formula $\cos x = \sqrt{\frac{1}{2}\left(1 + \cos 2x\right)}$ (see Formula 17a in Appendix D), we get

$$\frac{\sin \pi/2}{\pi/2} = \cos\tfrac{\pi}{4}\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}\sqrt{\frac{\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}+1}{2}}\sqrt{\frac{\sqrt{\frac{\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}+1}{2}}+1}{2}}\cdots \quad\Rightarrow$$

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}+1}{2}}\cdots = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\sqrt{\frac{\frac{\sqrt{2+\sqrt{2}}}{2}+1}{2}}\cdots$$

$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

11. $a_{n+1} = \dfrac{a_n + b_n}{2}$, $b_{n+1} = \sqrt{b_n a_{n+1}}$. So $a_1 = \cos\theta$, $b_1 = 1 \quad\Rightarrow\quad a_2 = \dfrac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2}$,

$b_2 = \sqrt{b_1 a_2} = \sqrt{\cos^2\frac{\theta}{2}} = \cos\frac{\theta}{2}$ since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then

$a_3 = \frac{1}{2}\left(\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) = \cos\frac{\theta}{2}\cdot\frac{1}{2}\left(1+\cos\frac{\theta}{2}\right) = \cos\frac{\theta}{2}\cos^2\frac{\theta}{4} \quad\Rightarrow$

$b_3 = \sqrt{b_2 a_3} = \sqrt{\cos\frac{\theta}{2}\cos\frac{\theta}{2}\cos^2\frac{\theta}{4}} = \cos\frac{\theta}{2}\cos\frac{\theta}{4} \quad\Rightarrow$

$a_4 = \frac{1}{2}\left(\cos\frac{\theta}{2}\cos^2\frac{\theta}{4} + \cos\frac{\theta}{2}\cos\frac{\theta}{4}\right) = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cdot\frac{1}{2}\left(1+\cos\frac{\theta}{4}\right) = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos^2\frac{\theta}{8} \quad\Rightarrow$

$b_4 = \sqrt{\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos^2\frac{\theta}{8}} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}$. By now we see the pattern:

$b_n = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdot\cdots\cdot\cos\frac{\theta}{2^{n-1}}$ and $a_n = b_n\cos\frac{\theta}{2^{n-1}}$. (This could be proved by mathematical induction.) By Exercise 10(a), $\sin\theta = 2^{n-1}\sin\frac{\theta}{2^{n-1}}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^{n-1}}$. So

$b_n = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^{n-1}} \to \frac{\sin\theta}{\theta}$ as $n\to\infty$ by Exercise 10(b), and

$a_n = b_n\cos\frac{\theta}{2^{n-1}} \to \frac{\sin\theta}{\theta}\cdot 1 = \frac{\sin\theta}{\theta}$ as $n\to\infty$. So $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \frac{\sin\theta}{\theta}$.

12. Let's first try the case $k=1$: $a_0 + a_1 = 0 \quad\Rightarrow\quad a_1 = -a_0 \quad\Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1}\right) = \lim_{n\to\infty}\left(a_0\sqrt{n} - a_0\sqrt{n+1}\right) = a_0\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right)\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}$$

$$= a_0\lim_{n\to\infty}\frac{-1}{\sqrt{n}+\sqrt{n+1}} = 0$$

In general we have $a_0 + a_1 + \cdots + a_k = 0 \quad\Rightarrow\quad a_k = -a_0 - a_1 - \cdots - a_{k-1} \quad\Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right)$$

$$= \lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_{k-1}\sqrt{n+k-1} - a_0\sqrt{n+k} - a_1\sqrt{n+k} - \cdots - a_{k-1}\sqrt{n+k}\right)$$

$$= a_0\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+k}\right) + a_1\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n+k}\right) + \cdots + a_{k-1}\lim_{n\to\infty}\left(\sqrt{n+k-1}-\sqrt{n+k}\right)$$

Each of these limits is 0 by the same type of simplification as in the case $k=1$. So we have

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = a_0(0) + a_1(0) + \cdots + a_{k-1}(0) = 0$$

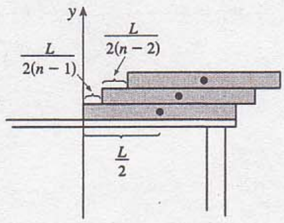
13. Let $f(x) = \sum_{m=0}^{\infty} c_m x^m$ and $g(x) = e^{f(x)} = \sum_{n=0}^{\infty} d_n x^n$. Then $g'(x) = \sum_{n=0}^{\infty} n d_n x^{n-1}$, so $n d_n$ occurs as the coefficient of x^{n-1}. But also

$$\begin{aligned} g'(x) &= e^{f(x)} f'(x) = \left(\textstyle\sum_{n=0}^{\infty} d_n x^n\right)\left(\textstyle\sum_{m=1}^{\infty} m c_m x^{m-1}\right) \\ &= \left(d_0 + d_1 x + d_2 x^2 + \cdots + d_{n-1} x^{n-1} + \cdots\right)\left(c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1} + \cdots\right) \end{aligned}$$

so the coefficient of x^{n-1} is $c_1 d_{n-1} + 2c_2 d_{n-2} + 3c_3 d_{n-3} + \cdots + n c_n d_0 = \sum_{i=1}^{n} i c_i d_{n-i}$. Therefore, $n d_n = \sum_{i=1}^{n} i c_i d_{n-i}$.

14. Place the y-axis as shown and let the length of each book be L. We want to show that the center of mass of the system of n books lies above the table, that is, $\overline{x} < L$. The x-coordinates of the centers of mass of the books are $x_1 = \dfrac{L}{2}$, $x_2 = \dfrac{L}{2(n-1)} + \dfrac{L}{2}$,

$x_3 = \dfrac{L}{2(n-1)} + \dfrac{L}{2(n-2)} + \dfrac{L}{2}$, and so on.



Each book has the same mass m, so if there are n books, then

$$\begin{aligned} \overline{x} &= \frac{mx_1 + mx_2 + \cdots + mx_n}{mn} = \frac{x_1 + x_2 + \cdots + x_n}{n} \\ &= \frac{1}{n}\left[\frac{L}{2} + \left(\frac{L}{2(n-1)} + \frac{L}{2}\right) + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2}\right) + \cdots \right. \\ &\qquad \left. + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \cdots + \frac{L}{4} + \frac{L}{2} + \frac{L}{2}\right)\right] \\ &= \frac{L}{n}\left[\frac{n-1}{2(n-1)} + \frac{n-2}{2(n-2)} + \cdots + \frac{2}{4} + \frac{1}{2} + \frac{n}{2}\right] = \frac{L}{n}\left[(n-1)\frac{1}{2} + \frac{n}{2}\right] = \frac{2n-1}{2n} L < L \end{aligned}$$

This shows that, no matter how many books are added according to the given scheme, the center of mass lies above the table. It remains to observe that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{1}{2}\sum(1/n)$ is divergent (harmonic series), so we can make the top book extend as far as we like beyond the edge of the table if we add enough books.

15. $u = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots$, $v = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots$, $w = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots$. The key idea is to differentiate: $\dfrac{du}{dx} = \dfrac{3x^2}{3!} + \dfrac{6x^5}{6!} + \dfrac{9x^8}{9!} + \cdots = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots = w$. Similarly, $\dfrac{dv}{dx} = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots = u$, and $\dfrac{dw}{dx} = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots = v$. So $u' = w$, $v' = u$, and $w' = v$. Now differentiate the left hand side of the desired equation:

$$\begin{aligned} \frac{d}{dx}\left(u^3 + v^3 + w^3 - 3uvw\right) &= 3u^2u' + 3v^2v' + 3w^2w' - 3\left(u'vw + uv'w + uvw'\right) \\ &= 3u^2w + 3v^2u + 3w^2v - 3\left(vw^2 + u^2w + uv^2\right) = 0 \quad \Rightarrow \end{aligned}$$

$u^3 + v^3 + w^3 - 3uvw = C$. To find the value of the constant C, we put $x = 0$ in the last equation and get $1^3 + 0^3 + 0^3 - 3(1 \cdot 0 \cdot 0) = C \;\Rightarrow\; C = 1$, so $u^3 + v^3 + w^3 - 3uvw = 1$.

16. First notice that both series are absolutely convergent (p-series with $p > 1$.) Let the given expression be called x. Then

$$x = \frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \frac{1 + \left(2 \cdot \frac{1}{2^p} - \frac{1}{2^p}\right) + \frac{1}{3^p} + \left(2 \cdot \frac{1}{4^p} - \frac{1}{4^p}\right) + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

$$= \frac{\left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots\right) + \left(2 \cdot \frac{1}{2^p} + 2 \cdot \frac{1}{4^p} + 2 \cdot \frac{1}{6^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

$$= 1 + \frac{2\left(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \frac{1}{8^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + \frac{\frac{1}{2^{p-1}}\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + 2^{1-p}x$$

Therefore, $x = 1 + 2^{1-p}x \;\Leftrightarrow\; x - 2^{1-p}x = 1 \;\Leftrightarrow\; x\left(1 - 2^{1-p}\right) = 1 \;\Leftrightarrow\; x = \dfrac{1}{1 - 2^{1-p}}$.

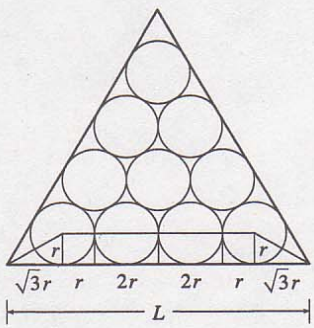
17. If L is the length of a side of the equilateral triangle, then the area is $A = \frac{1}{2}L \cdot \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$ and so $L^2 = \frac{4}{\sqrt{3}}A$. Let r be the radius of one of the circles. When there are n rows of circles, the figure shows that $L = \sqrt{3}r + r + (n-2)(2r) + r + \sqrt{3}r = r\left(2n - 2 + 2\sqrt{3}\right)$, so $r = \dfrac{L}{2\left(n + \sqrt{3} - 1\right)}$. The number of circles is $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ and so the total area of the circles is

$$A_n = \frac{n(n+1)}{2}\pi r^2 = \frac{n(n+1)}{2}\pi \frac{L^2}{4\left(n + \sqrt{3} - 1\right)^2} = \frac{n(n+1)}{2}\pi \frac{4A/\sqrt{3}}{4\left(n + \sqrt{3} - 1\right)^2}$$

$$= \frac{n(n+1)}{\left(n + \sqrt{3} - 1\right)^2}\frac{\pi A}{2\sqrt{3}} \quad \Rightarrow$$

$$\frac{A_n}{A} = \frac{n(n+1)}{\left(n + \sqrt{3} - 1\right)^2}\frac{\pi}{2\sqrt{3}}$$

$$= \frac{1 + 1/n}{\left[1 + \left(\sqrt{3} - 1\right)/n\right]^2}\frac{\pi}{2\sqrt{3}} \to \frac{\pi}{2\sqrt{3}} \text{ as } n \to \infty$$



18. Given $a_0 = a_1 = 1$ and $a_n = \dfrac{(n-1)(n-2)\,a_{n-1} - (n-3)\,a_{n-2}}{n\,(n-1)}$, we calculate the next few terms of the sequence: $a_2 = \dfrac{1 \cdot 0 \cdot a_1 - (-1)\,a_0}{2 \cdot 1} = \dfrac{1}{2}$, $a_3 = \dfrac{2 \cdot 1 \cdot a_2 - 0a_1}{3 \cdot 2} = \dfrac{1}{6}$, $a_4 = \dfrac{3 \cdot 2 \cdot a_3 - 1a_2}{4 \cdot 3} = \dfrac{1}{24}$. It seems that $a_n = \dfrac{1}{n!}$, so we try to prove this by induction. The first step is done, so assume $a_k = \dfrac{1}{k!}$ and $a_{k-1} = \dfrac{1}{(k-1)!}$. Then

$$a_{k+1} = \frac{k\,(k-1)\,a_k - (k-2)\,a_{k-1}}{(k+1)\,k} = \frac{\dfrac{k\,(k-1)}{k!} - \dfrac{k-2}{(k-1)!}}{(k+1)\,k} = \frac{(k-1)-(k-2)}{[(k+1)\,(k)]\,(k-1)!} = \frac{1}{(k+1)!}$$

and the induction is complete. Therefore, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 1/n! = e$.

19. Call the series S. We group the terms according to the number of digits in their denominators:

$$S = \underbrace{\left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + \tfrac{1}{9}\right)}_{g_1} + \underbrace{\left(\tfrac{1}{11} + \cdots + \tfrac{1}{99}\right)}_{g_2} + \underbrace{\left(\tfrac{1}{111} + \cdots + \tfrac{1}{999}\right)}_{g_3} + \cdots$$

Now in the group g_n, there are 9^n terms, since we have 9 choices for each of the n digits in the denominator. Furthermore, each term in g_n is less than $\frac{1}{10^{n-1}}$. So $g_n < 9^n \cdot \frac{1}{10^{n-1}} = 9\left(\frac{9}{10}\right)^{n-1}$. Now $\sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1}$ is a geometric series with $a = 9$ and $r = \frac{9}{10} < 1$. Therefore, by the Comparison Test, $S = \sum_{n=1}^{\infty} g_n < \sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1} = \frac{9}{1 - 9/10} = 90$.

20. (a) Since P_n is defined as the midpoint of $P_{n-4}P_{n-3}$, $x_n = \frac{1}{2}\left(x_{n-4} + x_{n-3}\right)$ for $n \geq 5$. So we prove by induction that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$. The case $n = 1$ is immediate, since $\frac{1}{2}0 + 1 + 1 + 0 = 2$. Assume that the result holds for $n = k - 1$, that is, $\frac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2$. Then for $n = k$,

$$\begin{aligned} \tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + x_{k+3} &= \tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + \tfrac{1}{2}\left(x_{k+3-4} + x_{k+3-3}\right) \quad \text{(by above)} \\ &= \tfrac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2 \quad \text{(by the induction hypothesis)} \end{aligned}$$

Similarly, for $n \geq 5$, $y_n = \frac{1}{2}\left(y_{n-4} + y_{n-3}\right)$, so the same argument as above holds for y, with 2 replaced by $\frac{1}{2}y_1 + y_2 + y_3 + y_4 = \frac{1}{2}1 + 1 + 0 + 0 = \frac{3}{2}$. So $\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2}$ for all n.

(b) $\lim\limits_{n\to\infty}\left(\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}\right) = \frac{1}{2}\lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} x_{n+1} + \lim\limits_{n\to\infty} x_{n+2} + \lim\limits_{n\to\infty} x_{n+3} = 2$. Since all the limits on the left hand side are the same, we get $\frac{7}{2}\lim\limits_{n\to\infty} x_n = 2 \;\Rightarrow\; \lim\limits_{n\to\infty} x_n = \frac{4}{7}$. In the same way, $\lim\limits_{n\to\infty} y_n = \frac{3}{7}$, so $P = \left(\frac{4}{7}, \frac{3}{7}\right)$.

13 Vectors and the Geometry of Space

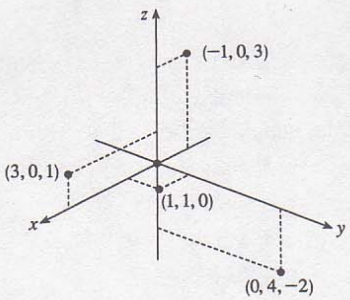
ET 12

13.1 Three-Dimensional Coordinate Systems

ET 12.1

1. We start at the origin, which has coordinates $(0,0,0)$. First we move 4 units along the positive x-axis, affecting only the x-coordinate, bringing us to the point $(4,0,0)$. We then move 3 units straight downward, in the negative z-direction. Thus only the z-coordinate is affected, and we arrive at $(4,0,-3)$.

2.

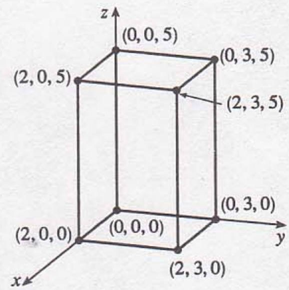


3. The distance from a point to the xz-plane is the absolute value of the y-coordinate of the point. $Q(-5,-1,4)$ has the y-coordinate with the smallest absolute value, so Q is the point closest to the xz-plane. $R(0,3,8)$ must lie in the yz-plane since the distance from R to the yz-plane, given by the x-coordinate of R, is 0.

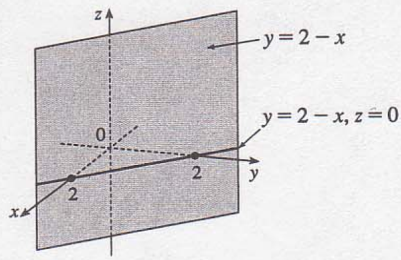
4. The projection of $(2,3,5)$ on the xy-plane is $(2,3,0)$; on the yz-plane, $(0,3,5)$; on the xz-plane, $(2,0,5)$.

The length of the diagonal of the box is the distance between the origin and $(2,3,5)$, given by

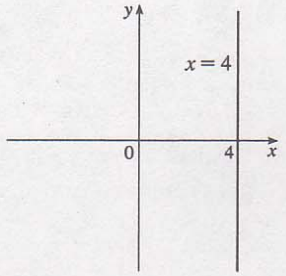
$$\sqrt{(2-0)^2+(3-0)^2+(5-0)^2}=\sqrt{38}\approx 6.16.$$

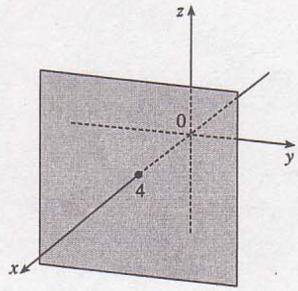


5. The equation $x+y=2$ represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates have a sum of 2, or equivalently where $y=2-x$. This is the set $\{(x,2-x,z) \mid x\in\mathbb{R}, z\in\mathbb{R}\}$ which is a vertical plane that intersects the xy-plane in the line $y=2-x$, $z=0$.

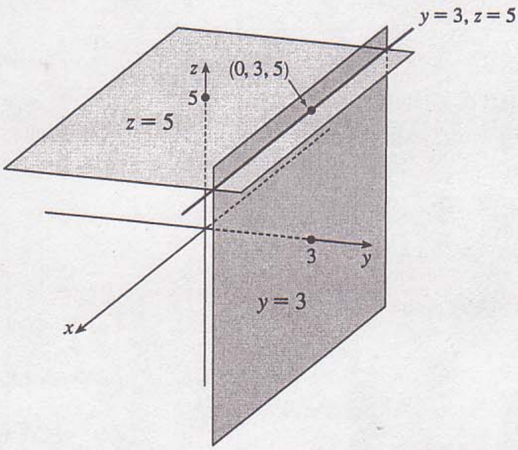


6. (a) In $\mathbb{R}^2$, the equation $x = 4$ represents a line parallel to the y-axis. In $\mathbb{R}^3$, the equation $x = 4$ represents the set $\{(x, y, z) \mid x = 4\}$, the set of all points whose x-coordinate is 4. This is the vertical plane that is parallel to the yz-plane and 4 units in front of it.





(b) In $\mathbb{R}^3$, the equation $y = 3$ represents a vertical plane that is parallel to the xz-plane and 3 units to the right of it. The equation $z = 5$ represents a horizontal plane parallel to the xy-plane and 5 units above it. The pair of equations $y = 3$, $z = 5$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y = 3$, $z = 5$. This line can also be described as the set $\{(x, 3, 5) \mid x \in \mathbb{R}\}$, which is the set of all points in $\mathbb{R}^3$ whose x-coordinate may vary but whose y- and z-coordinates are fixed at 3 and 5, respectively. Thus the line is parallel to the x-axis and intersects the yz-plane in the point $(0, 3, 5)$.



7. We first find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{[1 - (-2)]^2 + (2 - 4)^2 + (-1 - 0)^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$$

$$|QR| = \sqrt{(-1 - 1)^2 + (1 - 2)^2 + [2 - (-1)]^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|PR| = \sqrt{[-1 - (-2)]^2 + (1 - 4)^2 + (2 - 0)^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$$

Since all three sides have the same length, PQR is an equilateral triangle.

8. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|AB| = \sqrt{(5-3)^2 + [-3-(-4)]^2 + (0-1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|BC| = \sqrt{(6-5)^2 + [-7-(-3)]^2 + (4-0)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$|CA| = \sqrt{(3-6)^2 + [-4-(-7)]^2 + (1-4)^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$

Since the Pythagorean Theorem is satisfied by $|AB|^2 + |CA|^2 = |BC|^2$, ABC is a right triangle. ABC is not isosceles, as no two sides have the same length.

9. (a) First we find the distances between points:

$$|AB| = \sqrt{(7-5)^2 + (9-1)^2 + (-1-3)^2} = \sqrt{84} = 2\sqrt{21}$$

$$|BC| = \sqrt{(1-7)^2 + (-15-9)^2 + [11-(-1)]^2} = \sqrt{756} = 6\sqrt{21}$$

$$|AC| = \sqrt{(1-5)^2 + (-15-1)^2 + (11-3)^2} = \sqrt{336} = 4\sqrt{21}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $|AB| + |AC| = |BC|$, the three points lie on a straight line.

(b) The distances between points are

$$|KL| = \sqrt{(1-0)^2 + (2-3)^2 + [-2-(-4)]^2} = \sqrt{6}$$

$$|LM| = \sqrt{(3-1)^2 + (0-2)^2 + [1-(-2)]^2} = \sqrt{17}$$

$$|KM| = \sqrt{(3-0)^2 + (0-3)^2 + [1-(-4)]^2} = \sqrt{43}$$

Since $\sqrt{6} + \sqrt{17} \neq \sqrt{43}$, the three points do not lie on a straight line.

10. (a) The distance from a point to the xy-plane is the absolute value of the z-coordinate of the point. Thus, the distance is $|-5| = 5$.

(b) Similarly, the distance is the absolute value of the x-coordinate of the point: $|3| = 3$.

(c) The distance is the absolute value of the y-coordinate of the point: $|7| = 7$.

(d) The point on the x-axis closest to $(3, 7, -5)$ is the point $(3, 0, 0)$. (Approach the x-axis perpendicularly.) The distance from $(3, 7, -5)$ to the x-axis is the distance between these two points:
$\sqrt{(3-3)^2 + (7-0)^2 + (-5-0)^2} = \sqrt{74} \approx 8.60$.

(e) The point on the y-axis closest to $(3, 7, -5)$ is $(0, 7, 0)$. The distance between these points is
$\sqrt{(3-0)^2 + (7-7)^2 + (-5-0)^2} = \sqrt{34} \approx 5.83$.

(f) The point on the z-axis closest to $(3, 7, -5)$ is $(0, 0, -5)$. The distance between these points is
$\sqrt{(3-0)^2 + (7-0)^2 + [-5-(-5)]^2} = \sqrt{58} \approx 7.62$.

11. An equation of the sphere with center $(0, 1, -1)$ and radius 4 is $(x-0)^2 + (y-1)^2 + [z-(-1)]^2 = 4^2$ or $x^2 + (y-1)^2 + (z+1)^2 = 16$. The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ in the equation, we have $(y-1)^2 + (z+1)^2 = 16$, $x = 0$, which represents a circle in the yz-plane with center $(0, 1, -1)$ and radius 4.

12. An equation of the sphere with center $(6, 5, -2)$ and radius $\sqrt{7}$ is $(x-6)^2 + (y-5)^2 + [z-(-2)]^2 = \left(\sqrt{7}\right)^2$ or $(x-6)^2 + (y-5) + (z+2)^2 = 7$. The intersection of this sphere with the xy-plane is the set of points on the sphere whose z-coordinate is 0. Putting $z = 0$ into the equation, we have $(x-6)^2 + (y-5)^2 = 3, z = 0$ which represents a circle in the xy-plane with center $(6, 5, 0)$ and radius $\sqrt{3}$. To find the intersection with the xz-plane, we set $y = 0$: $(x-6)^2 + (z+2)^2 = -18$. Since no points satisfy this equation, the sphere does not intersect the xz-plane. (Also note that the distance from the center of the sphere to the xz-plane is greater than the radius of the sphere.) Similarly, the sphere does not intersect the yz-plane since substituting $x = 0$ into the equation gives $(y-5)^2 + (z+2)^2 = -29$.

13. The radius of the sphere is the distance between $(4, 3, -1)$ and $(3, 8, 1)$:
$r = \sqrt{(3-4)^2 + (8-3)^2 + [1-(-1)]^2} = \sqrt{30}$. Thus, an equation of the sphere is
$(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

14. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1, 2, 3)$: $r = \sqrt{(1-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{14}$. Then an equation of the sphere is
$(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$.

15. Completing squares in the equation gives
$(x^2 + 2x + 1) + (y^2 + 8y + 16) + (z^2 - 4z + 4) = 28 + 1 + 16 + 4 \quad \Rightarrow$
$(x+1)^2 + (y+4)^2 + (z-2)^2 = 49$ which we recognize as an equation of a sphere with center $(-1, -4, 2)$ and radius 7.

16. Completing squares in the equation gives
$(x^2 - 4x + 4) + (y^2 + 2y + 1) + z^2 = 0 + 4 + 1 \quad \Rightarrow \quad (x-2)^2 + (y+1)^2 + z^2 = 5$ which we recognize as an equation of a sphere with center $(2, -1, 0)$ and radius $\sqrt{5}$.

17. Completing squares in the equation gives $\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) + \left(z^2 - z + \frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \quad \Rightarrow$
$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{3}{4}$ which we recognize as an equation of a sphere with center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and radius $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

18. Completing squares in the equation gives $2x^2 + 2\left(y^2 + 2y + 1\right) + 2\left(z^2 - z + \frac{1}{4}\right) = 1 + 2 + \frac{1}{2} = \frac{7}{2}$
$\Rightarrow \quad (x-0)^2 + (y+1)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{7}{4}$ which we recognize as an equation of a sphere with center $\left(0, -1, \frac{1}{2}\right)$ and radius $\frac{\sqrt{7}}{2}$.

19. (a) If the midpoint of the line segment from $P_1\,(x_1, y_1, z_1)$ to $P_2\,(x_2, y_2, z_2)$ is $Q = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$, then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$. We verify that this is the case:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\begin{aligned}
|P_1Q| &= \sqrt{\left[\tfrac{1}{2}(x_1 + x_2) - x_1\right]^2 + \left[\tfrac{1}{2}(y_1 + y_2) - y_1\right]^2 + \left[\tfrac{1}{2}(z_1 + z_2) - z_1\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} \\
&= \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]} \\
&= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \tfrac{1}{2}\,|P_1P_2|
\end{aligned}$$

$$\begin{aligned}
|QP_2| &= \sqrt{\left[x_2 - \tfrac{1}{2}(x_1 + x_2)\right]^2 + \left[y_2 - \tfrac{1}{2}(y_1 + y_2)\right]^2 + \left[z_2 - \tfrac{1}{2}(z_1 + z_2)\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} \\
&= \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]} \\
&= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \tfrac{1}{2}\,|P_1P_2|
\end{aligned}$$

So Q is indeed the midpoint of P_1P_2.

(b) By part (a), the midpoints of sides AB, BC and CA are $P_1\left(-\frac{1}{2}, 1, 4\right)$, $P_2\left(1, \frac{1}{2}, 5\right)$ and $P_3\left(\frac{5}{2}, \frac{3}{2}, 4\right)$. (Recall that a median of a triangle is a line segment from a vertex to the mipoint of the opposite side.) Then the lengths of the medians are:

$$\begin{aligned}
|AP_2| &= \sqrt{0^2 + \left(\tfrac{1}{2} - 2\right)^2 + (5 - 3)^2} = \sqrt{\tfrac{9}{4} + 4} = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2} \\
|BP_3| &= \sqrt{\left(\tfrac{5}{2} + 2\right)^2 + \left(\tfrac{3}{2}\right)^2 + (4 - 5)^2} = \sqrt{\tfrac{81}{4} + \tfrac{9}{4} + 1} = \sqrt{\tfrac{94}{4}} = \tfrac{1}{2}\sqrt{94} \\
|CP_1| &= \sqrt{\left(-\tfrac{1}{2} - 4\right)^2 + (1 - 1)^2 + (4 - 5)^2} = \sqrt{\tfrac{81}{4} + 1} = \tfrac{1}{2}\sqrt{85}
\end{aligned}$$

20. By Exercise 19(a), the midpoint of the diameter (and thus the center of the sphere) is $C\,(3, 2, 7)$. The radius is half the diameter, so $r = \frac{1}{2}\sqrt{(4 - 2)^2 + (3 - 1)^2 + (10 - 4)^2} = \frac{1}{2}\sqrt{44} = \sqrt{11}$. Therefore an equation of the sphere is $(x - 3)^2 + (y - 2)^2 + (z - 7)^2 = 11$.

21. (a) Since the sphere touches the xy-plane, its radius is the distance from its center, $(2, -3, 6)$, to the xy-plane, namely 6. Therefore $r = 6$ and an equation of the sphere is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 6^2 = 36$.

(b) The radius of this sphere is the distance from its center $(2, -3, 6)$ to the yz-plane, which is 2. Therefore, an equation is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 4$.

(c) Here the radius is the distance from the center $(2, -3, 6)$ to the xz-plane, which is 3. Therefore, an equation is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 9$.

22. The largest sphere contained in the first octant must have a radius equal to the minimum distance from the center $(5, 4, 9)$ to any of the three coordinate planes. The shortest such distance is to the xz-plane, a distance of 4. Thus an equation of the sphere is $(x-5)^2 + (y-4)^2 + (z-9)^2 = 16$.

23. The equation $x = 9$ represents a plane parallel to the yz-plane and 9 units in front of it.

24. The equation $z = -8$ represents a plane parallel to the xy-plane and 8 units below it.

25. The inequality $y > 2$ represents a half-space containing all points to the right of the plane $y = 2$.

26. The inequality $z \leq 0$ represents a half-space containing all points on and below the xy-plane.

27. The inequality $|z| \leq 2$ is equivalent to $-2 \leq z \leq 2$, so it represents all points on and between the two horizontal planes $z = 2$ and $z = -2$.

28. The equation $y = z$ represents a plane perpendicular to the yz-plane and intersecting the yz-plane in the line $y = z$, $x = 0$.

29. The inequality $x^2 + y^2 + z^2 > 1$ is equivalent to $\sqrt{x^2 + y^2 + z^2} > 1$, so the region consists of those points whose distance from the origin is greater than 1. This is the set of all points outside the sphere with radius 1 and center $(0, 0, 0)$.

30. The inequality $1 \leq x^2 + y^2 + z^2 \leq 25$ is equivalent to $1 \leq \sqrt{x^2 + y^2 + z^2} \leq 5$, so the region consists of those points whose distance from the origin is at least 1 and at most 5. This is the set of all points on and between the concentric spheres with radii 1 and 5 and center $(0, 0, 0)$.

31. Completing the square in z gives $x^2 + y^2 + \left(z^2 - 2z + 1\right) < 3 + 1$ or $x^2 + y^2 + (z-1)^2 < 4$, which is equivalent to $\sqrt{x^2 + y^2 + (z-1)^2} < 2$. Thus the region consists of those points whose distance from the point $(0, 0, 1)$ is less than 2. This is the set of all points inside the sphere with radius 2 and center $(0, 0, 1)$.

32. The equation $x^2 + y^2 = 1$ represents the set of all points in $\mathbb{R}^3$ where $x^2 + y^2 = 1$, a surface that intersects the xy-plane in the circle $x^2 + y^2 = 1$, $z = 0$. Since z can vary, the surface is a circular cylinder of radius 1. Thus, the equation represents the region consisting of all points on a circular cylinder of radius 1 with axis the z-axis.

33. Here $x^2 + z^2 \leq 9$ or equivalently $\sqrt{x^2 + z^2} \leq 3$ which describes the set of all points in $\mathbb{R}^3$ whose distance from the y-axis is at most 3. Thus, the inequality represents the region consisting of all points on and inside a circular cylinder of radius 3 with axis the y-axis.

34. The equation $xyz = 0$ is satisfied when any of x, y, or z is 0. Thus, the equation represents the region consisting of all points on the three coordinate planes $x = 0$, $y = 0$, and $z = 0$.

35. This describes all points with negative y-coordinates, that is, $y < 0$.

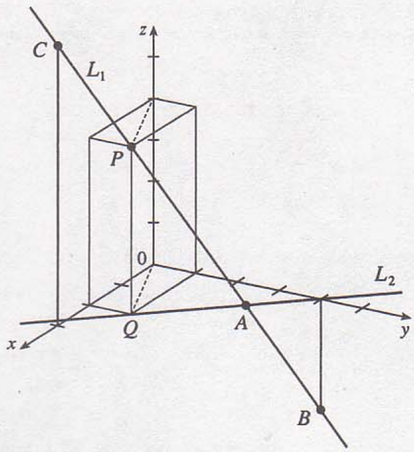
36. Because the box lies in the first quadrant, each point must comprise only non-negative coordinates. So inequalities describing the region are $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$.

37. This describes a region all of whose points have a distance to the origin which is greater than r, but smaller than R. So inequalities describing the region are $r < \sqrt{x^2 + y^2 + z^2} < R$, or $r^2 < x^2 + y^2 + z^2 < R^2$.

38. The solid sphere itself is represented by $\sqrt{x^2 + y^2 + z^2} \leq 2$. Since we want only the upper hemisphere, we restrict the z-coordinate to non-negative values. Then inequalities describing the region are $\sqrt{x^2 + y^2 + z^2} \leq 2$, $z \geq 0$, or $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$.

39. (a) To find the x- and y-coordinates of the point P, we project it onto L_2 and project the resulting point Q onto the x- and y-axes. To find the z-coordinate, we project P onto either the xz-plane or the yz-plane (using our knowledge of its x- or y-coordinate) and then project the resulting point onto the z-axis. (Or, we could draw a line parallel to QO from P to the z-axis.) The coordinates of P are $(2, 1, 4)$.

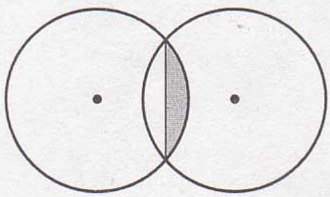
(b) A is the intersection of L_1 and L_2, B is directly below the y-intercept of L_2, and C is directly above the x-intercept of L_2.



40. Let $P = (x, y, z)$. Then $2\,|PB| = |PA| \;\Leftrightarrow\; 4\,|PB|^2 = |PA|^2 \;\Leftrightarrow$
$4\left((x-6)^2 + (y-2)^2 + (z+2)^2\right) = (x+1)^2 + (y-5)^2 + (z-3)^2 \;\Leftrightarrow$
$4\left(x^2 - 12x + 36\right) - x^2 - 2x + 4\left(y^2 - 4y + 4\right) - y^2 + 10y + 4\left(z^2 + 4z + 4\right) - z^2 + 6z = 35 \;\Leftrightarrow$
$3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = 35 - 144 - 16 - 16 \;\Leftrightarrow\; x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$. By completing the square three times we get $\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \frac{332}{9}$, which is an equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

41. We need to find a set of points $\left\{P(x, y, z) \mid |AP| = |BP|\right\}$.
$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2} \;\Rightarrow$
$(x+1)^2 + (y-5) + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2 \;\Rightarrow$
$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4 \;\Rightarrow$
$14x - 6y - 10z = 9$. Thus the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

42. Completing the square three times in the first equation gives $(x+2)^2 + (y-1)^2 + (z+2)^2 = 2^2$, a sphere with center $(-2, 1, 2)$ and radius 2. The second equation is that of a sphere with center $(0, 0, 0)$ and radius 2. The distance between the centers of the spheres is $\sqrt{(-2-0)^2 + (1-0)^2 + (-2-0)^2} = \sqrt{4+1+4} = 3$.



Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h = 2 - \frac{3}{2} = \frac{1}{2}$. From Exercise 6.2.47 [ET 6.2.47], the volume of a cap of a sphere is $V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi\left(\frac{1}{2}\right)^2\left(3 \cdot 2 - \frac{1}{2}\right) = \frac{11\pi}{24}$. So the total volume is $2 \cdot \frac{11\pi}{24} = \frac{11\pi}{12}$.

13.2 Vectors

ET 12.2

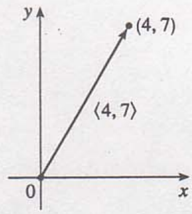
1. (a) The cost of a theater ticket is a scalar, because it has only magnitude.

(b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.

(c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.

(d) The population of the world is a scalar, because it has only magnitude.

2. If the initial point of the vector $\langle 4, 7 \rangle$ is placed at the origin, then $\langle 4, 7 \rangle$ is the position vector of the point $(4, 7)$.



3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, and $\overrightarrow{EA} = \overrightarrow{CE}$.

4. Geometrically, by the Triangle Law, we can see that $\mathbf{u} + \mathbf{w} = \mathbf{v}$, thus $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Alternately, $\mathbf{w}$ can be visualized directly as the difference of $\mathbf{v}$ and $\mathbf{u}$ (see Figure 8 in the text).

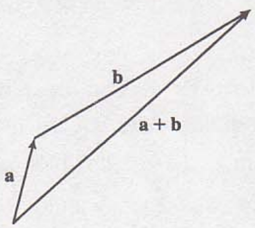
5. (a) By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC}$ is the vector with initial point A and terminal point C, namely $\overrightarrow{AC}$.

(b) By the Triangle Law, $\overrightarrow{CD} + \overrightarrow{DA}$ is the vector with initial point C and terminal point A, namely $\overrightarrow{CA}$.

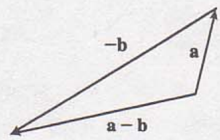
(c) First we consider $\overrightarrow{BC} - \overrightarrow{DC}$ as $\overrightarrow{BC} + \left(-\overrightarrow{DC}\right)$. Then since $-\overrightarrow{DC}$ has the same length as $\overrightarrow{CD}$ but points in the opposite direction, we have $-\overrightarrow{DC} = \overrightarrow{CD}$ and so $\overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$.

(d) We use the Triangle Law twice: $\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \left(\overrightarrow{BC} + \overrightarrow{CD}\right) + \overrightarrow{DA} = \overrightarrow{BD} + \overrightarrow{DA} = \overrightarrow{BA}$.

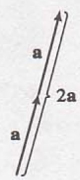
6. (a)



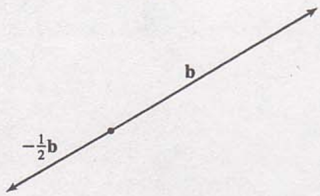
(b)



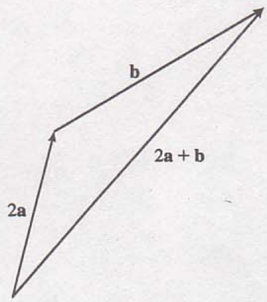
(c)



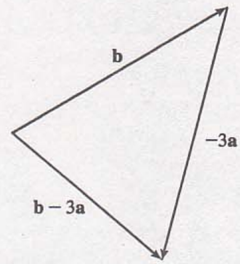
(d)



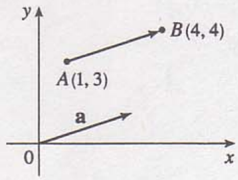
(e)



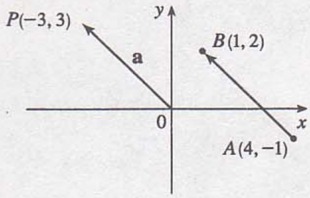
(f)



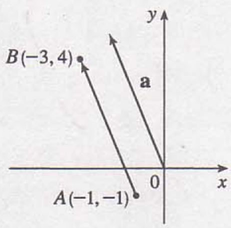
7. $\mathbf{a} = \langle 4-1, 4-3 \rangle = \langle 3, 1 \rangle$



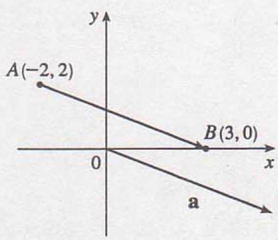
8. $\mathbf{a} = \langle 1-4, 2+1 \rangle = \langle -3, 3 \rangle$



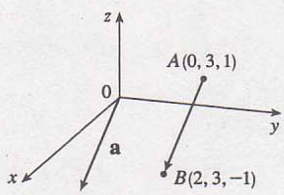
9. $\mathbf{a} = \langle -3-(-1), 4-(-1) \rangle = \langle -2, 5 \rangle$



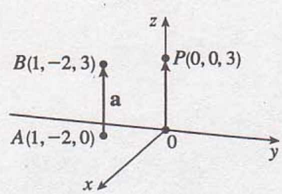
10. $\mathbf{a} = \langle 3-(-2), 0-2 \rangle = \langle 5, -2 \rangle$



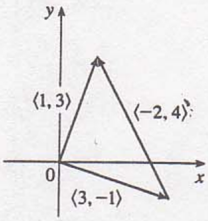
11. $\mathbf{a} = \langle 2-0, 3-3, -1-1 \rangle = \langle 2, 0, -2 \rangle$



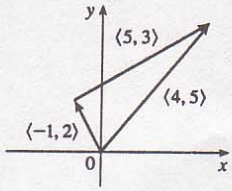
12. $\mathbf{a} = \langle 1-1, -2+2, 3-0 \rangle = \langle 0, 0, 3 \rangle$



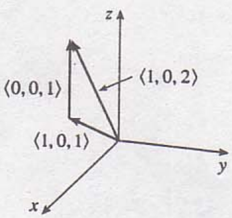
13. $\langle 3,-1\rangle + \langle -2,4\rangle = \langle 3+(-2), -1+4\rangle = \langle 1,3\rangle$



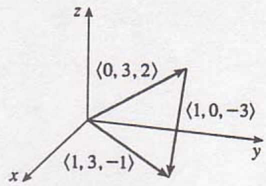
14. $\langle -1,2\rangle + \langle 5,3\rangle = \langle -1+5, 2+3\rangle = \langle 4,5\rangle$



15. $\langle 1,0,1\rangle + \langle 0,0,1\rangle = \langle 1+0, 0+0, 1+1\rangle$
$= \langle 1,0,2\rangle$



16. $\langle 0,3,2\rangle + \langle 1,0,-3\rangle = \langle 1,3,-1\rangle$



17. $|\mathbf{a}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

$\mathbf{a}+\mathbf{b} = \langle -4+6, 3+2\rangle = \langle 2,5\rangle$

$\mathbf{a}-\mathbf{b} = \langle -4-6, 3-2\rangle = \langle -10,1\rangle$

$2\mathbf{a} = \langle 2(-4), 2(3)\rangle = \langle -8,6\rangle$

$3\mathbf{a}+4\mathbf{b} = \langle -12,9\rangle + \langle 24,8\rangle = \langle 12,17\rangle$

18. $|\mathbf{a}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\mathbf{a}+\mathbf{b} = (2\mathbf{i}-3\mathbf{j}) + (\mathbf{i}+5\mathbf{j}) = 3\mathbf{i}+2\mathbf{j}$

$\mathbf{a}-\mathbf{b} = (2\mathbf{i}-3\mathbf{j}) - (\mathbf{i}+5\mathbf{j}) = \mathbf{i}-8\mathbf{j}$

$2\mathbf{a} = 2(2\mathbf{i}-3\mathbf{j}) = 4\mathbf{i}-6\mathbf{j}$

$3\mathbf{a}+4\mathbf{b} = 3(2\mathbf{i}-3\mathbf{j}) + 4(\mathbf{i}+5\mathbf{j})$
$= 6\mathbf{i}-9\mathbf{j}+4\mathbf{i}+20\mathbf{j} = 10\mathbf{i}+11\mathbf{j}$

19. $|\mathbf{a}| = \sqrt{6^2+2^2+3^2} = \sqrt{49} = 7$

$\mathbf{a}+\mathbf{b} = \langle 6+(-1), 2+5, 3+(-2)\rangle = \langle 5,7,1\rangle$

$\mathbf{a}-\mathbf{b} = \langle 6-(-1), 2-5, 3-(-2)\rangle = \langle 7,-3,5\rangle$

$2\mathbf{a} = \langle 2(6), 2(2), 2(3)\rangle = \langle 12,4,6\rangle$

$3\mathbf{a}+4\mathbf{b} = \langle 18,6,9\rangle + \langle -4,20,-8\rangle = \langle 14,26,1\rangle$

20. $|\mathbf{a}| = \sqrt{(-3)^2+(-4)^2+(-1)^2} = \sqrt{26}$

$\mathbf{a}+\mathbf{b} = \langle -3+6, -4+2, -1+(-3)\rangle = \langle 3,-2,-4\rangle$

$\mathbf{a}-\mathbf{b} = \langle -3-6, -4-2, -1-(-3)\rangle = \langle -9,-6,2\rangle$

$2\mathbf{a} = \langle 2(-3), 2(-4), 2(-1)\rangle = \langle -6,-8,-2\rangle$

$3\mathbf{a}+4\mathbf{b} = \langle -9,-12,-3\rangle + \langle 24,8,-12\rangle$
$= \langle 15,-4,-15\rangle$

21. $|\mathbf{a}| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$

$\mathbf{a}+\mathbf{b} = (\mathbf{i}-2\mathbf{j}+\mathbf{k}) + (\mathbf{j}+2\mathbf{k}) = \mathbf{i}-\mathbf{j}+3\mathbf{k}$

$\mathbf{a}-\mathbf{b} = (\mathbf{i}-2\mathbf{j}+\mathbf{k}) - (\mathbf{j}+2\mathbf{k}) = \mathbf{i}-3\mathbf{j}-\mathbf{k}$

$2\mathbf{a} = 2(\mathbf{i}-2\mathbf{j}+\mathbf{k}) = 2\mathbf{i}-4\mathbf{j}+2\mathbf{k}$

$3\mathbf{a}+4\mathbf{b} = 3(\mathbf{i}-2\mathbf{j}+\mathbf{k}) + 4(\mathbf{j}+2\mathbf{k})$
$= 3\mathbf{i}-6\mathbf{j}+3\mathbf{k}+4\mathbf{j}+8\mathbf{k} = 3\mathbf{i}-2\mathbf{j}+11\mathbf{k}$

22. $|\mathbf{a}| = \sqrt{3^2+0^2+(-2)^2} = \sqrt{13}$

$\mathbf{a}+\mathbf{b} = (3\mathbf{i}-2\mathbf{k}) + (\mathbf{i}-\mathbf{j}+\mathbf{k}) = 4\mathbf{i}-\mathbf{j}-\mathbf{k}$

$\mathbf{a}-\mathbf{b} = (3\mathbf{i}-2\mathbf{k}) - (\mathbf{i}-\mathbf{j}+\mathbf{k}) = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$

$2\mathbf{a} = 2(3\mathbf{i}-2\mathbf{k}) = 6\mathbf{i}-4\mathbf{k}$

$3\mathbf{a}+4\mathbf{b} = 3(3\mathbf{i}-2\mathbf{k}) + 4(\mathbf{i}-\mathbf{j}+\mathbf{k})$
$= 9\mathbf{i}-6\mathbf{k}+4\mathbf{i}-4\mathbf{j}+4\mathbf{k} = 13\mathbf{i}-4\mathbf{j}-2\mathbf{k}$

23. $|\langle 9,-5\rangle| = \sqrt{9^2+(-5)^2} = \sqrt{106}$, so $\mathbf{u} = \frac{1}{\sqrt{106}}\langle 9,-5\rangle = \left\langle \frac{9}{\sqrt{106}}, \frac{-5}{\sqrt{106}} \right\rangle$.

24. $|12\mathbf{i}-5\mathbf{j}| = \sqrt{12^2+(-5)^2} = \sqrt{169} = 13$, so $\mathbf{u} = \frac{1}{13}(12\mathbf{i}-5\mathbf{j}) = \frac{12}{13}\mathbf{i} - \frac{5}{13}\mathbf{j}$.

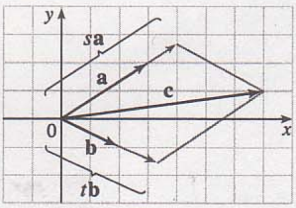
25. $|8\mathbf{i}-\mathbf{j}+4\mathbf{k}| = \sqrt{8^2+(-1)^2+4^2} = \sqrt{81} = 9$, so $\mathbf{u} = \frac{1}{9}(8\mathbf{i}-\mathbf{j}+4\mathbf{k}) = \frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}$.

26. $|\langle 1,-2,3\rangle| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{14}$, so $\mathbf{u} = \frac{1}{\sqrt{14}}\langle 1,-2,3\rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$.

27. By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA}$, but $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AC} + \left(-\overrightarrow{AC}\right) = \mathbf{0}$. So $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

28. $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{BA}$. $\mathbf{c} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{1}{3}\overrightarrow{AB} \quad\Rightarrow\quad \overrightarrow{AB} = 3\mathbf{c} - 3\mathbf{a}$.
$\mathbf{c} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{BA} \quad\Rightarrow\quad \overrightarrow{BA} = \frac{3}{2}\mathbf{c} - \frac{3}{2}\mathbf{b}$. $\overrightarrow{BA} = -\overrightarrow{AB}$, so $\frac{3}{2}\mathbf{c} - \frac{3}{2}\mathbf{b} = 3\mathbf{a} - 3\mathbf{c} \quad\Leftrightarrow$
$\mathbf{c} + 2\mathbf{c} = 2\mathbf{a} + \mathbf{b} \quad\Leftrightarrow\quad \mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

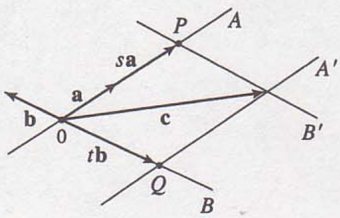
29. (a), (b)



(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.

(d) $\mathbf{c} = s\,\mathbf{a} + t\,\mathbf{b} \quad\Leftrightarrow\quad 7 = 3s + 2t$ and $1 = 2s - t$.
Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

30. Draw $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ emanating from the origin. Extend $\mathbf{a}$ and $\mathbf{b}$ to form lines A and B, and draw lines A' and B' parallel to these two lines through the terminal point of $\mathbf{c}$.



Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that $\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{c}$, so if $s = \dfrac{\left|\overrightarrow{OP}\right|}{|\mathbf{a}|}$ $\left(\text{or its negative, if } \mathbf{a} \text{ points in the direction opposite } \overrightarrow{OP}\right)$ and $t = \dfrac{\left|\overrightarrow{OQ}\right|}{|\mathbf{b}|}$ (or its negative, as in the diagram), then $c = s\mathbf{a} + t\mathbf{b}$, as required.

Argument using components: Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{b} = \langle b_1, b_2\rangle$, and $\mathbf{c} = \langle c_1, c_2\rangle$. We need $sa_1 + tb_1 = c_1$ and $sa_2 + tb_2 = c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t = \dfrac{c_2a_1 - c_1a_2}{b_2a_1 - b_1a_2}$. Similarly $s = \dfrac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}$. Since $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, the denominator is not zero.

31. $|\mathbf{F}_1| = 10$ lb and $|\mathbf{F}_2| = 12$ lb.

$$\begin{aligned}\mathbf{F}_1 &= -|\mathbf{F}_1|\cos 45^\circ\,\mathbf{i} + |\mathbf{F}_1|\sin 45^\circ\,\mathbf{j} = -10\cos 45^\circ\,\mathbf{i} + 10\sin 45^\circ\,\mathbf{j}\\ &= -5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j}\end{aligned}$$

$$\mathbf{F}_2 = |\mathbf{F}_2|\cos 30^\circ\,\mathbf{i} + |\mathbf{F}_2|\sin 30^\circ\,\mathbf{j} = 12\cos 30^\circ\,\mathbf{i} + 12\sin 30^\circ\,\mathbf{j} = 6\sqrt{3}\,\mathbf{i} + 6\,\mathbf{j}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \left(6\sqrt{3} - 5\sqrt{2}\right)\mathbf{i} + \left(6 + 5\sqrt{2}\right)\mathbf{j} \approx 3.32\mathbf{i} + 13.07\mathbf{j}$$

$$|\mathbf{F}| \approx \sqrt{(3.32)^2 + (13.07)^2} \approx 13.5 \text{ lb. } \tan\theta = \frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}} \quad\Rightarrow\quad \theta = \tan^{-1}\frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}} \approx 76^\circ.$$

32. Set up the coordinate axes so that north is the positive y-direction, and east is the positive x-direction. The wind is blowing at 50 km/h from the direction N 45° W, so that its velocity vector is 50 km/h S 45°E, which can be written as $\mathbf{v}_{\text{wind}} = 50\left(\cos 45^\circ\,\mathbf{i} - \sin 45^\circ\,\mathbf{j}\right)$. With respect to the still air, the velocity vector of the plane is 250 km/h N 60° E, or equivalently $\mathbf{v}_{\text{plane}} = 250\left(\cos 30^\circ\,\mathbf{i} + \sin 30^\circ\,\mathbf{j}\right)$. The velocity of the plane relative to the ground is

$$\begin{aligned}\mathbf{v} = \mathbf{v}_{\text{wind}} + \mathbf{v}_{\text{plane}} &= \left(50\cos 45^\circ + 250\cos 30^\circ\right)\mathbf{i} + \left(-50\sin 45^\circ + 250\sin 30^\circ\right)\mathbf{j}\\ &= \left(25\sqrt{2} + 125\sqrt{3}\right)\mathbf{i} + \left(125 - 25\sqrt{2}\right)\mathbf{j} \approx 251.9\,\mathbf{i} + 89.6\,\mathbf{j}\end{aligned}$$

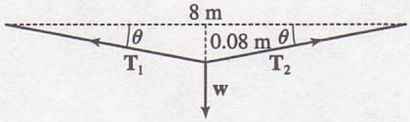
The ground speed is $|\mathbf{v}| \approx \sqrt{(251.9)^2 + (89.6)^2} \approx 267$ km/h. The angle the velocity vector makes with the x-axis is $\theta \approx \tan^{-1}\frac{89.6}{251.9} \approx 20^\circ$. Therefore, the true course of the plane is about N $(90 - 20)^\circ$ E $=$ N 70° E.

33. With respect to the water's surface, the woman's velocity is the vector sum of the velocity of the ship with respect to the water, and the woman's velocity with respect to the ship. If we let north be the positive y-direction, then $\mathbf{v} = \langle 0, 22\rangle + \langle -3, 0\rangle = \langle -3, 22\rangle$. The woman's speed is $|\mathbf{v}| = \sqrt{9 + 484} \approx 22.2$ mi/h. The vector $\mathbf{v}$ makes an angle θ with the east, where $\theta = \tan^{-1}\frac{22}{-3} \approx 98^\circ$. Therefore, the woman's direction is about N $(98 - 90)^\circ$ W $=$ N 8° W.

34. Call the two tensile forces $\mathbf{T}_3$ and $\mathbf{T}_5$, corresponding to the ropes of length 3 m and 5 m. In terms of vertical and horizontal components, $\mathbf{T}_3 = -|\mathbf{T}_3|\cos 52^\circ\,\mathbf{i} + |\mathbf{T}_3|\sin 52^\circ\,\mathbf{j}$ (1) and $\mathbf{T}_5 = |\mathbf{T}_5|\cos 40^\circ\,\mathbf{i} + |\mathbf{T}_5|\sin 40^\circ\,\mathbf{j}$ (2). The resultant of these forces, $\mathbf{T}_3 + \mathbf{T}_5$, counterbalances the force of gravity acting on the decoration [which is $-5g\mathbf{j} \approx -5\,(9.8)\,\mathbf{j} = -49\mathbf{j}$]. So $\mathbf{T}_3 + \mathbf{T}_5 = 49\mathbf{j}$. Hence $\mathbf{T}_3 + \mathbf{T}_5 = \left(-|\mathbf{T}_3|\cos 52^\circ + |\mathbf{T}_5|\cos 40^\circ\right)\mathbf{i} + \left(|\mathbf{T}_3|\sin 52^\circ + |\mathbf{T}_5|\sin 40^\circ\right)\mathbf{j} = 49\mathbf{j}$. Thus $-|\mathbf{T}_3|\cos 52^\circ + |\mathbf{T}_5|\cos 40^\circ = 0$ and $|\mathbf{T}_3|\sin 52^\circ + |\mathbf{T}_5|\sin 40^\circ = 49$.

From the first of these two equations $|\mathbf{T}_3| = |\mathbf{T}_5|\dfrac{\cos 40^\circ}{\cos 52^\circ}$. Substituting this into the second equation gives $|\mathbf{T}_5| = \dfrac{49}{\cos 40^\circ\tan 52^\circ + \sin 40^\circ} \approx 30$ N. Therefore, $|\mathbf{T}_3| = |\mathbf{T}_5|\dfrac{\cos 40^\circ}{\cos 52^\circ} \approx 38$ N. Finally, from (1) and (2), $\mathbf{T}_3 \approx -23\mathbf{i} + 30\mathbf{j}$, and $\mathbf{T}_5 \approx 23\mathbf{i} + 19\mathbf{j}$.

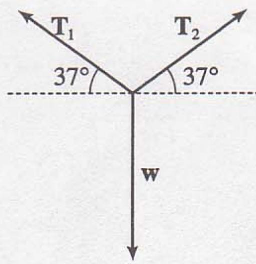
35. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ represent the tension vectors in each side of the clothesline as shown in the figure. $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal vertical components and opposite horizontal components, so we can write $\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j}$ $(a, b > 0)$.



By similar triangles, $\dfrac{b}{a} = \dfrac{0.08}{4} \Rightarrow a = 50b$. The force due to gravity acting on the shirt has magnitude $0.8g \approx (0.8)(9.8) = 7.84$ N, hence we have $\mathbf{w} = -7.84\,\mathbf{j}$. The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensile forces counterbalances $\mathbf{w}$, so $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} \Rightarrow (-a\,\mathbf{i} + b\,\mathbf{j}) + (a\,\mathbf{i} + b\,\mathbf{j}) = 7.84\,\mathbf{j} \Rightarrow$ $(-50b\,\mathbf{i} + b\,\mathbf{j}) + (50b\,\mathbf{i} + b\,\mathbf{j}) = 2b\,\mathbf{j} = 7.84\,\mathbf{j} \Rightarrow b = \frac{7.84}{2} = 3.92$ and $a = 50b = 196$. Thus the tensions are $\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j} = -196\,\mathbf{i} + 3.92\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j} = 196\,\mathbf{i} + 3.92\,\mathbf{j}$.

Alternatively, we can find the value of θ and proceed as in Example 5.

36. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\mathbf{T}_1$, $\mathbf{T}_2$ in each end of the chain and the weight $\mathbf{w}$, as shown in the figure. We know $|\mathbf{T}_1| = |\mathbf{T}_2| = 25$ N so, in terms of vertical and horizontal components, we have



$$\mathbf{T}_1 = -25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}$$
$$\mathbf{T}_2 = 25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}$$

The resultant vector $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$, giving $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w}$. Since $\mathbf{w} = -|\mathbf{w}|\,\mathbf{j}$, we have $(-25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) + (25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) = |\mathbf{w}|\,\mathbf{j} \Rightarrow$ $50\sin 37^\circ\,\mathbf{j} = |\mathbf{w}|\,\mathbf{j} \Rightarrow |\mathbf{w}| = 50\sin 37^\circ \approx 30.1$. So the weight is 30.1 N, and since $w = mg$, the mass is $\frac{30.1}{9.8} \approx 3.07$ kg.

37. $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the points (x, y, z) and (x_0, y_0, z_0), so the set of points is a sphere with radius 1 and center (x_0, y_0, z_0).

Alternate Method: $|\mathbf{r} - \mathbf{r}_0| = 1 \Leftrightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = 1 \Leftrightarrow$ $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$, which is the equation of a sphere with radius 1 and center (x_0, y_0, z_0).

38. Let P_1 and P_2 be the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2|$ is the sum of the distances from (x, y) to P_1 and P_2. Since this sum is constant, the set of points (x, y) represents an ellipse with foci P_1 and P_2. The condition $k > |\mathbf{r}_1 - \mathbf{r}_2|$ assures us that the ellipse is not degenerate.

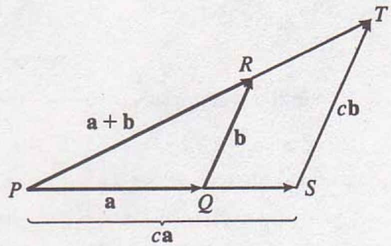
39.
$$\begin{aligned}\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2\rangle + (\langle b_1, b_2\rangle + \langle c_1, c_2\rangle) = \langle a_1, a_2\rangle + \langle b_1 + c_1, b_2 + c_2\rangle \\ &= \langle a_1 + b_1 + c_1, a_2 + b_2 + c_2\rangle = \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2\rangle \\ &= \langle a_1 + b_1, a_2 + b_2\rangle + \langle c_1, c_2\rangle = (\langle a_1, a_2\rangle + \langle b_1, b_2\rangle) + \langle c_1, c_2\rangle \\ &= (\mathbf{a} + \mathbf{b}) + \mathbf{c}\end{aligned}$$

40. *Algebraically:*

$$\begin{aligned} c(\mathbf{a}+\mathbf{b}) &= c(\langle a_1,a_2,a_3\rangle + \langle b_1,b_2,b_3\rangle) = c\langle a_1+b_1, a_2+b_2, a_3+b_3\rangle \\ &= \langle c(a_1+b_1), c(a_2+b_2), c(a_3+b_3)\rangle = \langle ca_1+cb_1, ca_2+cb_2, ca_3+cb_3\rangle \\ &= \langle ca_1, ca_2, ca_3\rangle + \langle cb_1, cb_2, cb_3\rangle = c\mathbf{a}+c\mathbf{b} \end{aligned}$$

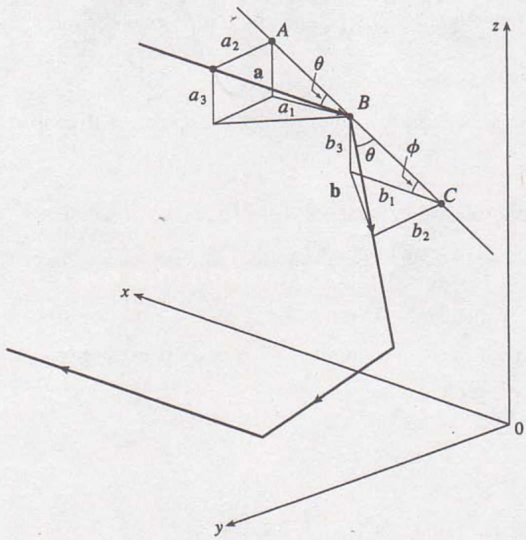
Geometrically:

According to the Triangle Law, if $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{QR}$, then $\mathbf{a}+\mathbf{b} = \overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS} = c\mathbf{a}$ and $\overrightarrow{ST} = c\mathbf{b}$. (We have drawn the case where $c > 1$.) By the Triangle Law, $\overrightarrow{PT} = c\mathbf{a} + c\mathbf{b}$. But triangle PQR and triangle PST are similar triangles because $c\mathbf{b}$ is parallel to $\mathbf{b}$. Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT} = c\overrightarrow{PR}$. Thus, $c\mathbf{a}+c\mathbf{b} = c(\mathbf{a}+\mathbf{b})$.



41. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (1) and $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ (2). However, $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into (2) gives $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$. Comparing this with (1) gives $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

42.



The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Let $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, as in the diagram. We can let $|\mathbf{b}| = |\mathbf{a}|$, since only its direction is important. Then $\dfrac{|b_2|}{|\mathbf{b}|} = \sin\theta = \dfrac{|a_2|}{|\mathbf{a}|} \Rightarrow |b_2| = |a_2|$. From the diagram $b_2\,\mathbf{j}$ and $a_2\,\mathbf{j}$ point in opposite directions, so $b_2 = -a_2$. $|AB| = |BC|$, so $|b_3| = \sin\phi\,|BC| = \sin\phi\,|AB| = |a_3|$, and $|b_1| = \cos\phi\,|BC| = \cos\phi\,|AB| = |a_1|$. $b_3\,\mathbf{k}$ and $a_3\,\mathbf{k}$ have the same direction, as do $b_1\,\mathbf{i}$ and $a_1\,\mathbf{i}$, so $\mathbf{b} = \langle a_1, -a_2, a_3\rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\langle -a_1, -a_2, -a_3\rangle = -\mathbf{a}$, which is parallel to $\mathbf{a}$.

13.3 The Dot Product

ET 12.3

1. (a) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ has no meaning.

(b) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|\mathbf{a}|$ and $\mathbf{b} \cdot \mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b} \cdot \mathbf{c})$ is an ordinary product of real numbers, and has meaning.

(d) Both $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$ are vectors, so the dot product $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ has meaning.

(e) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and this expression has no meaning.

(f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$ has no meaning.

2. Let the vectors be $\mathbf{a}$ and $\mathbf{b}$. Then by Theorem 3, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (6)\left(\frac{1}{3}\right)\cos\frac{\pi}{4} = \frac{6}{3\sqrt{2}} = \sqrt{2}$.

3. $\mathbf{a} \cdot \mathbf{b} = \langle 4, -1\rangle \cdot \langle 3, 6\rangle = (4)(3) + (-1)(6) = 6$

4. $\mathbf{a} \cdot \mathbf{b} = \left\langle \frac{1}{2}, 4\right\rangle \cdot \langle -8, -3\rangle = \left(\frac{1}{2}\right)(-8) + (4)(-3) = -16$

5. $\mathbf{a} \cdot \mathbf{b} = \langle 5, 0, -2\rangle \cdot \langle 3, -1, 10\rangle = (5)(3) + (0)(-1) + (-2)(10) = -5$

6. $\mathbf{a} \cdot \mathbf{b} = \langle 2, 6, -3\rangle \cdot \langle 8, -2, -1\rangle = (2)(8) + (6)(-2) + (-3)(-1) = 7$

7. $\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (5\mathbf{i} + 9\mathbf{k}) = (1)(5) + (-2)(0) + (3)(9) = 32$

8. $\mathbf{a} \cdot \mathbf{b} = (4\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) = (0)(2) + (4)(4) + (-3)(6) = -2$

9. Use Theorem 3: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (12)(15)\cos\frac{\pi}{6} = 180 \cdot \frac{\sqrt{3}}{2} = 90\sqrt{3} \approx 155.9$

10. Use Theorem 3: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (4)(10)\cos 120^\circ = 40\left(-\frac{1}{2}\right) = -20$

11. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 60^\circ = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}|\,|\mathbf{w}| \cos 120^\circ = (1)(1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$.

12. $\mathbf{u}$ is a unit vector, so $\mathbf{w}$ is also a unit vector, and $|\mathbf{v}|$ can be determined by examining the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$. Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is 45°, we have $|\mathbf{v}| = |\mathbf{u}| \cos 45^\circ = \frac{\sqrt{2}}{2}$. Then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{1}{2}$. Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, $\mathbf{u} \cdot \mathbf{w} = 0$.

13. (a) $\mathbf{i} \cdot \mathbf{j} = \langle 1, 0, 0\rangle \cdot \langle 0, 1, 0\rangle = (1)(0) + (0)(1) + (0)(0) = 0$. Similarly $\mathbf{j} \cdot \mathbf{k} = (0)(0) + (1)(0) + (0)(1) = 0$ and $\mathbf{k} \cdot \mathbf{i} = (0)(1) + (0)(0) + (1)(0) = 0$.

Another Method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product is $\cos\frac{\pi}{2} = 0$.

(b) By Property #1 of the Dot Product, $\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}|^2 = 1$.

14. The dot product $\mathbf{A} \cdot \mathbf{P}$ is

$$\begin{aligned}\langle a, b, c\rangle \cdot \langle 2, 1.5, 1\rangle &= a(2) + b(1.5) + c(1)\\ &= (\text{number of hamburgers sold})\,(\text{price per hamburger})\\ &\quad + (\text{number of hot dogs sold})\,(\text{price per hot dog})\\ &\quad + (\text{number of soft drinks sold})\,(\text{price per soft drink})\end{aligned}$$

so it is equal to the vendor's total revenue for that.day.

15. $|\mathbf{a}| = \sqrt{3^2+4^2} = 5$, $|\mathbf{b}| = \sqrt{5^2+12^2} = 13$, and $\mathbf{a}\cdot\mathbf{b} = (3)(5)+(4)(12) = 63$. Using Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{63}{5\cdot 13} = \dfrac{63}{65}$. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\frac{63}{65} \approx 14^\circ$.

16. $|\mathbf{a}| = \sqrt{3^2+1^2} = \sqrt{10}$, $|\mathbf{b}| = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$, and $\mathbf{a}\cdot\mathbf{b} = (3)(2)+(1)(4) = 10$. Using Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{10}{\sqrt{10}\cdot 2\sqrt{5}} = \dfrac{\sqrt{2}}{2}$. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\frac{\sqrt{2}}{2} = 45^\circ$.

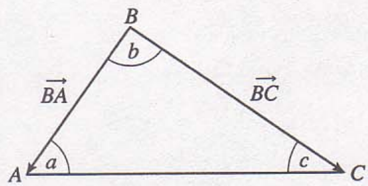
17. $|\mathbf{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, $|\mathbf{b}| = \sqrt{4^2+0^2+(-1)^2} = \sqrt{17}$, and $\mathbf{a}\cdot\mathbf{b} = (1)(4)+(2)(0)+(3)(-1) = 1$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{1}{\sqrt{14}\cdot\sqrt{17}} = \dfrac{1}{\sqrt{238}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\frac{1}{\sqrt{238}} \approx 86^\circ$.

18. $|\mathbf{a}| = \sqrt{6^2+(-3)^2+2^2} = 7$, $|\mathbf{b}| = \sqrt{2^2+1^2+(-2)^2} = 3$, and $\mathbf{a}\cdot\mathbf{b} = (6)(2)+(-3)(1)+(2)(-2) = 5$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{7\cdot 3} = \dfrac{5}{21}$ and $\theta = \cos^{-1}\frac{5}{21} \approx 76^\circ$.

19. $|\mathbf{a}| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$, $|\mathbf{b}| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$, and $\mathbf{a}\cdot\mathbf{b} = (0)(1)+(1)(2)+(1)(-3) = -1$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{2}\cdot\sqrt{14}} = \dfrac{-1}{2\sqrt{7}}$ and $\theta = \cos^{-1}\left(-\frac{1}{2\sqrt{7}}\right) \approx 101^\circ$.

20. $|\mathbf{a}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$, $|\mathbf{b}| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$, and $\mathbf{a}\cdot\mathbf{b} = (2)(3)+(-1)(2)+(1)(-1) = 3$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{3}{\sqrt{6}\cdot\sqrt{14}} = \dfrac{3}{2\sqrt{21}}$ and $\theta = \cos^{-1}\frac{3}{2\sqrt{21}} \approx 71^\circ$.

21. Let a, b and c be the angles at vertices A, B and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. Thus



$$\cos a = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{1}{\sqrt{30\cdot 29}}\langle 5,-1,2\rangle\cdot\langle -2,-4,-3\rangle = \frac{1}{\sqrt{870}}(-10+4-6) = -\frac{12}{\sqrt{870}}$$ and

$$a = \cos^{-1}\left(-\frac{12}{\sqrt{870}}\right) \approx 114^\circ.$$ Similarly

$$\cos b = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \frac{1}{\sqrt{30\cdot 83}}\langle 5,-1,2\rangle\cdot\langle -7,-3,-5\rangle = \frac{1}{\sqrt{2490}}(35-3+10) = \frac{42}{\sqrt{2490}},$$ so

$$b = \cos^{-1}\frac{42}{\sqrt{2490}} \approx 33^\circ,$$ and

$$\cos c = \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|} = \frac{1}{\sqrt{29\cdot 83}}\langle 2,4,3\rangle\cdot\langle 7,3,5\rangle = \frac{1}{\sqrt{2407}}(14+12+15) = \frac{41}{\sqrt{2407}},$$ so

$$c = \cos^{-1}\frac{41}{\sqrt{2407}} \approx 33^\circ.$$

Alternate Solution: Apply the Law of Cosines three times as follows: $\cos a = \dfrac{\left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$,

$\cos b = \dfrac{\left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|}$, and $\cos c = \dfrac{\left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|}$.

22. As in Exercise 21, let p, q and r be the angles at vertices P, Q and R. Then p is the angle between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, q is the angle between vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and r is the angle between vectors $\overrightarrow{RP}$ and $\overrightarrow{RQ}$. Thus

$$\cos p = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|} = \frac{\langle 2,2,-9\rangle\cdot\langle 5,5,-4\rangle}{\sqrt{89}\sqrt{66}} = \frac{56}{\sqrt{5874}}, \text{ so } p = \cos^{-1}\frac{56}{\sqrt{5874}} \approx 43\,^\circ;$$

$$\cos q = \frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|} = \frac{\langle -2,-2,9\rangle\cdot\langle 3,3,5\rangle}{\sqrt{89}\sqrt{43}} = \frac{33}{\sqrt{3827}}, \text{ so } q = \cos^{-1}\frac{33}{\sqrt{3827}} \approx 58\,^\circ; \text{ and}$$

$$\cos r = \frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{\left|\overrightarrow{RP}\right|\left|\overrightarrow{RQ}\right|} = \frac{\langle -5,-5,4\rangle\cdot\langle -3,-3,-5\rangle}{\sqrt{66}\sqrt{43}} = \frac{10}{\sqrt{2838}}, \text{ so } r = \cos^{-1}\frac{10}{\sqrt{2838}} \approx 79\,^\circ.$$

Alternate Solution: Apply the Law of Cosines three times as follows: $\cos p = \dfrac{\left|\overrightarrow{QR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2}{2\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|}$,

$\cos q = \dfrac{\left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PQ}\right|\left|\overrightarrow{QR}\right|}$, and $\cos r = \dfrac{\left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PR}\right|\left|\overrightarrow{QR}\right|}$.

23. Since $\mathbf{a} = -2\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors (and thus not orthogonal).

24. $\mathbf{a}\cdot\mathbf{b} = 8 + (-8) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

25. $\mathbf{a}\cdot\mathbf{b} = (-5)(6) + (3)(-8) + (7)(2) = -40 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

26. $\mathbf{a}\cdot\mathbf{b} = (6)(4) + (-1)(9) + (5)(-3) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

27. $\mathbf{a}\cdot\mathbf{b} = (-1)(3) + (2)(4) + (5)(-1) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

28. Because $\mathbf{a} = -\frac{2}{3}\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

29. For the two vectors to be orthogonal, we need $\langle -3x, 2x\rangle\cdot\langle 4, x\rangle = 0 \quad\Leftrightarrow\quad (-3x)(4) + (2x)(x) = 0 \quad\Leftrightarrow$ $-12x + 2x^2 = 0 \quad\Leftrightarrow\quad 2x(x-6) = 0 \quad\Leftrightarrow\quad x = 0$ or $x = 6$.

30. $\langle -6, b, 2\rangle$ and $\langle b, b^2, b\rangle$ are orthogonal when

$\langle -6, b, 2\rangle\cdot\langle b, b^2, b\rangle = 0 \quad\Leftrightarrow\quad (-6)(b) + (b)(b^2) + (2)(b) = 0$

$\Leftrightarrow\quad b^3 - 4b = 0 \quad\Leftrightarrow\quad b(b+2)(b-2) = 0 \quad\Leftrightarrow\quad b = 0$ or $b = \pm 2$.

31. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ be a vector orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$. Then $\mathbf{a}\cdot(\mathbf{i}+\mathbf{j}) = 0 \quad\Leftrightarrow\quad a_1 + a_2 = 0$ and $\mathbf{a}\cdot(\mathbf{i}+\mathbf{k}) = 0 \quad\Leftrightarrow\quad a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm\frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ are two such unit vectors.

32. Using Theorem 3, we need

$\langle 1,2,1\rangle \cdot \langle 1,0,c\rangle = |\langle 1,2,1\rangle|\,|\langle 1,0,c\rangle| \cos 60^\circ \quad\Leftrightarrow\quad 1+c=\sqrt{6}\sqrt{1+c^2}\cdot\frac{1}{2} \quad\Leftrightarrow\quad 2\,(1+c)=\sqrt{6}\sqrt{1+c^2}$.

Squaring both sides gives $6\left(1+c^2\right)=4\left(1+2c+c^2\right)$. Thus $6+6c^2=4+8c+4c^2$ or $2c^2-8c+2=0$ and $c=\dfrac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$. Each of these values for c can be checked to show it gives a solution.

33. Since $|\langle 1,2,2\rangle|=\sqrt{1+4+4}=3$, using (8) and (9) we have $\cos\alpha=\frac{1}{3}$, $\cos\beta=\frac{2}{3}$ and $\cos\gamma=\frac{2}{3}$, while $\alpha=\cos^{-1}\frac{1}{3}\approx 71^\circ$ and $\beta=\gamma=\cos^{-1}\frac{2}{3}\approx 48^\circ$.

34. $|\langle -4,-1,2\rangle|=\sqrt{16+1+4}=\sqrt{21}$, so $\cos\alpha=-\frac{4}{\sqrt{21}}$, $\cos\beta=-\frac{1}{\sqrt{21}}$ and $\cos\gamma=\frac{2}{\sqrt{21}}$, while $\alpha=\cos^{-1}\frac{-4}{\sqrt{21}}\approx 151^\circ$, $\beta=\cos^{-1}\frac{-1}{\sqrt{21}}\approx 103^\circ$ and $\gamma=\cos^{-1}\frac{2}{\sqrt{21}}\approx 64^\circ$.

35. $|-8\mathbf{i}+3\mathbf{j}+2\mathbf{k}|=\sqrt{64+9+4}=\sqrt{77}$, so $\cos\alpha=-\frac{8}{\sqrt{77}}$, $\cos\beta=\frac{3}{\sqrt{77}}$ and $\cos\gamma=\frac{2}{\sqrt{77}}$, while $\alpha=\cos^{-1}\frac{-8}{\sqrt{77}}\approx 156^\circ$, $\beta=\cos^{-1}\frac{3}{\sqrt{77}}\approx 70^\circ$ and $\gamma=\cos^{-1}\frac{2}{\sqrt{77}}\approx 77^\circ$.

36. $|3\mathbf{i}+5\mathbf{j}-4\mathbf{k}|=\sqrt{9+25+16}=5\sqrt{2}$, so $\cos\alpha=\frac{3}{5\sqrt{2}}$, $\cos\beta=\frac{1}{\sqrt{2}}$ and $\cos\gamma=-\frac{4}{5\sqrt{2}}$, while $\alpha=\cos^{-1}\frac{3}{5\sqrt{2}}\approx 65^\circ$, $\beta=\cos^{-1}\frac{1}{\sqrt{2}}=45^\circ$ and $\gamma=\cos^{-1}\frac{-4}{5\sqrt{2}}\approx 124^\circ$.

37. $|\langle 2,1.2,0.8\rangle|=\sqrt{4+1.44+0.64}=\frac{5}{5}\sqrt{6.08}=\frac{\sqrt{152}}{5}$, so $\cos\alpha=\frac{10}{\sqrt{152}}=\frac{5}{\sqrt{38}}$, $\cos\beta=\frac{6}{\sqrt{152}}=\frac{3}{\sqrt{38}}$ and $\cos\gamma=\frac{4}{\sqrt{152}}=\frac{2}{\sqrt{38}}$, while $\alpha=\cos^{-1}\frac{5}{\sqrt{38}}\approx 36^\circ$, $\beta=\cos^{-1}\frac{3}{\sqrt{38}}\approx 61^\circ$ and $\gamma=\cos^{-1}\frac{2}{\sqrt{38}}\approx 71^\circ$.

38. Since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$,

$\cos^2\gamma=1-\cos^2\alpha-\cos^2\beta=1-\cos^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}=1-\left(\frac{1}{\sqrt{2}}\right)^2-\left(\frac{1}{2}\right)^2=\frac{1}{4}$. Thus $\cos\gamma=\pm\frac{1}{2}$ and $\gamma=\frac{\pi}{3}$ or $\gamma=\frac{2\pi}{3}$.

39. $|\mathbf{a}|=\sqrt{4+9}=\sqrt{13}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=\dfrac{2\cdot 4+3\cdot 1}{\sqrt{13}}=\dfrac{11}{\sqrt{13}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b}=\dfrac{11}{\sqrt{13}}\dfrac{\mathbf{a}}{|\mathbf{a}|}=\frac{11}{\sqrt{13}}\cdot\frac{1}{\sqrt{13}}\langle 2,3\rangle=\frac{11}{13}\langle 2,3\rangle=\left\langle\frac{22}{13},\frac{33}{13}\right\rangle$.

40. $|\mathbf{a}|=\sqrt{9+1}=\sqrt{10}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=\dfrac{3\cdot 2-1\cdot 3}{\sqrt{10}}=\dfrac{3}{\sqrt{10}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b}=\frac{3}{\sqrt{10}}\dfrac{\mathbf{a}}{|\mathbf{a}|}=\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}}\langle 3,-1\rangle=\frac{3}{10}\langle 3,-1\rangle=\left\langle\frac{9}{10},-\frac{3}{10}\right\rangle$.

41. $|\mathbf{a}|=\sqrt{16+4+0}=2\sqrt{5}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=\dfrac{1}{2\sqrt{5}}(4+2+0)=\dfrac{3}{\sqrt{5}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b}=\dfrac{3}{\sqrt{5}}\dfrac{\mathbf{a}}{|\mathbf{a}|}=\dfrac{3}{\sqrt{5}}\cdot\dfrac{1}{2\sqrt{5}}\langle 4,2,0\rangle=\frac{1}{5}\langle 6,3,0\rangle=\left\langle\frac{6}{5},\frac{3}{5},0\right\rangle$.

42. $|\mathbf{a}|=\sqrt{1+4+4}=3$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=\dfrac{-3+(-6)+8}{3}=-\dfrac{1}{3}$, while the vector projection is $\operatorname{proj}_{\mathbf{a}}\mathbf{b}=-\dfrac{1}{3}\dfrac{\mathbf{a}}{|\mathbf{a}|}=-\dfrac{1}{3}\cdot\dfrac{\langle -1,-2,2\rangle}{3}=\left\langle\frac{1}{9},\frac{2}{9},-\frac{2}{9}\right\rangle$.

43. $|\mathbf{a}|=\sqrt{1+0+1}=\sqrt{2}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=\frac{1}{\sqrt{2}}(1+0+0)=\frac{1}{\sqrt{2}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b}=\dfrac{1}{\sqrt{2}}\dfrac{\mathbf{a}}{|\mathbf{a}|}=\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})=\frac{1}{2}(\mathbf{i}+\mathbf{k})$.

44. $|\mathbf{a}| = \sqrt{4+9+1} = \sqrt{14}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{2-18-2}{\sqrt{14}} = -\dfrac{18}{\sqrt{14}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

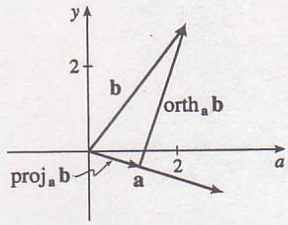
$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = -\frac{18}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = -\frac{18}{\sqrt{14}}\cdot\frac{2\mathbf{i}-3\mathbf{j}+\mathbf{k}}{\sqrt{14}} = -\tfrac{9}{7}\,(2\mathbf{i}-3\mathbf{j}+\mathbf{k}).$$

45. $$(\text{orth}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a} = (\mathbf{b}-\text{proj}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - (\text{proj}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\cdot\mathbf{a}$$
$$= \mathbf{b}\cdot\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,|\mathbf{a}|^2 = \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} = 0$$

So they are orthogonal by (7).

46. Using the result of Exercise 40, we have

$$\begin{aligned}\text{orth}_{\mathbf{a}}\,\mathbf{b} &= \mathbf{b} - \text{proj}_{\mathbf{a}}\,\mathbf{b}\\ &= \langle 2,3\rangle - \left\langle \tfrac{9}{10}, -\tfrac{3}{10}\right\rangle\\ &= \langle 1.1, 3.3\rangle\end{aligned}$$



47. $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = 2 \;\Leftrightarrow\; \mathbf{a}\cdot\mathbf{b} = 2\,|\mathbf{a}| = 2\sqrt{10}$. If $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then we need $3b_1 + 0b_2 - 1b_3 = 2\sqrt{10}$.

One possible solution is obtained by taking $b_1 = 0$, $b_2 = 0$, $b_3 = -2\sqrt{10}$.

In general, $\mathbf{b} = \left\langle s, t, 3s - 2\sqrt{10}\right\rangle$, $s, t \in \mathbb{R}$.

48. (a) $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \text{comp}_{\mathbf{b}}\,\mathbf{a} \;\Leftrightarrow\; \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|} \;\Leftrightarrow\; \dfrac{1}{|\mathbf{a}|} = \dfrac{1}{|\mathbf{b}|}$ or $\mathbf{a}\cdot\mathbf{b} = 0 \;\Leftrightarrow\; |\mathbf{b}| = |\mathbf{a}|$ or $\mathbf{a}\cdot\mathbf{b} = 0$. That is, if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or if they have the same length.

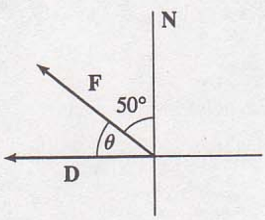
(b) $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \text{proj}_{\mathbf{b}}\,\mathbf{a} \;\Leftrightarrow\; \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\mathbf{b} \;\Leftrightarrow\; \mathbf{a}\cdot\mathbf{b} = 0$ or $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2}$. But $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2} \;\Rightarrow$

$\dfrac{|\mathbf{a}|}{|\mathbf{a}|^2} = \dfrac{|\mathbf{b}|}{|\mathbf{b}|^2} \;\Rightarrow\; |\mathbf{a}| = |\mathbf{b}|$. Substituting this into the previous equation gives $\mathbf{a} = \mathbf{b}$. So $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \text{proj}_{\mathbf{b}}\,\mathbf{a}$ $\Leftrightarrow$ $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, or they are equal.

49. Here $\mathbf{D} = (4-2)\,\mathbf{i} + (9-3)\,\mathbf{j} + (15-0)\,\mathbf{k} = 2\mathbf{i} + 6\mathbf{j} + 15\mathbf{k}$ so using (12) we have $W = \mathbf{F}\cdot\mathbf{D} = 20 + 108 - 90 = 38$ joules.

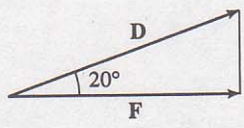
50. $W = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (20)\,(4)\cos 40^\circ$

≈ 61 ft-lb



51. $W = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (25)\,(10)\cos 20^\circ$

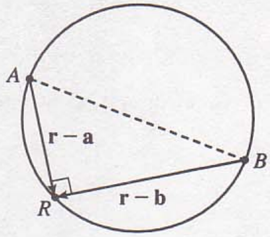
≈ 235 ft-lb



52. Here $|\mathbf{D}| = 100$ m, $|\mathbf{F}| = 50$ N, and $\theta = 30^\circ$. Thus $W = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (50)\,(100)\left(\frac{\sqrt{3}}{2}\right) = 2500\sqrt{3}$ joules.

53. First note that $\mathbf{n} = \langle a, b\rangle$ is perpendicular to the line, because if $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$ lie on the line, then $\mathbf{n} \cdot \overrightarrow{Q_1Q_2} = aa_2 - aa_1 + bb_2 - bb_1 = 0$, since $aa_2 + bb_2 = -c = aa_1 + bb_1$ from the equation of the line. Let $P_2 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. $\operatorname{comp}_{\mathbf{n}}\left(\overrightarrow{P_1P_2}\right) = \dfrac{|\mathbf{n} \cdot \langle x_2 - x_1, y_2 - y_1\rangle|}{|\mathbf{n}|} = \dfrac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2+b^2}} = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ since $ax_2 + by_2 = -c$. The required distance is $\dfrac{|3 \cdot -2 + -4 \cdot 3 + 5|}{\sqrt{3^2+4^2}} = \dfrac{13}{5}$.

54. $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ implies that the vectors $\mathbf{r} - \mathbf{a}$ and $\mathbf{r} - \mathbf{b}$ are orthogonal. From the diagram (in which A, B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B. The center of this circle is the midpoint of AB, that is, $\frac{1}{2}(\mathbf{a} + \mathbf{b}) = \left\langle \frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2), \frac{1}{2}(a_3 + b_3)\right\rangle$, and its radius is $\frac{1}{2}|\mathbf{a} - \mathbf{b}| = \frac{1}{2}\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$.



Or: Expand the given equation, substitute $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$ and complete the squares.

55. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1, 1, 1)$ has vector representation $\langle 1, 1, 1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x-axis [that is, $\langle 1, 0, 0\rangle$] is given by $\cos\theta = \dfrac{\langle 1,1,1\rangle \cdot \langle 1,0,0\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,0,0\rangle|} = \dfrac{1}{\sqrt{3}} \quad\Rightarrow\quad \theta = \cos^{-1}\frac{1}{\sqrt{3}} \approx 55^\circ$.

56. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then $\cos\theta = \dfrac{(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j})}{|\mathbf{i} + \mathbf{j} + \mathbf{k}|\,|\mathbf{i} + \mathbf{j}|} = \dfrac{1+1}{\sqrt{3}\sqrt{2}} = \sqrt{\dfrac{2}{3}} \quad\Rightarrow$ $\theta = \cos^{-1}\sqrt{\frac{2}{3}} \approx 35^\circ$.

57. Consider the H-C-H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1, 0, 0)$ and $(0, 1, 0)$ (or any H-C-H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\left\langle 1 - \frac{1}{2}, 0 - \frac{1}{2}, 0 - \frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$ and $\left\langle 0 - \frac{1}{2}, 1 - \frac{1}{2}, 0 - \frac{1}{2}\right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$. The bond angle, θ, is therefore given by

$$\cos\theta = \frac{\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle \cdot \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle}{\left|\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle\right|\,\left|\left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle\right|} = \frac{-\frac{1}{4} - \frac{1}{4} + \frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = -\frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\left(-\tfrac{1}{3}\right) \approx 109.5^\circ.$$

58. Let α be the angle between $\mathbf{a}$ and $\mathbf{c}$ and β be the angle between $\mathbf{c}$ and $\mathbf{b}$. We need to show that $\alpha = \beta$. Now $\cos\alpha = \dfrac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a} \cdot |\mathbf{a}|\,\mathbf{b} + \mathbf{a} \cdot |\mathbf{b}|\,\mathbf{a}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2\,|\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{c}|}$. Similarly, $\cos\beta = \dfrac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,|\mathbf{b}| + \mathbf{b} \cdot \mathbf{a}}{|\mathbf{c}|}$. Thus $\cos\alpha = \cos\beta$. However $0^\circ \le \alpha \le 180^\circ$ and $0^\circ \le \beta \le 180^\circ$, so $\alpha = \beta$ and $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

59. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $= \langle b_1, b_2, b_3 \rangle$.

Property 2: $\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$= b_1 a_1 + b_2 a_2 + b_3 a_3 = \langle b_1, b_2, b_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = \mathbf{b} \cdot \mathbf{a}$$

Property 4: $(c\mathbf{a}) \cdot \mathbf{b} = \langle ca_1, ca_2, ca_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = (ca_1)\, b_1 + (ca_2)\, b_2 + (ca_3)\, b_3$

$$= c\,(a_1 b_1 + a_2 b_2 + a_3 b_3) = c\,(\mathbf{a} \cdot \mathbf{b}) = a_1\,(cb_1) + a_2\,(cb_2) + a_3\,(cb_3)$$

$$= \langle a_1, a_2, a_3 \rangle \cdot \langle cb_1, cb_2, cb_3 \rangle = \mathbf{a} \cdot (c\mathbf{b})$$

Property 5: $\mathbf{0} \cdot \mathbf{a} = \langle 0, 0, 0 \rangle \cdot \langle a_1, a_2, a_3 \rangle = (0)\,(a_1) + (0)\,(a_2) + (0)\,(a_3) = 0$

60. Let the figure be called quadrilateral $ABCD$. The diagonals can be represented by $\overrightarrow{AC}$ and $\overrightarrow{BD}$. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{BC} - \overrightarrow{AB}$ (Since opposite sides of the object are of the same length and parallel, $\overrightarrow{AB} = \overrightarrow{DC}$.) Thus

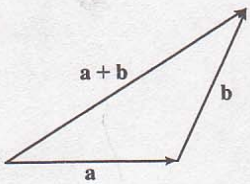
$$\overrightarrow{AC} \cdot \overrightarrow{BD} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(\overrightarrow{BC} - \overrightarrow{AB}\right) = \overrightarrow{AB} \cdot \left(\overrightarrow{BC} - \overrightarrow{AB}\right) + \overrightarrow{BC} \cdot \left(\overrightarrow{BC} - \overrightarrow{AB}\right)$$

$$= \overrightarrow{AB} \cdot \overrightarrow{BC} - \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 - \overrightarrow{AB} \cdot \overrightarrow{BC} = \left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2$$

But $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2$ because all sides of the quadrilateral are equal in length. Therefore $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$, and since both of these vectors are nonzero this tells us that the diagonals of the quadrilateral are perpendicular.

61. $|\mathbf{a} \cdot \mathbf{b}| = \big|\, |\mathbf{a}|\, |\mathbf{b}| \cos\theta \big| = |\mathbf{a}|\, |\mathbf{b}|\, |\cos\theta|$. Since $|\cos\theta| \le 1$, $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}|\, |\mathbf{b}|\, |\cos\,\theta| \le |\mathbf{a}|\, |\mathbf{b}|$.
Note: We have equality in the case of $\cos\theta = \pm 1$, so $\theta = 0$ or $\theta = \pi$, thus equality when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

62. (a)



The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.

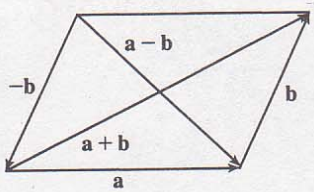
(b) $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a}) + 2\,(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{b}) = |\mathbf{a}|^2 + 2\,(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2$

$$\le |\mathbf{a}|^2 + 2\,|\mathbf{a}|\,|\mathbf{b}| + |\mathbf{b}|^2 \quad \text{(by the Cauchy-Schwartz Inequality)}$$

$$= (|\mathbf{a}| + |\mathbf{b}|)^2$$

Thus, taking the square root of both sides, $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$.

63. (a)



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

(b) $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + 2\,(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2$ and
$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - 2\,(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2$. Adding these two equations gives
$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2$.

13.4 The Cross Product ET 12.4

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}\mathbf{k} = (2-0)\,\mathbf{i} - (1-0)\,\mathbf{j} + (3-0)\,\mathbf{k} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

2. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{k}$

$= (2-0)\,\mathbf{i} - [10-(-4)]\,\mathbf{j} + [0-(-1)]\,\mathbf{k} = 2\mathbf{i} - 14\mathbf{j} + \mathbf{k}$

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}\mathbf{k} = (-1-0)\,\mathbf{i} - (1-0)\,\mathbf{j} + [2-(-3)]\,\mathbf{k} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 2 \\ 6 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 2 \\ 6 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ 6 & 3 \end{vmatrix}\mathbf{k}$

$= (2-6)\,\mathbf{i} - (-3-12)\,\mathbf{j} + (-9-12)\,\mathbf{k} = -4\mathbf{i} + 15\mathbf{j} - 21\mathbf{k}$

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k} = [2-(-1)]\,\mathbf{i} - (4-0)\,\mathbf{j} + (2-0)\,\mathbf{k} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

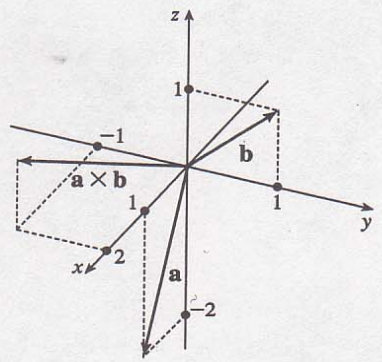
6. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k} = (-1-1)\,\mathbf{i} - (1-1)\,\mathbf{j} + [1-(-1)]\,\mathbf{k} = -2\mathbf{i} + 2\mathbf{k}$

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{k}$

$= [-6-(-8)]\,\mathbf{i} - (-9-4)\,\mathbf{j} + (-6-2)\,\mathbf{k} = 2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}$

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= 2\mathbf{i} - \mathbf{j} + \mathbf{k}$



9. (a) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the dot product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.

(c) Since $\mathbf{b} \times \mathbf{c}$ is a vector, the cross product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is meaningful and results in another vector.

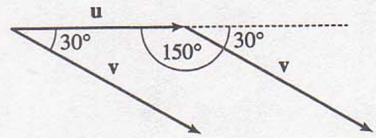
(d) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, so the cross product $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ is meaningless.

(e) Since $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{c} \cdot \mathbf{d})$ are both scalars, the cross product $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ is meaningless.

(f) $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are both vectors, so the dot product $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ is meaningful and is a scalar.

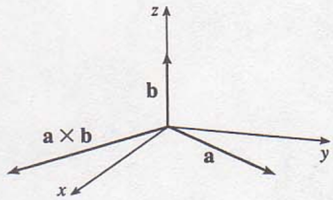
10. Using Theorem 6, we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = (5)\,(10) \sin 60^\circ = 25\sqrt{3}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed into the page.

11. If we sketch $\mathbf{u}$ and $\mathbf{v}$ starting from the same initial point, we see that the angle between them is 30°. Using Theorem 6, we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 30^\circ = (6)\,(8)\left(\frac{1}{2}\right) = 24$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed into the page.



12. (a) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta = 3 \cdot 2 \cdot \sin\frac{\pi}{2} = 6$

(b) $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{k}$, so it lies in the xy-plane, and its z-coordinate is 0. By the right-hand rule, its y-component is negative and its x-component is positive.



13. $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{k} = -2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$

(and notice $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ here, as we know is always true by Theorem 8.)

14. $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 0 & 2 & 5 \end{vmatrix} = \mathbf{a} \times \left[\begin{vmatrix} -1 & 0 \\ 2 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} \mathbf{k} \right] = \mathbf{a} \times (-5\mathbf{i} - 10\mathbf{j} + 4\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 3 \\ -5 & -10 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ -10 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -4 & 3 \\ -5 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -4 & 0 \\ -5 & -10 \end{vmatrix} \mathbf{k} = 30\mathbf{i} + \mathbf{j} + 40\mathbf{k}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 3 \\ 2 & -1 & 0 \end{vmatrix} \times \mathbf{c} = \left[\begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -4 & 3 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -4 & 0 \\ 2 & -1 \end{vmatrix} \mathbf{k} \right] \times \mathbf{c} = (3\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}) \times \mathbf{c}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 6 & 4 \\ 0 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 2 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} \mathbf{k} = 22\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$$

Thus $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

15. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\langle 1,-1,1\rangle \times \langle 0,4,4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}\mathbf{k} = -8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$. So two unit vectors orthogonal to both are $\pm\dfrac{\langle -8,-4,4\rangle}{\sqrt{64+16+16}} = \pm\dfrac{\langle -8,-4,4\rangle}{4\sqrt{6}}$, that is, $\left\langle -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle$ and $\left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\rangle$.

16. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{k} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Thus, two unit vectors orthogonal to both are $\pm\frac{1}{\sqrt{6}}\langle 1,-1,-2\rangle$, that is, $\left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right\rangle$ and $\left\langle -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\rangle$.

17. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$. Then $\mathbf{0} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix}\mathbf{k} = \mathbf{0}$,

$\mathbf{a} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix}\mathbf{k} = \mathbf{0}$.

18. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} &= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle b_1, b_2, b_3\rangle = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3 \\ &= (a_2b_3b_1 - a_3b_2b_1) - (a_1b_3b_2 - a_3b_1b_2) + (a_1b_2b_3 - a_2b_1b_3) = 0\end{aligned}$$

19.
$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle \\ &= \langle (-1)(b_2a_3 - b_3a_2), (-1)(b_3a_1 - b_1a_3), (-1)(b_1a_2 - b_2a_1)\rangle \\ &= -\langle b_2a_3 - b_3a_2, b_3a_1 - b_1a_3, b_1a_2 - b_2a_1\rangle = -\mathbf{b}\times\mathbf{a}\end{aligned}$$

20. $c\mathbf{a} = \langle ca_1, ca_2, ca_3\rangle$, so

$$\begin{aligned}(c\mathbf{a})\times\mathbf{b} &= \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1\rangle \\ &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle = c\,(\mathbf{a}\times\mathbf{b}) \\ &= \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1\rangle \\ &= \langle a_2(cb_3) - a_3(cb_2), a_3(cb_1) - a_1(cb_3), a_1(cb_2) - a_2(cb_1)\rangle = \mathbf{a}\times c\mathbf{b}\end{aligned}$$

21.
$$\begin{aligned}\mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\times\langle b_1+c_1, b_2+c_2, b_3+c_3\rangle \\ &= \langle a_2(b_3+c_3) - a_3(b_2+c_2), a_3(b_1+c_1) - a_1(b_3+c_3), a_1(b_2+c_2) - a_2(b_1+c_1)\rangle \\ &= \langle a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\rangle \\ &= \langle (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2), (a_3b_1 - a_1b_3) + (a_3c_1 - a_1c_3), \\ &\qquad (a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1)\rangle \\ &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1\rangle \\ &= (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})\end{aligned}$$

22. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}+\mathbf{b})$ by Property 1 of Theorem 8

$= -(\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b})$ by Property 3 of Theorem 8

$= -(-\mathbf{a}\times\mathbf{c}+(-\mathbf{b}\times\mathbf{c}))$ by Property 1 of Theorem 8

$= \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$ by Property 2 of Theorem 8

23. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. The vectors corresponding to $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are $\mathbf{a} = \langle 3,-1,0\rangle$ and $\mathbf{b} = \langle 2,-2,0\rangle$, so the area of parallelogram $ABCD$ is $|\mathbf{a}\times\mathbf{b}| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 2 & -2 & 0 \end{matrix} \right\| = |(0)\,\mathbf{i}-(0)\,\mathbf{j}+(-6+2)\,\mathbf{k}| = |-4\mathbf{k}| = 4.$

24. $\overrightarrow{PQ} = \langle 5,0,0\rangle$ and $\overrightarrow{PR} = \langle 2,6,6\rangle$, so the area of parallelogram $PQRS$ is

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 0 \\ 2 & 6 & 6 \end{matrix} \right\| = |(0)\,\mathbf{i}-(30)\,\mathbf{j}+(30)\,\mathbf{k}| = |-30\mathbf{j}+30\mathbf{k}| = 30\sqrt{2}.$$

25. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ} = \langle -1,2,0\rangle$ and $\overrightarrow{PR} = \langle -1,0,3\rangle$, so

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (2)\,(3)-(0)\,(0)\,,(0)\,(-1)-(-1)\,(3)\,,(-1)\,(0)-(2)\,(-1)\rangle = \langle 6,3,2\rangle$$

Therefore, $\langle 6,3,2\rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P, Q, and R.

(b) Note that the area of the triangle determined by P, Q, and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 6,3,2\rangle| = \sqrt{36+9+4} = 7$, so the area of the triangle is $\frac{1}{2}\,(7) = \frac{7}{2}$.

26. (a) $\overrightarrow{PQ} = \langle 1,4,6\rangle$ and $\overrightarrow{PR} = \langle 2,1,8\rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (4)\,(8)-(6)\,(1)\,,(6)\,(2)-(1)\,(8)\,,(1)\,(1)-(4)\,(2)\rangle = \langle 26,4,-7\rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 26,4,-7\rangle| = \sqrt{676+16+49} = \sqrt{741}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{741}$.

27. (a) $\overrightarrow{PQ} = \langle 1,-1,1\rangle$ and $\overrightarrow{PR} = \langle 4,3,7\rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (-1)\,(7)-(1)\,(3)\,,(1)\,(4)-(1)\,(7)\,,(1)\,(3)-(-1)\,(4)\rangle = \langle -10,-3,7\rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle -10,-3,7\rangle| = \sqrt{100+9+49} = \sqrt{158}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{158}$.

28. (a) $\overrightarrow{PQ} = \langle 1,1,3\rangle$ and $\overrightarrow{PR} = \langle 3,2,5\rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (1)\,(5)-(3)\,(2)\,,(3)\,(3)-(1)\,(5)\,,(1)\,(2)-(1)\,(3)\rangle = \langle -1,4,-1\rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is
$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle -1, 4, -1\rangle| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$, so the area of triangle PQR is
$\frac{1}{2} \cdot 3\sqrt{2} = \frac{3}{2}\sqrt{2}$.

29. We know that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the magnitude of their scalar triple product, which is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 0 & 6 \\ 2 & 3 & -8 \\ 8 & -5 & 6 \end{vmatrix} = 1 \begin{vmatrix} 3 & -8 \\ -5 & 6 \end{vmatrix} - 0 + 6 \begin{vmatrix} 2 & 3 \\ 8 & -5 \end{vmatrix}$$
$$= (18 - 40) + 6\,(-10 - 24) = -226$$

Thus the volume of the parallelepiped is $|-226| = 226$ cubic units.

30. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 2 \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -6 - 9 - 4 = -19$. So the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $|-19| = 19$ cubic units.

31. $\mathbf{a} = \overrightarrow{PQ} = \langle 1, -1, 2\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 3, 0, 6\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 2, -2, -3\rangle$.
$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} = 1 \begin{vmatrix} 0 & 6 \\ -2 & -3 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 6 \\ 2 & -3 \end{vmatrix} + 2 \begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix} = 12 - 21 - 12 = -21$, so the volume of the parallelepiped is 21 cubic units.

32. $\mathbf{a} = \overrightarrow{PQ} = \langle 2, 3, 3\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle -1, -1, -1\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 6, -2, 2\rangle$.
$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & 3 \\ -1 & -1 & -1 \\ 6 & -2 & 2 \end{vmatrix} = 2 \begin{vmatrix} -1 & -1 \\ -2 & 2 \end{vmatrix} - 3 \begin{vmatrix} -1 & -1 \\ 6 & 2 \end{vmatrix} + 3 \begin{vmatrix} -1 & -1 \\ 6 & -2 \end{vmatrix} = -8 - 12 + 24 = 4$, so the volume of the parallelepiped is 4 cubic units.

33. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = 2 \begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ 7 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 7 & 3 \end{vmatrix} = -4 - 6 + 10 = 0$, which says that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 0, and thus these three vectors are coplanar.

34. $\mathbf{a} = \overrightarrow{PQ} = \langle 1, 4, 5\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 2, -1, 1\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 5, 2, 7\rangle$.
$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & 2 & 7 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 2 & 7 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix} + 5 \begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = -9 - 36 + 45 = 0$, so the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 0, which says these vectors lie in the same plane. Therefore, their initial and terminal points P, Q, R and S also lie in the same plane.

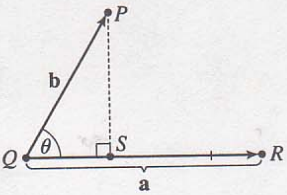
35. The magnitude of the torque is
$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.18\text{ m})\,(60\text{ N}) \sin(70 + 10)^\circ = 10.8 \sin 80^\circ \approx 10.6\text{ J}$.

36. $|\mathbf{r}| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ ft. A line drawn from the point P to the point of application of the force makes an angle of $180\,^\circ - (45 + 30)\,^\circ = 105\,^\circ$ with the force vector. Therefore, $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (4\sqrt{2})\,(36)\sin 105\,^\circ \approx 197$ ft-lb.

37. Using the notation of the text, $\mathbf{r} = \langle 0, 0.3, 0\rangle$ and $\mathbf{F}$ has direction $\langle 0, 3, -4\rangle$. The angle θ between them can be determined by $\cos\theta = \dfrac{\langle 0,0.3,0\rangle \cdot \langle 0,3,-4\rangle}{|\langle 0,0.3,0\rangle|\,|\langle 0,3,-4\rangle|} \;\Rightarrow\; \cos\theta = \dfrac{0.9}{(0.3)\,(5)} \;\Rightarrow\; \cos\theta = 0.6 \;\Rightarrow\; \theta \approx 53.1\,^\circ$. Then $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta \;\Rightarrow\; 100 = 0.3\,|\mathbf{F}|\sin 53.1\,^\circ \;\Rightarrow\; |\mathbf{F}| \approx 417$ N.

38. Since $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$, $0 \le \theta \le \pi$, $|\mathbf{u} \times \mathbf{v}|$ achieves its maximum value for $\sin\theta = 1 \;\Rightarrow\; \theta = \frac{\pi}{2}$, in which case $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| = 15$. The minimum value is zero, which occurs when $\sin\theta = 0 \;\Rightarrow\; \theta = 0$ or π, so when $\mathbf{u}$, $\mathbf{v}$ are parallel. Thus, when $\mathbf{u}$ points in the same direction as $\mathbf{v}$, so $\mathbf{u} = 3\mathbf{j}$, $|\mathbf{u} \times \mathbf{v}| = 0$. As $\mathbf{u}$ rotates counterclockwise, $\mathbf{u} \times \mathbf{v}$ is directed in the negative z-direction (by the right-hand rule) and the length increases until $\theta = \frac{\pi}{2}$, in which case $\mathbf{u} = -3\mathbf{i}$ and $|\mathbf{u} \times \mathbf{v}| = 15$. As $\mathbf{u}$ rotates to the negative y-axis, $\mathbf{u} \times \mathbf{v}$ remains pointed in the negative z-direction and the length of $\mathbf{u} \times \mathbf{v}$ decreases to 0, after which the direction of $\mathbf{u} \times \mathbf{v}$ reverses to point in the positive z-direction and $|\mathbf{u} \times \mathbf{v}|$ increases. When $\mathbf{u} = 3\mathbf{i}$ (so $\theta = \frac{\pi}{2}$), $|\mathbf{u} \times \mathbf{v}|$ again reaches its maximum of 15, after which $|\mathbf{u} \times \mathbf{v}|$ decreases to 0 as $\mathbf{u}$ rotates to the positive y-axis.

39. (a)



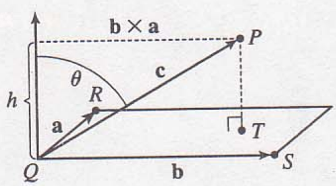
The distance between a point and a line is the length of the perpendicular from the point to the line, here $\left|\overrightarrow{PS}\right| = d$. But referring to triangle PQS, $d = \left|\overrightarrow{PS}\right| = \left|\overrightarrow{QP}\right|\sin\theta = |\mathbf{b}|\sin\theta$. But θ is the angle between $\overrightarrow{QP} = \mathbf{b}$ and $\overrightarrow{QR} = \mathbf{a}$. Thus by Theorem 6, $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$ and so

$$d = |\mathbf{b}|\sin\theta = \frac{|\mathbf{b}|\,|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}.$$

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, -2, -1\rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 1, -5, -7\rangle$. Then

$\mathbf{a} \times \mathbf{b} = \langle (-2)\,(-7) - (-1)\,(-5), (-1)\,(1) - (-1)\,(-7), (-1)\,(-5) - (-2)\,(1)\rangle = \langle 9, -8, 7\rangle$. Thus the distance is $d = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{1}{\sqrt{6}}\sqrt{81 + 64 + 49} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$.

40. (a)



The distance between a point and a plane is the length of the perpendicular from the point to the plane, here $\left|\overrightarrow{TP}\right| = d$. But $\overrightarrow{TP}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (because $\mathbf{b} \times \mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{a}$) and $d = \left|\overrightarrow{TP}\right| =$ the absolute value of the scalar projection of $\mathbf{c}$ along $\mathbf{b} \times \mathbf{a}$, which is $|\mathbf{c}|\,|\cos\theta|$. (Notice that this is the same setup as the development of the volume of a parallelepiped with $h = |\mathbf{c}|\,|\cos\theta|$). Thus $d = |\mathbf{c}|\,|\cos\theta| = h = V/A$ where $A = |\mathbf{a} \times \mathbf{b}|$, the area of the base. So finally $d = \dfrac{V}{A} = \dfrac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$.

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, 2, 0\rangle$, $\mathbf{b} = \overrightarrow{QS} = \langle -1, 0, 3\rangle$ and $\mathbf{c} = \overrightarrow{QP} = \langle 1, 1, 4\rangle$. Then

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} + 0 = 17$$

and
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}\mathbf{k} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

Thus $d = \dfrac{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}{|\mathbf{a}\times\mathbf{b}|} = \dfrac{17}{\sqrt{36+9+4}} = \dfrac{17}{7}$.

41.
$$\begin{aligned}
(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) &= (\mathbf{a}-\mathbf{b})\times\mathbf{a} + (\mathbf{a}-\mathbf{b})\times\mathbf{b} && \text{by Theorem 8 \#3} \\
&= \mathbf{a}\times\mathbf{a} + (-\mathbf{b})\times\mathbf{a} + \mathbf{a}\times\mathbf{b} + (-\mathbf{b})\times\mathbf{b} && \text{by Theorem 8 \#4} \\
&= (\mathbf{a}\times\mathbf{a}) - (\mathbf{b}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{b}) && \text{by Theorem 8 \#2} \\
&= \mathbf{0} - (\mathbf{b}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b}) - \mathbf{0} && \text{by Example 2} \\
&= (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{b}) && \text{by Theorem 8 \#1} \\
&= 2\,(\mathbf{a}\times\mathbf{b})
\end{aligned}$$

42. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$, so $\mathbf{b}\times\mathbf{c} = \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1\rangle$ and

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \langle a_2\,(b_1c_2 - b_2c_1) - a_3\,(b_3c_1 - b_1c_3)\,, a_3\,(b_2c_3 - b_3c_2) - a_1\,(b_1c_2 - b_2c_1)\,, \\
&\qquad a_1\,(b_3c_1 - b_1c_3) - a_2\,(b_2c_3 - b_3c_2)\rangle \\
&= \langle a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3, a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1, \\
&\qquad a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2\rangle \\
&= \langle (a_2c_2 + a_3c_3)\,b_1 - (a_2b_2 + a_3b_3)\,c_1, (a_1c_1 + a_3c_3)\,b_2 - (a_1b_1 + a_3b_3)\,c_2, \\
&\qquad (a_1c_1 + a_2c_2)\,b_3 - (a_1b_1 + a_2b_2)\,c_3\rangle \\
(\bigstar) \quad &= \langle (a_2c_2 + a_3c_3)\,b_1 - (a_2b_2 + a_3b_3)\,c_1 + a_1b_1c_1 - a_1b_1c_1, \\
&\qquad (a_1c_1 + a_3c_3)\,b_2 - (a_1b_1 + a_3b_3)\,c_2 + a_2b_2c_2 - a_2b_2c_2, \\
&\qquad (a_1c_1 + a_2c_2)\,b_3 - (a_1b_1 + a_2b_2)\,c_3 + a_3b_3c_3 - a_3b_3c_3\rangle \\
&= \langle (a_1c_1 + a_2c_2 + a_3c_3)\,b_1 - (a_1b_1 + a_2b_2 + a_3b_3)\,c_1, \\
&\qquad (a_1c_1 + a_2c_2 + a_3c_3)\,b_2 - (a_1b_1 + a_2b_2 + a_3b_3)\,c_2, \\
&\qquad (a_1c_1 + a_2c_2 + a_3c_3)\,b_3 - (a_1b_1 + a_2b_2 + a_3b_3)\,c_3\rangle \\
&= (a_1c_1 + a_2c_2 + a_3c_3)\,\langle b_1, b_2, b_3\rangle - (a_1b_1 + a_2b_2 + a_3b_3)\,\langle c_1, c_2, c_3\rangle \\
&= (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}
\end{aligned}$$

($\bigstar$) Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don't change the value of the component.

43. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$$=[(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]+[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}]+[(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}] \qquad \text{by Exercise 42}$$

$$=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}+(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}=\mathbf{0}$$

44. Let $\mathbf{c}\times\mathbf{d}=\mathbf{v}$. Then

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})&=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{v}) && \text{by Theorem 8 \#5}\\ &=\mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]\\ &=\mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{d}] && \text{by Exercise 42}\\ &=(\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) && \text{by Theorem 8 \#2}\\ &=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}\end{aligned}$$

45. (a) No. If $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$, then $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, which can happen if $\mathbf{b}\neq\mathbf{c}$. For example, let $\mathbf{a}=\langle 1,1,1\rangle$, $\mathbf{b}=\langle 1,0,0\rangle$ and $\mathbf{c}=\langle 0,1,0\rangle$.

(b) No. If $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ then $\mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{0}$, which implies that $\mathbf{a}$ is parallel to $\mathbf{b}-\mathbf{c}$, which of course can happen if $\mathbf{b}\neq\mathbf{c}$.

(c) Yes. Since $\mathbf{a}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, by part (a). From part (b), $\mathbf{a}$ is also parallel to $\mathbf{b}-\mathbf{c}$. Thus since $\mathbf{a}\neq\mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b}-\mathbf{c}$, we have $\mathbf{b}-\mathbf{c}=\mathbf{0}$, so $\mathbf{b}=\mathbf{c}$.

46. (a) $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_i$ if $i\neq j$ by the definition of $\mathbf{k}_i$ and Theorem 5.

(b) $$\mathbf{k}_1\cdot\mathbf{v}_1=\frac{\mathbf{v}_2\times\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_1=\frac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1$$

$$\mathbf{k}_2\cdot\mathbf{v}_2=\frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_2=\frac{\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{(\mathbf{v}_2\times\mathbf{v}_3)\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1 \text{ by Theorem 8 \#5}$$

$$\mathbf{k}_3\cdot\mathbf{v}_3=\frac{(\mathbf{v}_1\times\mathbf{v}_2)\cdot\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1 \text{ by Theorem 8 \#5}$$

(c) $$\mathbf{k}_1\cdot(\mathbf{k}_2\times\mathbf{k}_3)=\mathbf{k}_1\cdot\left(\frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\times\frac{\mathbf{v}_1\times\mathbf{v}_2}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\right)=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot[(\mathbf{v}_3\times\mathbf{v}_1)\times(\mathbf{v}_1\times\mathbf{v}_2)]$$

$$=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot([(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_2]\,\mathbf{v}_1-[(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_1]\,\mathbf{v}_2) \text{ by Exercise 42.}$$

But $(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_1=0$ since $\mathbf{v}_3\times\mathbf{v}_1$ is orthogonal to $\mathbf{v}_1$, and $(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_2=\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)=(\mathbf{v}_2\times\mathbf{v}_3)\cdot\mathbf{v}_1=\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)$. Thus

$$\begin{aligned}\mathbf{k}_1\cdot(\mathbf{k}_2\times\mathbf{k}_3)&=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]\,\mathbf{v}_1=\frac{\mathbf{k}_1\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\\ &=\frac{1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} \text{ by part (b).}\end{aligned}$$

Discovery Project □ The Geometry of a Tetrahedron

1. Set up a coordinate system so that vertex S is at the origin, $R = (0, y_1, 0)$, $Q = (x_2, y_2, 0)$, $P = (x_3, y_3, z_3)$. Then $\overrightarrow{SR} = \langle 0, y_1, 0\rangle$, $\overrightarrow{SQ} = \langle x_2, y_2, 0\rangle$, $\overrightarrow{SP} = \langle x_3, y_3, z_3\rangle$, $\overrightarrow{QR} = \langle -x_2, y_1 - y_2, 0\rangle$, and $\overrightarrow{QP} = \langle x_3 - x_2, y_3 - y_2, z_3\rangle$.
Let

$$\begin{aligned}\mathbf{v}_S &= \overrightarrow{QR} \times \overrightarrow{QP}\\ &= (y_1 z_3 - y_2 z_3)\,\mathbf{i} + x_2 z_3 \mathbf{j} + (-x_2 y_3 - x_3 y_1 + x_3 y_2 + x_2 y_1)\,\mathbf{k}\end{aligned}$$

Then $\mathbf{v}_S$ is an outward normal to the face opposite vertex S. Similarly,
$\mathbf{v}_R = \overrightarrow{SQ} \times \overrightarrow{SP} = y_2 z_3 \mathbf{i} - x_2 z_3 \mathbf{j} + (x_2 y_3 - x_3 y_2)\,\mathbf{k}$, $\mathbf{v}_Q = \overrightarrow{SP} \times \overrightarrow{SR} = -y_1 z_3 \mathbf{i} + x_3 y_1 \mathbf{k}$, and
$\mathbf{v}_P = \overrightarrow{SR} \times \overrightarrow{SQ} = -x_2 y_1 \mathbf{k} \quad\Rightarrow\quad \mathbf{v}_S + \mathbf{v}_R + \mathbf{v}_Q + \mathbf{v}_P = \mathbf{0}$. Now

$$\begin{aligned}|\mathbf{v}_S| &= \text{area of the parallelogram determined by } \overrightarrow{QR} \text{ and } \overrightarrow{QP}\\ &= 2\,(\text{area of triangle } RQP)\\ &= 2|\mathbf{v}_1|\end{aligned}$$

So $\mathbf{v}_S = 2\mathbf{v}_1$, and similarly $\mathbf{v}_R = 2\mathbf{v}_2$, $\mathbf{v}_Q = 2\mathbf{v}_3$, $\mathbf{v}_P = 2\mathbf{v}_4$. Thus $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$.

2. (a) Let $S = (x_0, y_0, z_0)$, $R = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, $P = (x_3, y_3, z_3)$ be the four vertices. Then

$$\begin{aligned}\text{Volume} &= \tfrac{1}{3}\,(\text{distance from } S \text{ to plane } RQP) \times (\text{area of triangle } RQP)\\ &= \frac{1}{3}\,\frac{\left|\mathbf{N}\cdot\overrightarrow{SR}\right|}{|\mathbf{N}|}\cdot\frac{1}{2}\left|\overrightarrow{RQ}\times\overrightarrow{RP}\right|\end{aligned}$$

where $\mathbf{N}$ is a vector which is normal to the face RQP. Thus $\mathbf{N} = \overrightarrow{RQ} \times \overrightarrow{RP}$. Therefore

$$V = \left|\tfrac{1}{6}\left(\overrightarrow{RQ}\times\overrightarrow{RP}\right)\cdot\overrightarrow{SR}\right| = \frac{1}{6}\left\|\begin{matrix} x_0 - x & y_0 - y_1 & z_0 - z_1\\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1\\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1\end{matrix}\right\|.$$

(b) Using the formula from part (a), $V = \dfrac{1}{6}\left\|\begin{matrix} 1-1 & 1-2 & 1-3\\ 1-1 & 1-2 & 2-3\\ 3-1 & -1-2 & 2-3\end{matrix}\right\| = \dfrac{1}{6}\,|2\,(1-2)| = \dfrac{1}{3}$.

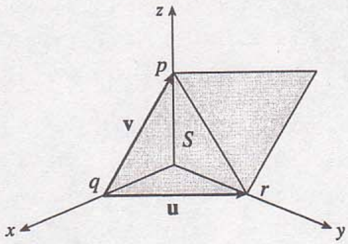
3. We define a vector $\mathbf{v}_1$ to have length equal to the area of the face opposite vertex P, so we can say $|\mathbf{v}_1| = A$, and direction perpendicular to the face and pointing outward, as in Problem 1. Similarly, we define $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ so that $|\mathbf{v}_2| = B$, $|\mathbf{v}_3| = C$, and $|\mathbf{v}_4| = D$ and with the analogous directions. From Problem 1, we know
$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0} \quad\Rightarrow\quad \mathbf{v}_4 = -(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \quad\Rightarrow\quad |\mathbf{v}_4| = |-(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)| = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|$
$\Rightarrow\quad |\mathbf{v}_4|^2 = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|^2 \quad\Rightarrow$

$$\begin{aligned}\mathbf{v}_4\cdot\mathbf{v}_4 &= (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\cdot(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\\ &= \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_1\cdot\mathbf{v}_2 + \mathbf{v}_1\cdot\mathbf{v}_3 + \mathbf{v}_2\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2 + \mathbf{v}_2\cdot\mathbf{v}_3 + \mathbf{v}_3\cdot\mathbf{v}_1 + \mathbf{v}_3\cdot\mathbf{v}_2 + \mathbf{v}_3\cdot\mathbf{v}_3\end{aligned}$$

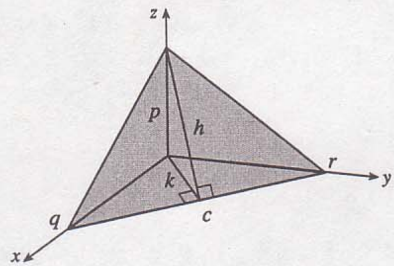
Since the vertex S is trirectangular, we know the three faces meeting at S are mutually perpendicular, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are also mutually perpendicular. Therefore, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Thus we have $\mathbf{v}_4 \cdot \mathbf{v}_4 = \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3 \quad \Rightarrow \quad |\mathbf{v}_4|^2 = |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 + |\mathbf{v}_3|^2 \quad \Rightarrow \quad D^2 = A^2 + B^2 + C^2$.

Another Method: We introduce a coordinate system, as shown. Recall that the area of the parallelogram spanned by two vectors is equal to the length of their cross product, so since $\mathbf{u} \times \mathbf{v} = \langle -q, r, 0 \rangle \times \langle -q, 0, p \rangle = \langle pr, pq, qr \rangle$, we have $|\mathbf{u} \times \mathbf{v}| = \sqrt{(pr)^2 + (pq)^2 + (qr)^2}$, and therefore

$$D^2 = \left(\tfrac{1}{2}\,|\mathbf{u} \times \mathbf{v}|\right)^2 = \tfrac{1}{4}\left[(pr)^2 + (pq)^2 + (qr)^2\right]$$
$$= \left(\tfrac{1}{2}pr\right)^2 + \left(\tfrac{1}{2}pq\right)^2 + \left(\tfrac{1}{2}qr\right)^2 = A^2 + B^2 + C^2.$$



A Third Method: We draw a line from S perpendicular to QR, as shown. Now $D = \frac{1}{2}ch$, so $D^2 = \frac{1}{4}c^2h^2$. Substituting $h^2 = p^2 + k^2$, we get $D^2 = \frac{1}{4}c^2\left(p^2 + k^2\right) = \frac{1}{4}c^2p^2 + \frac{1}{4}c^2k^2$. But $C = \frac{1}{2}ck$, so $D^2 = \frac{1}{4}c^2p^2 + C^2$. Now substituting $c^2 = q^2 + r^2$ gives $D^2 = \frac{1}{4}p^2q^2 + \frac{1}{4}q^2r^2 + C^2 = A^2 + B^2 + C^2$.



13.5 Equations of Lines and Planes — ET 12.5

1. (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.

   (b) False; for example, the x- and y-axes are both perpendicular to the z-axis, yet the x- and y-axes are not parallel.

   (c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.

   (d) False; for example, the xy- and yz-planes are not parallel, yet they are both perpendicular to the xz-plane.

   (e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane $z = 1$.

   (f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

   (g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x-axis.

   (h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.

   (i) True; see Figure 9 and the accompanying discussion.

   (j) False; they can be skew, as in Example 3.

(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ, $0\,^\circ \leq \theta < 90\,^\circ$, and the line will intersect the plane at an angle $90\,^\circ - \theta$.

2. For this line, we have $\mathbf{r}_0 = \mathbf{i} - 3\,\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (\mathbf{i} - 3\mathbf{k}) + t\,(2\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}) = (1 + 2t)\,\mathbf{i} - 4t\,\mathbf{j} + (-3 + 5t)\,\mathbf{k}$ and parametric equations are $x = 1 + 2t$, $y = -4t$, $z = -3 + 5t$.

3. For this line, we have $\mathbf{r}_0 = -2\,\mathbf{i} + 4\,\mathbf{j} + 10\,\mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} + \mathbf{j} - 8\,\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (-2\,\mathbf{i} + 4\,\mathbf{j} + 10\,\mathbf{k}) + t\,(3\,\mathbf{i} + \mathbf{j} - 8\,\mathbf{k}) = (-2 + 3t)\,\mathbf{i} + (4 + t)\,\mathbf{j} + (10 - 8t)\,\mathbf{k}$ and parametric equations are $x = -2 + 3t$, $y = 4 + t$, $z = 10 - 8t$.

4. This line has the same direction as the given line, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Here $\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, so a vector equation is $\mathbf{r} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t\,(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2t\mathbf{i} - t\mathbf{j} + 3t\mathbf{k}$ and parametric equations are $x = 2t$, $y = -t$, $z = 3t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1, 3, 1\rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t\,(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = (1 + t)\,\mathbf{i} + 3t\mathbf{j} + (6 + t)\,\mathbf{k}$, and parametric equations are $x = 1 + t, y = 3t, z = 6 + t$.

6. The vector $\mathbf{v} = \langle 1 - 0, 2 - 0, 3 - 0\rangle = \langle 1, 2, 3\rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 0 + 1 \cdot t = t$, $y = 0 + 2 \cdot t = 2t$, $z = 0 + 3 \cdot t = 3t$, while symmetric equations are $x = \dfrac{y}{2} = \dfrac{z}{3}$.

7. $\mathbf{v} = \langle 3 - 3, 2 - 1, -6 - (-1)\rangle = \langle 0, 1, -5\rangle$, and letting $P_0 = (3, 1, -1)$, parametric equations are $x = 3$, $y = 1 + t$, $z = -1 - 5t$, while symmetric equations are $x = 3, y - 1 = (z + 1)\,/\,(-5)$. Notice here that the direction number $a = 0$, so rather than writing $(x - 3)\,/0$ in the symmetric equation we must write the equation $x = 3$ separately.

8. $\mathbf{v} = \langle 4 - (-1)\,, -3 - 0, 3 - 5\rangle = \langle 5, -3, -2\rangle$, and letting $P_0 = (-1, 0, 5)$, parametric equations are $x = -1 + 5t$, $y = -3t$, $z = 5 - 2t$, while symmetric equations are $\dfrac{x + 1}{5} = \dfrac{y}{-3} = \dfrac{z - 5}{-2}$.

9. $\mathbf{v} = \left\langle 2 - 0, 1 - \frac{1}{2}, -3 - 1\right\rangle = \left\langle 2, \frac{1}{2}, -4\right\rangle$, and letting $P_0 = (2, 1, -3)$, parametric equations are $x = 2 + 2t$, $y = 1 + \frac{1}{2}t$, $z = -3 - 4t$, while symmetric equations are $\dfrac{x - 2}{2} = \dfrac{y - 1}{1/2} = \dfrac{z + 3}{-4}$ or $\dfrac{x - 2}{2} = 2y - 2 = \dfrac{z + 3}{-4}$.

10. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1\rangle \times \langle 1, 0, 1\rangle = \langle 1, 0, -1\rangle$ is the direction of this line. Taking the point $(0, 1, 0)$ as P_0, parametric equations are $x = t$, $y = 1$, $z = -t$, and symmetric equations are $x = -z$, $y = 1$.

11. Direction vectors of the lines are $\mathbf{v}_1 = \langle 6, 9, 12\rangle$ and $\mathbf{v}_2 = \langle 4, 6, 8\rangle$, and since $\mathbf{v}_1 = \frac{3}{2}\mathbf{v}_2$, the direction vectors and thus the lines are parallel.

12. Direction vectors of the lines are $\mathbf{v}_1 = \langle 1, -2, 5\rangle$ and $\mathbf{v}_2 = \langle 3, 4, 1\rangle$. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 - 8 + 5 = 0$, the direction vectors and thus the lines are perpendicular.

13. (a) A direction vector of the line with parametric equations $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$ is $\mathbf{v} = \langle 2, 3, -7\rangle$ and the desired parallel line must also have $\mathbf{v}$ as a direction vector. Here $P_0 = (0, 2, -1)$, so symmetric equations for the line are $\dfrac{x}{2} = \dfrac{y - 2}{3} = \dfrac{z + 1}{-7}$.

(b) The line intersects the xy-plane when $z = 0$, so we need $\frac{x}{2} = \frac{y-2}{3} = \frac{1}{-7}$ or $x = -\frac{2}{7}$, $y = \frac{11}{7}$. Thus the point of intersection with the xy-plane is $\left(-\frac{2}{7}, \frac{11}{7}, 0\right)$. Similarly for the yz-plane, we need $x = 0 \Leftrightarrow$ $0 = \frac{y-2}{3} = \frac{z+1}{-7} \Leftrightarrow y = 2, z = -1$. Thus the line intersects the yz-plane at $(0, 2, -1)$. For the xz-plane, we need $y = 0 \Leftrightarrow \frac{x}{2} = -\frac{2}{3} = \frac{z+1}{-7} \Leftrightarrow x = -\frac{4}{3}, z = \frac{11}{3}$. So the line intersects the xz-plane at $\left(-\frac{4}{3}, 0, \frac{11}{3}\right)$.

14. (a) A vector normal to the plane $2x - y + z = 1$ is $\mathbf{n} = \langle 2, -1, 1\rangle$, and since the line is to be perpendicular to the plane, $\mathbf{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x = 5 + 2t$, $y = 1 - t$, $z = t$.

(b) On the xy-plane, $z = 0$. So $z = t = 0$ in the parametric equations of the line, and therefore $x = 5$ and $y = 1$, giving the point of intersection $(5, 1, 0)$. For the yz-plane, $x = 0$ which implies $t = -\frac{5}{2}$, so $y = \frac{7}{2}$ and $z = -\frac{5}{2}$ and the point is $\left(0, \frac{7}{2}, -\frac{5}{2}\right)$. For the xz-plane, $y = 0$ which implies $t = 1$, so $x = 7$ and $z = 1$ and the point of intersection is $(7, 0, 1)$.

15. The lines aren't parallel since the direction vectors $\langle 2, 4, -3\rangle$ and $\langle 1, 3, 2\rangle$ aren't parallel, so we check to see if the lines intersect. The parametric equations of the lines are L_1: $x = 4 + 2t$, $y = -5 + 4t$, $z = 1 - 3t$ and L_2: $x = 2 + s$, $y = -1 + 3s$, $z = 2s$. For the lines to intersect we must be able to find one value of t and one value of s satisfying the following three equations: $4 + 2t = 2 + s$, $-5 + 4t = -1 + 3s$, $1 - 3t = 2s$. Solving the first two equations we get $t = -5$, $s = -8$ and checking, we see that these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

16. Since the direction vectors $\langle 2, 1, 4\rangle$ and $\langle 1, 2, 3\rangle$ aren't parallel, the lines aren't parallel. Here the parametric equations are L_1: $x = 1 + 2t$, $y = t$, $z = 1 + 4t$; L_2: $x = s$, $y = -2 + 2s$, $z = -2 + 3s$. Thus, for the lines to intersect, the three equations $1 + 2t = s$, $t = -2 + 2s$ and $1 + 4t = -2 + 3s$ must be satisfied simultaneously. Solving the first two equations gives $t = 0$, $s = 1$ and, checking, we see these values do satisfy the third equation, so the lines intersect when $t = 0$ and $s = 1$, that is, at the point $(1, 0, 1)$.

17. Since the direction vectors are $\mathbf{v}_1 = \langle -6, 9, -3\rangle$ and $\mathbf{v}_2 = \langle 2, -3, 1\rangle$, we have $\mathbf{v}_1 = -3\mathbf{v}_2$ so the lines are parallel.

18. Since the direction vectors are $\langle 1, -1, 3\rangle$ and $\langle -1, 2, 1\rangle$, the lines aren't parallel. For the lines to intersect, the three equations $1 + t = 2 - s$, $2 - t = 1 + 2s$, $3t = 4 + s$ must be satisfied simultaneously. Solving the first two equations gives $t = 1$, $s = 0$ and, checking, we see these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

19. Since the plane is perpendicular to the vector $\langle -2, 1, 5\rangle$, we can take $\langle -2, 1, 5\rangle$ as a normal vector to the plane. $(6, 3, 2)$ is a point on the plane, so setting $a = -2$, $b = 1$, $c = 5$ and $x_0 = 6$, $y_0 = 3$, $z_0 = 2$ in Equation 6 gives $-2(x - 6) + 1(y - 3) + 5(z - 2) = 0$ or $-2x + y + 5z = 1$ to be an equation of the plane.

20. $\mathbf{j} + 2\mathbf{k} = \langle 0, 1, 2\rangle$ is a normal vector to the plane and $(4, 0, -3)$ is a point on the plane, so setting $a = 0$, $b = 1$, $c = 2$, $x_0 = 4$, $y_0 = 0$, $z_0 = -3$ in Equation 6 gives $0(x - 4) + 1(y - 0) + 2[z - (-3)] = 0$ or $y + 2z = -6$ to be an equation of the plane.

21. $\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle 1, 1, -1\rangle$ is a normal vector to the plane and $(1, -1, 1)$ is a point on the plane, so setting $a = 1$, $b = 1$, $c = -1$, $x_0 = 1$, $y_0 = -1$, $z_0 = 1$ in Equation 6 gives $1(x - 1) + 1[y - (-1)] - 1(z - 1) = 0$ or $x + y - z = -1$ to be an equation of the plane.

22. Since the line is perpendicular to the plane, its direction vector $\langle 1, 2, -3\rangle$ is a normal vector to the plane. An equation of the plane, then, is $1[x - (-2)] + 2(y - 8) - 3(z - 10) = 0$ or $x + 2y - 3z = -16$.

23. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 2, -1, 3 \rangle$, and an equation of the plane is $2(x-0) - 1(y-0) + 3(z-0) = 0$ or $2x - y + 3z = 0$.

24. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 1, 1, 1 \rangle$, and an equation of the plane is $1[x-(-1)] + 1(y-6) + 1[z-(-5)] = 0$ or $x + y + z = 0$.

25. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 3, 0, -7 \rangle$, and an equation of the plane is $3(x-4) + 0[y-(-2)] - 7(z-3) = 0$ or $3x - 7z = -9$.

26. First, a normal vector for the plane $2x + 4y + 8z = 17$ is $\mathbf{n} = \langle 2, 4, 8 \rangle$. A direction vector for the line is $\mathbf{v} = \langle 2, 1, -1 \rangle$, and since $\mathbf{n} \cdot \mathbf{v} = 0$ we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t = 0$, we know the point $(3, 0, 8)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n} = \langle 2, 4, 8 \rangle$, so an equation of the plane is $2(x-3) + 4(y-0) + 8(z-8) = 0$ or $x + 2y + 4z = 35$.

27. Here the vectors $\mathbf{a} = \langle 1-0, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$ and $\mathbf{b} = \langle 1-0, 1-1, 0-1 \rangle = \langle 1, 0, -1 \rangle$ lie in the plane, so $\mathbf{a} \times \mathbf{b}$ is a normal vector to the plane. Thus, we can take $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1-0, 0+1, 0+1 \rangle = \langle 1, 1, 1 \rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x-0) + 1(y-1) + 1(z-1) = 0$ or $x + y + z = 2$.

28. Here the vectors $\mathbf{a} = \langle 2, -4, 6 \rangle$ and $\mathbf{b} = \langle 5, 1, 3 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -12-6, 30-6, 2+20 \rangle = \langle -18, 24, 22 \rangle$ is a normal vector to the plane and an equation of the plane is $-18(x-0) + 24(y-0) + 22(z-0) = 0$ or $-18x + 24y + 22z = 0$.

29. Here the vectors $\mathbf{a} = \langle 8-3, 2-(-1), 4-2 \rangle = \langle 5, 3, 2 \rangle$ and $\mathbf{b} = \langle -1-3, -2-(-1), -3-2 \rangle = \langle -4, -1, -5 \rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -15+2, -8+25, -5+12 \rangle = \langle -13, 17, 7 \rangle$ and an equation of the plane is $-13(x-3) + 17[y-(-1)] + 7(z-2) = 0$ or $-13x + 17y + 7z = -42$.

30. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 3, 1, -1 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, 3)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(0, 1, 2)$ is on the line, so $\mathbf{b} = \langle 1-0, 2-1, 3-2 \rangle = \langle 1, 1, 1 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1+1, -1-3, 3-1 \rangle = \langle 2, -4, 2 \rangle$. Thus, an equation of the plane is $2(x-1) - 4(y-2) + 2(z-3) = 0$ or $2x - 4y + 2z = 0$. (Equivalently, we can write $x - 2y + z = 0$.)

31. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -2, 5, 4 \rangle$ is one vector in the plane. We can verify that the given point $(6, 0, -2)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, 3, 7)$ is on the line, so $\mathbf{b} = \langle 6-4, 0-3, -2-7 \rangle = \langle 2, -3, -9 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -45+12, 8-18, 6-10 \rangle = \langle -33, -10, -4 \rangle$. Thus, an equation of the plane is $-33(x-6) - 10(y-0) - 4[z-(-2)] = 0$ or $33x + 10y + 4z = 190$.

32. Since the line $x = 2y = 3z$, or $x = \dfrac{y}{1/2} = \dfrac{z}{1/3}$, lies in the plane, its direction vector $\mathbf{a} = \left\langle 1, \frac{1}{2}, \frac{1}{3} \right\rangle$ is parallel to the plane. The point $(0, 0, 0)$ is on the line (put $t = 0$), and we can verify that the given point $(1, -1, 1)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 1, -1, 1 \rangle$, is therefore parallel to the plane, but not parallel to $\langle 1, 2, 3 \rangle$. Then $\mathbf{a} \times \mathbf{b} = \left\langle \frac{1}{2} + \frac{1}{3}, \frac{1}{3} - 1, -1 - \frac{1}{2} \right\rangle = \left\langle \frac{5}{6}, -\frac{2}{3}, -\frac{3}{2} \right\rangle$ is a normal vector to the plane, and an equation of the plane is $\frac{5}{6}(x-0) - \frac{2}{3}(y-0) - \frac{3}{2}(z-0) = 0$ or $5x - 4y - 9z = 0$.

33. A direction vector for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, -1 \rangle \times \langle 2, -1, 3 \rangle = \langle 2, -5, -3 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. Setting $x = 0$, the equations of the planes reduce to $y - z = 2$ and $-y + 3z = 1$ with simultaneous solution $y = \frac{7}{2}$ and $z = \frac{3}{2}$. So a point on the line is $\left(0, \frac{7}{2}, \frac{3}{2}\right)$ and another vector parallel to the plane is $\left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle$. Then a normal vector to the plane is $\mathbf{n} = \langle 2, -5, -3 \rangle \times \left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle = \langle -2, 4, -8 \rangle$ and an equation of the plane is $-2(x+1) + 4(y-2) - 8(z-1) = 0$ or $x - 2y + 4z = -1$.

34. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -1, 2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, the normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = 3 \langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

35. Substituting the parametric equations of the line into the equation of the plane gives $x + y + z = 1 + t + 2t + 3t = 1 \quad \Rightarrow \quad t = 0$. This value of t corresponds to the point of intersection $(1, 0, 0)$, obtained by substitution of $t = 0$ into the equations of the line.

36. Substitute the parametric expressions for x, y and z into the equation of the plane: $2x - y + z = 2(5) - (4 - t) + 2t = 5 \quad \Rightarrow \quad t = -\frac{1}{3}$. Therefore, the point of intersection of the line and the plane is given by $x = 5$, $y = 4 - \left(-\frac{1}{3}\right) = \frac{13}{3}$ and $z = 2\left(-\frac{1}{3}\right) = -\frac{2}{3}$, that is, the point $\left(5, \frac{13}{3}, -\frac{2}{3}\right)$.

37. Substituting the parametric equations of the line into the equation of the plane gives $2x + y - z + 5 = 2(1 + 2t) + (-1) - t + 5 = 0 \quad \Rightarrow \quad 3t + 6 = 0 \quad \Rightarrow \quad t = -2$. Therefore, the point of intersection is $x = 1 + 2(-2) = -3$, $y = -1$ and $z = -2$ and the point of intersection is $(-3, -1, -2)$.

38. Substitution into the equation of the plane of the parametric expressions for x, y and z gives $z = 1 - 2x + y \quad \Rightarrow \quad (1 + t) = 1 - 2(1 - t) + t \quad \Rightarrow \quad -2 + 2t = 0 \quad \Rightarrow \quad t = 1$. Thus, $x = 1 - 1$, $y = 1$ and $z = 1 + 1$ and the point of intersection is $(0, 1, 2)$.

39. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are $1, 0, -1$.

40. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the two planes are $\langle 1, 1, 1 \rangle$ and $\langle 1, 2, 3 \rangle$. The cosine of the angle θ between these two planes is

$$\cos\theta = \frac{\langle 1, 1, 1 \rangle \cdot \langle 1, 2, 3 \rangle}{|\langle 1, 1, 1 \rangle| \, |\langle 1, 2, 3 \rangle|} = \frac{1 + 2 + 3}{\sqrt{1 + 1 + 1}\sqrt{1 + 4 + 9}} = \frac{6}{\sqrt{42}} = \sqrt{\frac{6}{7}}.$$

41. The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, 0, 1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 1 \rangle$. Thus the normal vectors (and consequently the planes) aren't parallel. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 \neq 0$ so the planes aren't perpendicular. Letting θ be the angle between the two planes, we have $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \, |\mathbf{n}_2|} = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$ and $\theta = \cos^{-1} \frac{1}{2} = 60^\circ$.

42. Here the normals are $\mathbf{n}_1 = \langle -8, -6, 2 \rangle$ and $\mathbf{n}_2 = \langle 4, 3, -1 \rangle$. Since $\mathbf{n}_1 = -2\mathbf{n}_2$, the normals (and thus the planes) are parallel.

43. The normals are $\mathbf{n}_1 = \langle 1, 4, -3 \rangle$ and $\mathbf{n}_2 = \langle -3, 6, 7 \rangle$, so the normals (and thus the planes) aren't parallel. But $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 + 24 - 21 = 0$, so the normals (and thus the planes) are perpendicular.

44. The normals are $\mathbf{n}_1 = \langle 2, 2, -1 \rangle$ and $\mathbf{n}_2 = \langle 6, -3, 2 \rangle$ so the planes aren't parallel. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 12 - 6 - 2 = 4 \neq 0$, so the planes aren't perpendicular. Then $\cos\theta = \dfrac{4}{\sqrt{9}\sqrt{49}} = \dfrac{4}{21}$ and $\theta = \cos^{-1} \frac{4}{21} \approx 79\,°$.

45. The normals are $\mathbf{n}_1 = \langle 2, 4, -2 \rangle$ and $\mathbf{n}_2 = \langle -3, -6, 3 \rangle$. Since $\mathbf{n}_1 = -\frac{2}{3}\mathbf{n}_2$, the normals (and thus the planes) are parallel.

46. The normals are $\mathbf{n}_1 = \langle 2, -5, 1 \rangle$ and $\mathbf{n}_2 = \langle 4, 2, 2 \rangle$.
$\mathbf{n}_1 \cdot \mathbf{n}_2 = \langle 2, -5, 1 \rangle \cdot \langle 4, 2, 2 \rangle = 8 - 10 + 2 = 0$, so the normals (and thus the planes) are perpendicular.

47. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z = 0$. (This will only work if the line of intersection crosses the xy-plane; otherwise, try setting x or y equal to 0.) Then the equations of the planes reduce to $x + y = 2$ and $3x - 4y = 6$. Solving these two equations gives $x = 2$, $y = 0$. So a point on the line of intersection is $(2, 0, 0)$. The direction of the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 5 - 4, -3 - 5, -4 - 3 \rangle = \langle 1, -8, -7 \rangle$, and symmetric equations for the line are $x - 2 = \dfrac{y}{-8} = \dfrac{z}{-7}$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{3 - 4 - 5}{\sqrt{3}\sqrt{50}} = -\dfrac{\sqrt{6}}{5}$. Therefore $\theta = \cos^{-1}\left(-\frac{\sqrt{6}}{5}\right) \approx 119°$ (or 61°).

48. (a) $x - 2y + z = 1 \quad \Rightarrow \quad \mathrm{n}_1 = \langle 1, -2, 1 \rangle$ and $2x + y + z = 1 \quad \Rightarrow \quad \mathrm{n}_2 = \langle 2, 1, 1 \rangle$. The vector that gives the direction of the line of intersection of these two planes is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -2 - 1, 2 - 1, 1 + 4 \rangle = \langle -3, 1, 5 \rangle$. Setting $x = y = 0$, we see that both planes contain $(0, 0, 1)$ so that this point must lie on their line of intersection. Then symmetric equations for this line are $\dfrac{x}{-3} = y = \dfrac{z - 1}{5}$.

(b) $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{2 - 2 + 1}{\sqrt{1 + 4 + 1}\sqrt{4 + 1 + 1}} = \dfrac{1}{6} \quad \Rightarrow \quad \theta = \cos^{-1} \frac{1}{6} \approx 80°$.

49. Setting $x = 0$, the equations of the two planes become $z = y$ and $5y + z = -1$, which intersect at $y = -\frac{1}{6}$ and $z = -\frac{1}{6}$. Thus we can choose $(x_0, y_0, z_0) = \left(0, -\frac{1}{6}, -\frac{1}{6}\right)$. The vector giving the direction of this intersecting line, $\mathbf{v}$, is perpendicular to the normal vectors of both planes. So $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -5, -1 \rangle \times \langle 1, 1, -1 \rangle = \langle 6, 1, 7 \rangle$. Therefore, by Equations 2, parametric equations for this line are $x = 6t$, $y = -\frac{1}{6} + t$, $z = -\frac{1}{6} + 7t$.

50. Setting $y = 0$, the equations of the two planes become $2x + 5z = -3$ and $x + z = -3$, which intersect at $x = -\frac{7}{3}$ and $z = \frac{1}{3}$. Thus we can choose $(x_0, y_0, z_0) = \left(-\frac{7}{3}, 0, \frac{1}{3}\right)$. The vector giving the direction of this intersecting line, $\mathbf{v}$, is perpendicular to the normal vectors of both planes. So $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, 0, 5 \rangle \times \langle 1, -3, 1 \rangle = \langle 15, 5 - 2, -6 \rangle = 3\,\langle 5, 1, -2 \rangle$. Therefore, by Equations 2, parametric equations of the line of intersection of the two planes are $x = -\frac{7}{3} + 5t$, $y = t$, $z = \frac{1}{3} - 2t$.

51. The plane contains all perpendicular bisectors of the line segment joining $(1, 1, 0)$ and $(0, 1, 1)$. All of these bisectors pass through the midpoint of this segment $\left(\dfrac{1}{2}, \dfrac{1 + 1}{2}, \dfrac{1}{2}\right) = \left(\dfrac{1}{2}, 1, \dfrac{1}{2}\right)$. The direction of this line segment $\langle 1 - 0, 1 - 1, 0 - 1 \rangle = \langle 1, 0, -1 \rangle$ is perpendicular to the plane so that we can choose this to be $\mathbf{n}$. Therefore the equation of the plane is $1\left(x - \frac{1}{2}\right) + 0\,(y - 1) - 1\left(z - \frac{1}{2}\right) = 0 \quad \Leftrightarrow \quad x = z$.

52. The plane will contain all perpendicular bisectors of the line segment joining the two points. Thus, a point in the plane is $P_0 = (-1, -1, 2)$, the midpoint of the line segment joining the two given points, and a normal to the plane is $\mathbf{n} = \langle 6, -6, 2 \rangle$, the vector connecting the two points. So an equation of the plane is $6(x+1) - 6(y+1) + 2(z-2) = 0$ or $3x - 3y + z = 2$.

53. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0 \rangle$ and $\mathbf{b} = \langle -a, 0, c \rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab \rangle = \langle bc, ac, ab \rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. This is a good equation to remember!

54. (a) For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1 + t = 2 - s$, $1 - t = s$ and $2t = 2$. From the third we get $t = 1$, and putting this in the second gives $s = 0$. These values of s and t do satisfy the first equation, so the lines intersect at the point $P_0 = (1 + 1, 1 - 1, 2(1)) = (2, 0, 2)$.

(b) The direction vectors of the lines are $\langle 1, -1, 2 \rangle$ and $\langle -1, 1, 0 \rangle$, so a normal vector for the plane is $\langle -1, 1, 0 \rangle \times \langle 1, -1, 2 \rangle = \langle 2, 2, 0 \rangle$ and it contains the point $(2, 0, 2)$. Then the equation of the plane is $2(x-2) + 2(y-0) + 0(z-2) = 0 \quad \Leftrightarrow \quad x + y = 2$.

55. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1, 1, 1 \rangle$, and a direction vector for the given line, $\langle 1, -1, 2 \rangle$. So a direction vector for the required line is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \langle 3, -1, -2 \rangle$. Thus L is given by $\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t \langle 3, -1, -2 \rangle$, or in parametric form, $x = 3t$, $y = 1 - t$, $z = 2 - 2t$.

56. Let L be the given line. Then $(1, 1, 0)$ is the point on L corresponding to $t = 0$. L is in the direction of $\mathbf{a} = \langle 1, -1, 2 \rangle$ and $\mathbf{b} = \langle -1, 0, 2 \rangle$ is the vector joining $(1, 1, 0)$ and $(0, 1, 2)$. Then

$\mathbf{b} - \text{proj}_{\mathbf{a}}\, \mathbf{b} = \langle -1, 0, 2 \rangle - \dfrac{\langle 1, -1, 2 \rangle \cdot \langle -1, 0, 2 \rangle}{1^2 + (-1)^2 + 2^2} \langle 1, -1, 2 \rangle = \langle -1, 0, 2 \rangle - \frac{1}{2} \langle 1, -1, 2 \rangle = \left\langle -\frac{3}{2}, \frac{1}{2}, 1 \right\rangle$ is a direction vector for the required line. Thus $2 \left\langle -\frac{3}{2}, \frac{1}{2}, 1 \right\rangle = \langle -3, 1, 2 \rangle$ is also a direction vector, and the line has parametric equations $x = -3t$, $y = 1 + t$, $z = 2 + 2t$. (Notice that this is the same line as in Exercise 55.)

57. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1 = \langle 4, -2, 6 \rangle$, $\mathbf{n}_2 = \langle 4, -2, -2 \rangle$, $\mathbf{n}_3 = \langle -6, 3, -9 \rangle$, $\mathbf{n}_4 = \langle 2, -1, -1 \rangle$. Now $\mathbf{n}_1 = -\frac{2}{3}\mathbf{n}_3$, so $\mathbf{n}_1$ and $\mathbf{n}_3$ are parallel, and hence P_1 and P_3 are parallel; similarly P_2 and P_4 are parallel because $\mathbf{n}_2 = 2\mathbf{n}_4$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. $\left(0, 0, \frac{1}{2}\right)$ lies on P_1, but not on P_3, so they are not the same plane, but both P_2 and P_4 contain the point $(0, 0, -3)$, so these two planes are identical.

58. Let L_i have direction vector $\mathbf{v}_i$. Then $\mathbf{v}_1 = \langle 1, 1, -5 \rangle$, $\mathbf{v}_2 = \langle 1, 1, -1 \rangle$, $\mathbf{v}_3 = \langle 1, 1, -1 \rangle$, $\mathbf{v}_4 = \langle 2, 2, -10 \rangle$. $\mathbf{v}_2$ and $\mathbf{v}_3$ are equal so they're parallel. $\mathbf{v}_4 = 2\mathbf{v}_1$, so L_4 and L_1 are parallel. L_3 contains the point $(1, 4, 1)$, but this point does not lie on L_2, so they're not equal. $(2, 1, -3)$ lies on L_4, and on L_1, with $t = 1$. So L_1 and L_4 are identical.

59. Let $Q = (2, 2, 0)$ and $R = (3, -1, 5)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (1, 2, 3)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1, -3, 5 \rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle -1, 0, 3 \rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -3, 5 \rangle \times \langle -1, 0, 3 \rangle|}{|\langle 1, -3, 5 \rangle|} = \frac{|\langle -9, -8, -3 \rangle|}{|\langle 1, -3, 5 \rangle|} = \frac{\sqrt{9^2 + 8^2 + 3^2}}{\sqrt{1^2 + 3^2 + 5^2}} = \frac{\sqrt{154}}{\sqrt{35}} = \sqrt{\frac{22}{5}}.$$

60. Let $Q = (5, 0, 1)$ and $R = (4, 3, 3)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (1, 0, -1)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle -1, 3, 2 \rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle -4, 0, -2 \rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle -1, 3, 2 \rangle \times \langle -4, 0, -2 \rangle|}{|\langle -1, 3, 2 \rangle|} = \frac{|\langle -6, -10, 12 \rangle|}{|\langle -1, 3, 2 \rangle|} = \frac{2\sqrt{3^2 + 5^2 + 6^2}}{\sqrt{1^2 + 3^2 + 2^2}} = \frac{2\sqrt{70}}{\sqrt{14}} = 2\sqrt{5}.$$

61. By Equation 8, the distance is $D = \dfrac{1}{\sqrt{1+4+4}}\left[(1)(2)+(-2)(8)+(-2)(5)-1\right] = \dfrac{25}{3}$.

62. By Equation 8, the distance is $D = \dfrac{1}{\sqrt{16+36+1}}\left[4(3)+(-6)(-2)+1(7)-5\right] = \dfrac{26}{\sqrt{53}}$.

63. Put $y = z = 0$ in the equation of the first plane, to get the point $(-1,0,0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(-1,0,0)$ to the second plane. By Equation 8, $D = \dfrac{|3(-1)+6(0)-3(0)-4|}{\sqrt{3^2+6^2+(-3)^2}} = \dfrac{7}{3\sqrt{6}}$.

64. Put $y = z = 0$ in the equation of the first plane to get the point $\left(\frac{4}{3},0,0\right)$ on the plane. Because the planes are parallel the distance D between them is the distance from $\left(\frac{4}{3},0,0\right)$ to the second plane. By Equation 8, $D = \dfrac{\left|1\left(\frac{4}{3}\right)+2(0)-3(0)-1\right|}{\sqrt{1^2+2^2+(-3)^2}} = \dfrac{1}{3\sqrt{14}}$.

65. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax+by+cz+d_1=0$. Then $ax_0+by_0+cz_0+d_1=0$ and the distance between P_0 and the plane given by $ax+by+cz+d_2=0$ is, from Equation 8, $D = \dfrac{|ax_0+by_0+cz_0+d_2|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|-d_1+d_2|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.

66. The planes must have parallel normal vectors, so if $ax+by+cz+d=0$ is such a plane, then for some $t \neq 0$, $\langle a,b,c\rangle = t\langle 1,2,-2\rangle = \langle t,2t,-2t\rangle$. So this plane is given by the equation $x+2y-2z+e=0$, where $e = d/t$. By Exercise 65, the distance between the planes is $2 = \dfrac{|1-e|}{\sqrt{1^2+2^2+(-2)^2}} \quad\Leftrightarrow\quad 6 = |1-e| \quad\Leftrightarrow\quad e = 7$ or -5.

So the desired planes have equations $x+2y-2z=7$ and $x+2y-2z=-5$.

67. L_1: $x = y = z \quad\Rightarrow\quad x = y$ (1). L_2: $x+1 = y/2 = z/3 \quad\Rightarrow\quad x+1 = y/2$ (2). The solution of (1) and (2) is $x = y = -2$. However, when $x=-2$, $x = z \quad\Rightarrow\quad z=-2$, but $x+1 = z/3 \quad\Rightarrow\quad z=-3$, a contradiction. Hence the lines do not intersect. For L_1, $\mathbf{v}_1 = \langle 1,1,1\rangle$, and for L_2, $\mathbf{v}_2 = \langle 1,2,3\rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$, the direction vectors of the two lines. So set $\mathbf{n} = \langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 3-2,-3+1,2-1\rangle = \langle 1,-2,1\rangle$. From above, we know that $(-2,-2,-2)$ and $(-2,-2,-3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 7, $1(-2)-2(-2)+1(-2)+d_1=0 \quad\Rightarrow\quad d_1=0$ and $1(-2)-2(-2)+1(-3)+d_2=0 \quad\Rightarrow\quad d_2=1$. By Exercise 65, the distance between these two skew lines is $D = \dfrac{|0-1|}{\sqrt{1+4+1}} = \dfrac{1}{\sqrt{6}}$.

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\mathbf{n} = \langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 1,-2,1\rangle$. Pick any point on each of the lines, say $(-2,-2,-2)$ and $(-2,-2,-3)$, and form the vector $\mathbf{b} = \langle 0,0,1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} = \dfrac{|1\cdot 0-2\cdot 0+1\cdot 1|}{\sqrt{1+4+1}} = \dfrac{1}{\sqrt{6}}$.

68. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$, the direction vectors of the two lines respectively. Thus set $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 36 - 30, 4 - 6, 15 - 12\rangle = \langle 6, -2, 3\rangle$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 7, $6 - 2 + 0 + d_1 = 0 \quad \Rightarrow \quad d_1 = -4$ and $6 - 10 - 6 + d_2 = 0 \quad \Rightarrow \quad d_2 = 10$. Then by Exercise 65, the distance between the two skew lines is given by

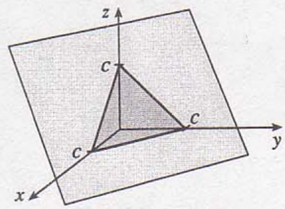
$$D = \frac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \frac{14}{7} = 2.$$

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 6, -2, 3\rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 1, 0)$ and $(1, 5, -2)$, and form the vector $\mathbf{b} = \langle 0, 4, -2\rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is,

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{36 + 4 + 9}} |0 - 8 - 6| = \frac{14}{7} = 2.$$

69. If $a \neq 0$, then $ax + by + cz + d = 0 \quad \Rightarrow \quad a(x + d/a) + b(y - 0) + c(z - 0) = 0$ which by (6) is the scalar equation of the plane through the point $(-d/a, 0, 0)$ with normal vector $\langle a, b, c\rangle$. Similarly, if $b \neq 0$ (or if $c \neq 0$) the equation of the plane can be rewritten as $a(x - 0) + b(y + d/b) + c(z - 0) = 0$ [or as $a(x - 0) + b(y - 0) + c(z + d/c) = 0$] which by (6) is the scalar equation of a plane through the point $(0, -d/b, 0)$ [or the point $(0, 0, -d/c)$] with normal vector $\langle a, b, c\rangle$.

70. (a) The planes $x + y + z = c$ have normal vector $\langle 1, 1, 1\rangle$, so they are all parallel. Their x-, y-, and z-intercepts are all c. When $c > 0$ their intersection with the first octant is an equilateral triangle and when $c < 0$ their intersection with the octant diagonally opposite the first is an equilateral triangle.



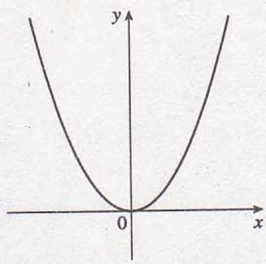
(b) The planes $x + y + cz = 1$ have x-intercept 1, y-intercept 1, and z-intercept $1/c$. The plane with $c = 0$ is parallel to the z-axis. As c gets larger, the planes get closer to the xy-plane.

(c) The planes $y \cos\theta + z \cos\theta = 1$ have normal vectors $\langle 0, \cos\theta, \sin\theta\rangle$, which are perpendicular to the x-axis, and so the planes are parallel to the x-axis. We look at their intersection with the yz-plane. These are lines that are perpendicular to $\langle \cos\theta, \sin\theta\rangle$ and pass through $(\cos\theta, \sin\theta)$, since $\cos^2\theta + \sin^2\theta = 1$. So these are the tangent lines to the unit circle. Thus the family consists of all planes tangent to the circular cylinder with radius 1 and axis the x-axis.

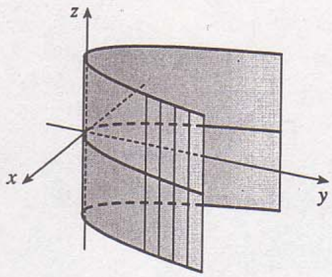
13.6 Cylinders and Quadric Surfaces

ET 12.6

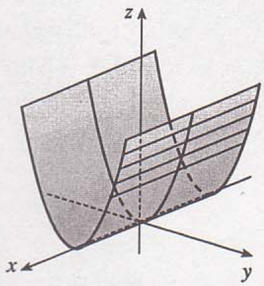
1. (a) In $\mathbb{R}^2$, the equation $y = x^2$ represents a parabola.



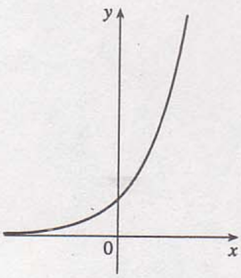
(b) In $\mathbb{R}^3$, the equation $y = x^2$ doesn't involve z, so any horizontal plane with equation $z = k$ intersects the graph in a curve with equation $y = x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z-axis.



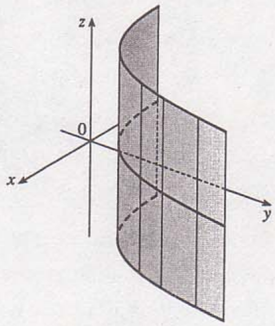
(c) In $\mathbb{R}^3$, the equation $z = y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z = y^2$ in the direction of the x-axis. Thus, the rulings of the cylinder are parallel to the x-axis.



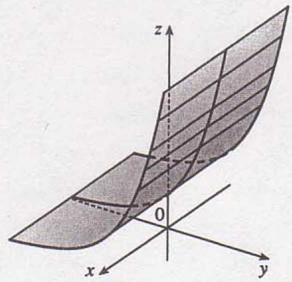
2. (a)



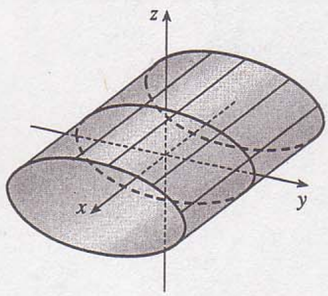
(b) Since the equation $y = e^x$ doesn't involve z, horizontal traces are copies of the curve $y = e^x$. The rulings are parallel to the z-axis.



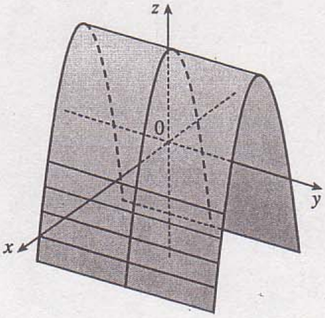
(c) The equation $z = e^y$ doesn't involve x, so vertical traces in $x = k$ (parallel to the yz-plane) are copies of the curve $z = e^y$. The rulings are parallel to the x-axis.



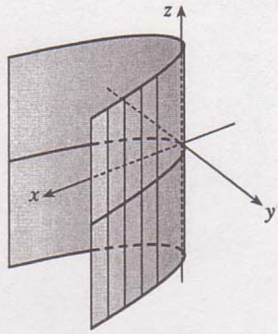
3. Since x is missing from the equation, the vertical traces $y^2 + 4z^2 = 4$, $x = k$, are copies of the same ellipse in the plane $x = k$. Thus, the surface $y^2 + 4z^2 = 4$ is an elliptic cylinder with rulings parallel to the x-axis.



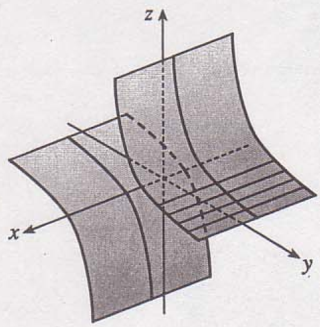
4. Since y is missing from the equation, each vertical trace $z = 4 - x^2$, $y = k$, is a copy of the same parabola in the plane $y = k$. Thus, the surface $z = 4 - x^2$ is a parabolic cylinder with rulings parallel to the y-axis.



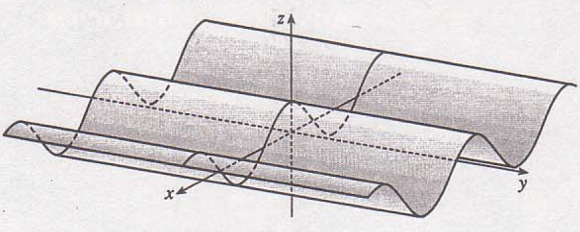
5. Since z is missing, each horizontal trace $x = y^2$, $z = k$, is a copy of the same parabola in the plane $z = k$. Thus, the surface $x - y^2 = 0$ is a parabolic cylinder with rulings parallel to the z-axis.



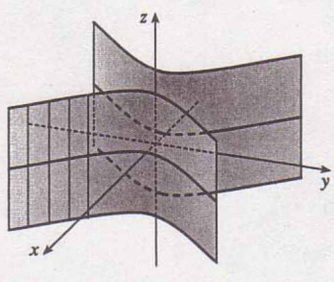
6. Since x is missing, each vertical trace $yz = 4$, $x = k$ is a copy of the same hyperbola in the plane $x = k$. Thus, the surface $yz = 4$ is a hyperbolic cylinder with rulings parallel to the x-axis.



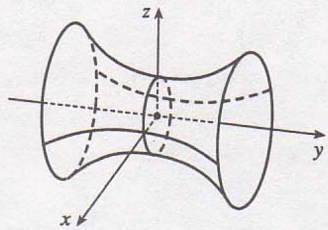
7. Since y is missing, each vertical trace $z = \cos x$, $y = k$ is a copy of a cosine curve in the plane $y = k$. Thus, the surface $z = \cos x$ is a cylindrical surface with rulings parallel to the y-axis.



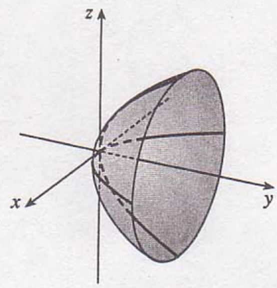
8. Since z is missing, each horizontal trace $x^2 - y^2 = 1$, $z = k$ is a copy of the same hyperbola in the plane $z = k$. Thus, the surface $x^2 - y^2 = 1$ is a hyperbolic cylinder with rulings parallel to the z-axis.



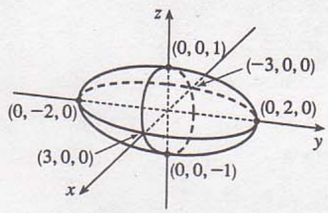
9. The trace in any plane $x = k$ is given by $z^2 - y^2 = 1 - k^2$, $x = k$ whose graph is a hyperbola. The trace in any plane $y = k$ is the circle given by $x^2 + z^2 = 1 + k^2$, $y = k$, and the trace in any plane $z = k$ is the hyperbola given by $x^2 - y^2 = 1 - k^2$, $z = k$. Thus the surface is a hyperboloid of one sheet with axis the y-axis.



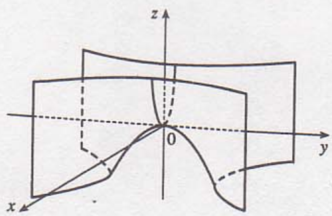
10. Traces: $x = k$, $4y = k^2 + z^2$, a parabola; $y = k$, $4k = x^2 + z^2$, a circle for $k > 0$; $z = k$, $4y = x^2 + k^2$ a parabola. Thus the surface is a circular paraboloid with axis the y-axis and vertex at $(0, 0, 0)$.



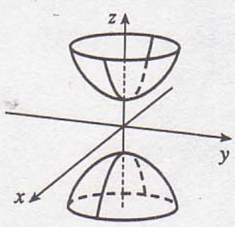
11. Traces: $x = k$, $9y^2 + 36z^2 = 36 - 4k^2$, an ellipse for $|k| < 3$; $y = k$, $4x^2 + 36z^2 = 36 - 9k^2$, an ellipse for $|k| < 2$; $z = k$, $4x^2 + 9y^2 = 36\left(1 - k^2\right)$, an ellipse for $|k| < 1$. Thus the surface is an ellipsoid with center at the origin and axes along the x-, y- and z-axes.



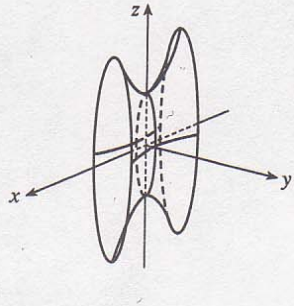
12. Traces: $x = k$, $z - k^2 = -y^2$, a parabola; $y = k$, $z + k^2 = x^2$, a parabola; $z = k$, $x^2 - y^2 = k$, a hyperbola. Thus the surface is a hyperbolic paraboloid with saddle point (0, 0, 0) (and since $c > 0$, the saddle is upside down).



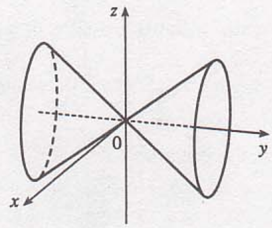
13. Traces: $x = k$, $4z^2 - y^2 = 1 + k^2$, a hyperbola; $y = k$, $4z^2 - x^2 = 1 + k^2$, a hyperbola; $z = k$, $-x^2 - y^2 = 1 - 4k^2$ or $x^2 + y^2 = 4k^2 - 1$, a circle for $k > \frac{1}{2}$ or $k < -\frac{1}{2}$. Thus the surface is a hyperboloid of two sheets with axis the z-axis.



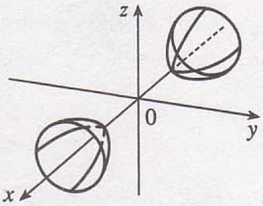
14. Traces: $x = k$, $25y^2 + z^2 = 100 + 4k^2$, an ellipse; $y = k$, $25k^2 + z^2 = 100 + 4x^2$ or $z^2 - 4x^2 = 100 - 25k^2$, a hyperbola for $|k| < 2$; $z = k$, $25y^2 + k^2 = 100 + 4x^2$ or $25y^2 - 4x^2 = 100 - k^2$, a hyperbola for $|k| < 10$. Thus the surface is a hyperboloid of one sheet with axis the x-axis.



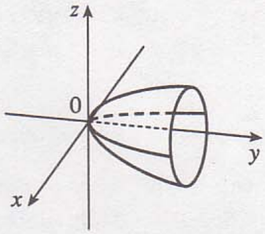
15. Traces: $x = k$, $y^2 = k^2 + z^2$ or $y^2 - z^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$; $y = k$, $x^2 + z^2 = k^2$, a circle for $k \neq 0$; $z = k$, $y^2 = x^2 + k^2$ or $y^2 - x^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$. Thus the surface is a cone (right circular) with axis the y-axis and vertex the origin.



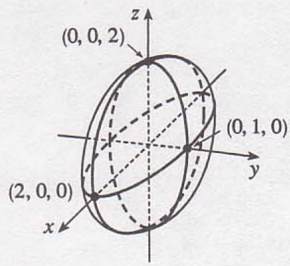
16. Traces: $x = k$, $y^2 + z^2 = 9\left(k^2 - 1\right)$, a circle for $|k| > 1$; $y = k$, $9x^2 - z^2 = 9 + k^2$, a hyperbola; $z = k$, $9x^2 - y^2 = 9 + k^2$, a hyperbola. Thus the surface is a hyperboloid of two sheets with axis the x-axis.



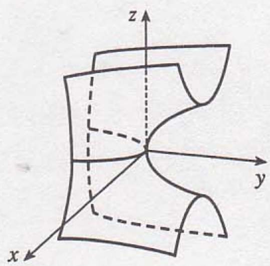
17. Traces: $x = k$, $k^2 + 4z^2 - y = 0$ or $y - k^2 = 4z^2$, a parabola; $y = k$, $x^2 + 4z^2 = k$, an ellipse for $k > 0$; $z = k$, $x^2 + 4k^2 - y = 0$ or $y - 4k^2 = x^2$, a parabola. Thus the surface is an elliptic paraboloid with axis the y-axis and vertex the origin.



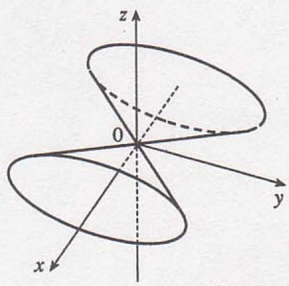
18. Traces: $x = k$, $|k| \leq 2 \quad \Rightarrow \quad y^2 + \dfrac{z^2}{4} = 1 - \dfrac{k^2}{4}$, ellipses; $y = k$, $|k| \leq 1 \quad \Rightarrow \quad x^2 + z^2 = 4\left(1 - k^2\right)$, circles; $z = k$, $|k| \leq 2 \quad \Rightarrow \quad \dfrac{x^2}{4} + y^2 = 1 - \dfrac{k^2}{4}$, ellipses. $x^2 + 4y^2 + z^2 = 4$ $\Leftrightarrow \quad \dfrac{x^2}{2^2} + \dfrac{y^2}{1^2} + \dfrac{z^2}{2^2} = 1$, which is the equation of an ellipsoid.



19. Traces: $x = k \quad \Rightarrow \quad y = z^2 - k^2$, parabolas; $y = k \quad \Rightarrow \quad k = z^2 - x^2$, hyperbolas on the z-axis for $k > 0$, and hyperbolas on the x-axis for $k < 0$; $z = k \quad \Rightarrow \quad y = k^2 - x^2$, parabolas. Thus, $\dfrac{y}{1} = \dfrac{z^2}{1^2} - \dfrac{x^2}{1^2}$ is a hyperbolic paraboloid.



20. Traces: $x = k \quad \Rightarrow \quad y^2 + 4z^2 = 16k^2$, ellipses; $y = k \quad \Rightarrow \quad 16x^2 - 4z^2 = k^2$, hyperbolas if $k \neq 0$ and two intersecting lines if $k = 0$; $z = k \quad \Rightarrow \quad 16x^2 - y^2 = 4k^2$, hyperbolas if $k \neq 0$ and two intersecting lines if $k = 0$. $16x^2 = y^2 + 4z^2 \quad \Leftrightarrow \quad x^2 = \dfrac{y^2}{4^2} + \dfrac{z^2}{2^2}$ is an elliptic cone with axis the x-axis and vertex the origin.



21. This is the equation of an ellipsoid: $x^2 + 4y^2 + 9z^2 = x^2 + \dfrac{y^2}{(1/2)^2} + \dfrac{z^2}{(1/3)^2} = 1$, with x-intercepts ± 1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

22. This is the equation of an ellipsoid: $9x^2 + 4y^2 + z^2 = \dfrac{x^2}{(1/3)^2} + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, with x-intercepts $\pm\frac{1}{3}$, y-intercepts $\pm\frac{1}{2}$ and z-intercepts ± 1. So the major axis is the z-axis and the only possible graph is IV.

23. This is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis, hence the correct graph is II.

24. This is a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz-plane at all, so the axis of the hyperboloid is the y-axis and the graph is III.

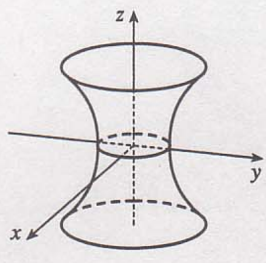
25. There are no real values of x and z that satisfy this equation for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

26. This is the equation of a cone with axis the y-axis, so the graph is I.

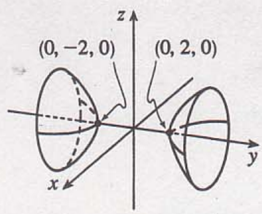
27. This surface is a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz-plane is an ellipse. So the graph is VIII.

28. This is the equation of a hyperbolic paraboloid. The trace in the xy-plane is the parabola $y = x^2$. So the correct graph is V.

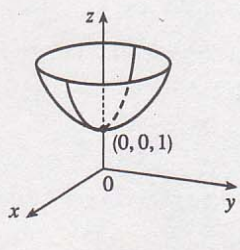
29. $z^2 = 3x^2 + 4y^2 - 12$ or $3x^2 + 4y^2 - z^2 = 12$ or $\dfrac{x^2}{4} + \dfrac{y^2}{3} - \dfrac{z^2}{12} = 1$ or $\dfrac{x^2}{2^2} + \dfrac{y^2}{(\sqrt{3})^2} - \dfrac{z^2}{(\sqrt{12})^2} = 1$ represents a hyperboloid of one sheet with axis the z-axis.



30. $4x^2 - 9y^2 + z^2 + 36 = 0$ or $-4x^2 + 9y^2 - z^2 = 36$ or $-\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} - \dfrac{z^2}{6^2} = 1$, a hyperboloid of two sheets with axis the y-axis.



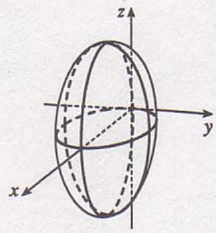
31. $z = x^2 + y^2 + 1$ or $z - 1 = x^2 + y^2$, a circular paraboloid with axis the z-axis and vertex $(0, 0, 1)$.



32. Completing the square in x gives $(x-1)^2 + 4y^2 + z^2 = 1$ or

$(x-1)^2 + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, an ellipsoid with center $(1, 0, 0)$ and

intercepts $(0, 0, 0)$, $(2, 0, 0)$.

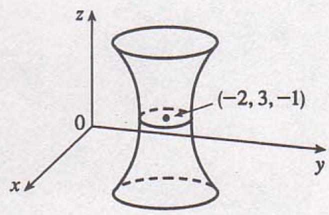


33. Completing the square in all three variables gives

$(x+2)^2 + (y-3)^2 - 4(z+1)^2 = 13 + 9$ or

$\dfrac{(x+2)^2}{(\sqrt{22})^2} + \dfrac{(y-3)^2}{(\sqrt{22})^2} - \dfrac{(z+1)^2}{(\frac{1}{2}\sqrt{22})^2} = 1$,

a hyperboloid of one sheet with center $(-2, 3, -1)$ and axis the vertical line $y = 3$, $x = -2$.

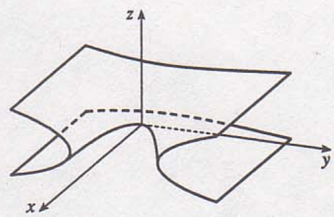


34. $4x = y^2 - 2z^2$ or $x = \dfrac{y^2}{2^2} - \dfrac{z^2}{(\sqrt{2})^2}$, a hyperbolic paraboloid with

saddle point $(0, 0, 0)$. The traces in the xy-, yz-, and xz-planes are

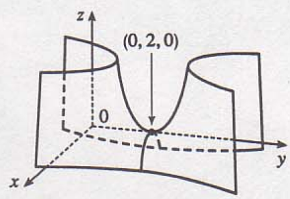
respectively $x = \dfrac{y^2}{2^2}$ (a parabola), $\dfrac{y^2}{2^2} = \dfrac{z^2}{(\sqrt{2})^2}$ (two inter-

secting lines), and $x = -\dfrac{z^2}{(\sqrt{2})^2}$ (a parabola).



35. Completing the square in y gives $x^2 - (y-2)^2 + z = 4 - 4 = 0$ or

$z = (y-2)^2 - x^2$, a hyperbolic paraboloid with center at $(0, 2, 0)$.

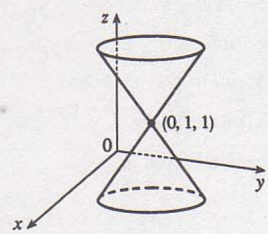


36. Completing the squares in y and z gives

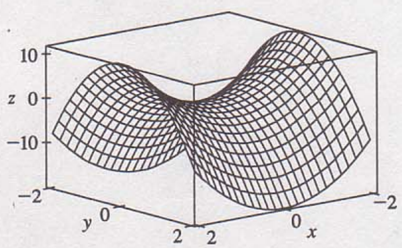
$9x^2 + (y-1)^2 - (z-1)^2 = 1 - 1 = 0$ or

$(z-1)^2 = \dfrac{x^2}{(1/3)^2} + (y-1)^2$, an elliptic cone with axis parallel to

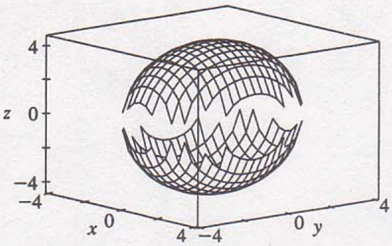
the z-axis and vertex $(0, 1, 1)$.



37.

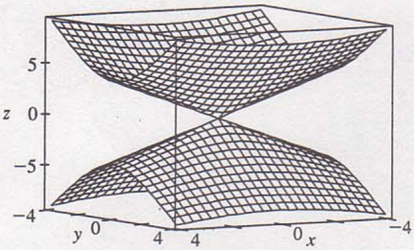


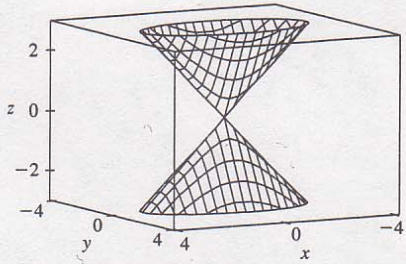
38.



In Section 17.6 [ET 16.6], we will be able to graph ellipsoids without gaps; see Exercise 17.6.45 [ET 16.6.45].

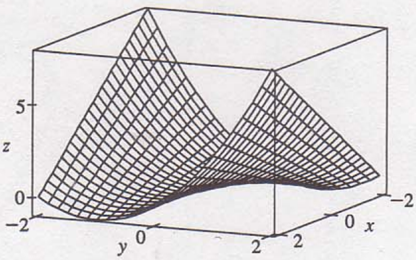
39.



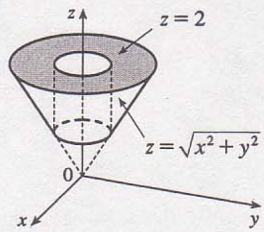


To restrict the z-range as in the second graph, we can use the option `view = -2..2` in Maple's `plot3d` command, or `PlotRange -> {-2,2}` in Mathematica's `Plot3D` command.

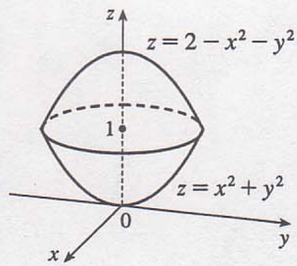
40.



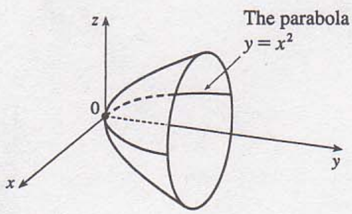
41.



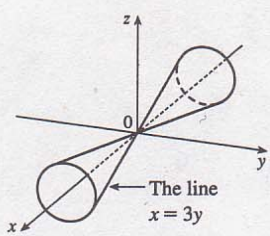
42.



43. The surface is a paraboloid of revolution (circular paraboloid) with vertex at the origin, axis the y-axis and opens to the right. Thus the trace in the yz-plane is also a parabola: $y = z^2$, $x = 0$. The equation is $y = x^2 + z^2$.



44. The surface is a right circular cone with vertex at $(0, 0, 0)$ and axis the x-axis. For $x = k \neq 0$, the trace is a circle with center $(k, 0, 0)$ and radius $r = y = \dfrac{x}{3} = \dfrac{k}{3}$. Thus the equation is $\frac{1}{3}x^2 = y^2 + z^2$.



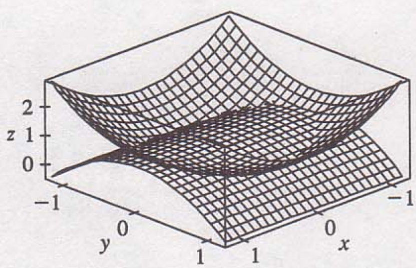
45. Let $P = (x, y, z)$ be an arbitrary point equidistant from $(-1, 0, 0)$ and the plane $x = 1$. Then the distance from P to $(-1, 0, 0)$ is $\sqrt{(x+1)^2 + y^2 + z^2}$ and the distance from P to the plane $x = 1$ is $|x-1|/\sqrt{1^2} = |x-1|$ (by Equation 13.5.8 [ET 12.5.8]). So $|x-1| = \sqrt{(x+1)^2 + y^2 + z^2} \quad\Leftrightarrow\quad (x-1)^2 = (x+1)^2 + y^2 + z^2 \quad\Leftrightarrow$ $x^2 - 2x + 1 = x^2 + 2x + 1 + y^2 + z^2 \quad\Leftrightarrow\quad -4x = y^2 + z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x-axis, which opens in the negative direction.

46. Let $P = (x, y, z)$ be an arbitrary point whose distance from the x-axis is twice its distance from the yz-plane. The distance from P to the x-axis is $\sqrt{(x-x)^2 + y^2 + z^2} = \sqrt{y^2 + z^2}$ and the distance from P to the yz-plance $(x = 0)$ is $|x|/1 = |x|$. Thus $\sqrt{y^2 + z^2} = 2|x| \quad\Leftrightarrow\quad y^2 + z^2 = 4x^2 \quad\Leftrightarrow\quad x^2 = (y^2/2^2) + (z^2/2^2)$. So the surface is a right circular cone with vertex the origin and axis the x-axis.

47. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. L_1: $x = a + t$, $y = b + t$, $z = c + 2(b-a)t$, L_2: $x = a + t$, $y = b - t$, $z = c - 2(b+a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \quad\Rightarrow$ $c + 2(b-a)t = (b+t)^2 - (a+t)^2 = b^2 - a^2 + 2(b-a)t \quad\Rightarrow\quad c = b^2 - a^2$. As this is true for all values of t, L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \quad\Rightarrow$ $c - 2(b+a)t = (b-t)^2 - (a+t)^2 = b^2 - a^2 - 2(b+a)t \quad\Rightarrow\quad c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

48. Any point on the curve of intersection must satisfy both $2x^2 + 4y^2 - 2z^2 + 6x = 2$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$. Subtracting, we get $6x + 5y = 2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

49.



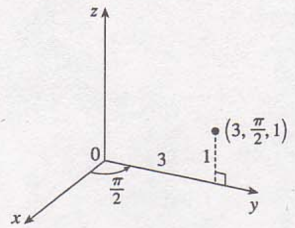
The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy-plane is the set of points $(x, y, 0)$ which satisfy $x^2 + y^2 = 1 - y^2 \quad\Leftrightarrow\quad x^2 + 2y^2 = 1 \quad\Leftrightarrow$ $x^2 + \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$. This is an equation of an ellipse.

13.7 Cylindrical and Spherical Coordinates ET 12.7

1. See Figure 1 and the accompanying discussion; see the paragraph accompanying Figure 3.

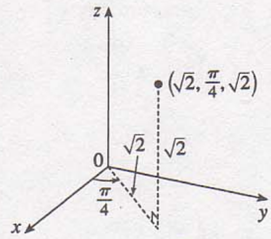
2. See Figure 5 and the accompanying discussion.

3.



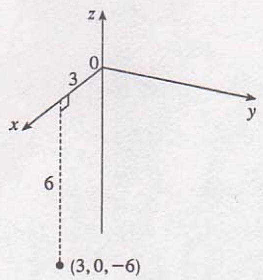
$x = 3\cos\frac{\pi}{2} = 0$, $y = 3\sin\frac{\pi}{2} = 3$, and $z = 1$, so the point is $(0, 3, 1)$ in rectangular coordinates.

4.



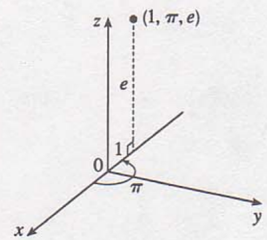
$x = \sqrt{2}\cos\frac{\pi}{4} = 1$, $y = \sqrt{2}\sin\frac{\pi}{4} = 1$, $z = \sqrt{2}$, so the point is $\left(1, 1, \sqrt{2}\right)$ in rectangular coordinates.

5.



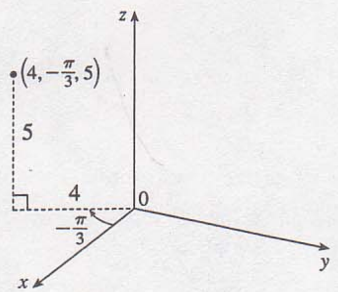
$x = 3\cos 0 = 3$, $y = 3\sin 0 = 0$, and $z = -6$, so the point is $(3, 0, -6)$ in rectangular coordinates.

6.



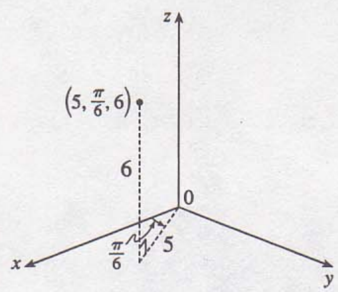
$x = 1\cos\pi = -1$, $y = 1\sin\pi = 0$, and $z = e$, so the point is $(-1, 0, e)$ in rectangular coordinates.

7.



$x = 4\cos\left(-\frac{\pi}{3}\right) = 2$,
$y = 4\sin\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$, and $z = 5$, so the point is $\left(2, -2\sqrt{3}, 5\right)$ in rectangular coordinates.

8.



$x = 5\cos\left(\frac{\pi}{6}\right) = \frac{5\sqrt{3}}{2}$, $y = 5\sin\left(\frac{\pi}{6}\right) = \frac{5}{2}$, and $z = 6$, so the point is $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 6\right)$ in rectangular coordinates.

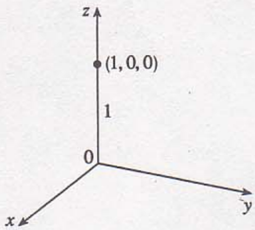
9. $r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$ so $r = \sqrt{2}$; $\tan\theta = \dfrac{y}{x} = \dfrac{-1}{1} = -1$ and the point $(1, -1)$ is in the fourth quadrant of the xy-plane, so $\theta = \frac{7\pi}{4} + 2n\pi$; $z = 4$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}, 4\right)$.

10. $r^2 = x^2 + y^2 = 3^2 + 3^2 = 18$ so $r = \sqrt{18} = 3\sqrt{2}$; $\tan\theta = \dfrac{y}{x} = \dfrac{3}{3} = 1$ and the point $(3, 3)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4} + 2n\pi$; $z = -2$. Thus, one set of cylindrical coordinates is $\left(3\sqrt{2}, \frac{\pi}{4}, -2\right)$.

11. $r^2 = (-1)^2 + \left(-\sqrt{3}\right)^2 = 4$ so $r = 2$; $\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$ and the point $\left(-1, -\sqrt{3}\right)$ is in the third quadrant of the xy-plane, so $\theta = \frac{4\pi}{3} + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $\left(2, \frac{4\pi}{3}, 2\right)$.

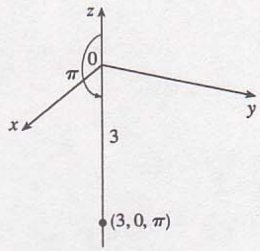
12. $r^2 = 3^2 + 4^2 = 25$ so $r = 5$; $\tan\theta = \frac{4}{3}$ and the point $(3, 4)$ is in the first quadrant of the xy-plane, so $\theta = \tan^{-1}\frac{4}{3} + 2n\pi \approx 0.93 + 2n\pi$; $z = 5$. Thus, one set of cylindrical coordinates is $\left(5, \tan^{-1}\frac{4}{3}, 5\right) \approx (5, 0.93, 5)$.

13.



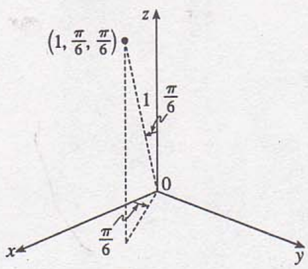
$x = \rho\sin\phi\cos\theta = (1)\sin 0\cos 0 = 0$,
$y = \rho\sin\phi\sin\theta = (1)\sin 0\sin 0 = 0$, and
$z = \rho\cos\phi = (1)\cos 0 = 1$ so the point is $(0, 0, 1)$ in rectangular coordinates.

14.



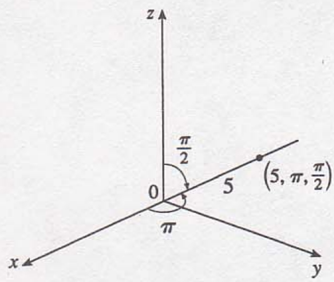
$x = 3\sin\pi\cos 0 = 0$, $y = 3\sin\pi\sin 0 = 0$, $z = 3\cos\pi = -3$ and in rectangular coordinates the point is $(0, 0, -3)$.

15.



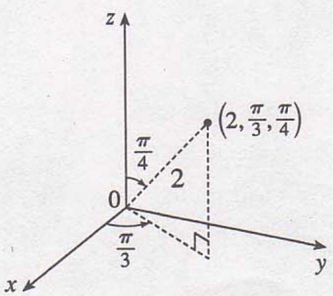
$x = \sin\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$, $y = \sin\frac{\pi}{6}\sin\frac{\pi}{6} = \frac{1}{4}$, and $z = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so the point is $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ in rectangular coordinates.

16.

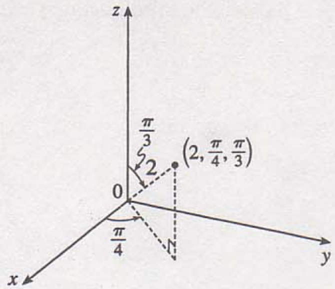


$x = 5\sin\frac{\pi}{2}\cos\pi = -5$, $y = 5\sin\frac{\pi}{2}\sin\pi = 0$, $z = 5\cos\frac{\pi}{2} = 0$ so the point is $(-5, 0, 0)$ in rectangular coordinates.

17.



$x = 2\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{\sqrt{2}}{2}$, $y = 2\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in rectangular coordinates.

18. 

$x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{3} = 1$ so the point is $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$ in rectangular coordinates.

19. $\rho = \sqrt{9+0+0} = 3$, $\cos\phi = \frac{0}{3} = 0$ so $\phi = \frac{\pi}{2}$, and $\cos\theta = \dfrac{-3}{3\sin\frac{\pi}{2}} = -1$ so $\theta = \pi$, thus spherical coordinates are $\left(3, \pi, \frac{\pi}{2}\right)$.

20. $\rho = \sqrt{1+1+2} = 2$, $\cos\phi = \dfrac{\sqrt{2}}{2}$ so $\phi = \frac{\pi}{4}$, and $\cos\theta = \dfrac{1}{2\sin\frac{\pi}{4}} = \dfrac{1}{\sqrt{2}}$ so $\theta = \frac{\pi}{4}$, thus in spherical coordinates the point is $\left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)$.

21. $\rho = \sqrt{3+1} = 2$, $\cos\phi = \frac{1}{2}$ so $\phi = \frac{\pi}{3}$, and $\cos\theta = \dfrac{\sqrt{3}}{2\sin\frac{\pi}{3}} = \dfrac{\sqrt{3}\cdot 2}{2\cdot\sqrt{3}} = 1$ so $\theta = 0$, thus the point is $\left(2, 0, \frac{\pi}{3}\right)$ in spherical coordinates.
Note: It is also apparent that $\theta = 0$ since the point is in the xz-plane and $x > 0$.

22. $\rho = \sqrt{3+9+4} = 4$, $\cos\phi = -\frac{2}{4} = -\frac{1}{2}$ so $\phi = \frac{2\pi}{3}$, and $\cos\theta = -\dfrac{\sqrt{3}}{4\sin\frac{5\pi}{6}} = -\dfrac{\sqrt{3}\cdot 2}{4\cdot\sqrt{3}} = -\frac{1}{2}$ and $y = -3$ so $\theta = \frac{4\pi}{3}$. Thus in spherical coordinates the point is $\left(4, \frac{4\pi}{3}, \frac{2\pi}{3}\right)$.

23. $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} = \sqrt{2+0} = \sqrt{2}$; $\theta = \frac{\pi}{4}$; $z = \rho\cos\phi = \sqrt{2}\cos\phi = 0$ so $\phi = \frac{\pi}{2}$ and the point is $\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

24. $\rho = \sqrt{r^2+z^2} = \sqrt{1+1} = \sqrt{2}$, $z = 1 = \sqrt{2}\cos\phi$, so $\phi = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$ and the point is $\left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$.

25. $\rho = \sqrt{r^2+z^2} = \sqrt{4^2+4^2} = 4\sqrt{2}$; $\theta = \frac{\pi}{3}$; $z = 4 = 4\sqrt{2}\cos\phi$ so $\cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$ and the point is $\left(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$.

26. $\rho = \sqrt{r^2+z^2} = \sqrt{12^2+5^2} = 13$, $z = 5 = 13\cos\phi$, so $\phi = \cos^{-1}\frac{5}{13}$, $\theta = \pi$ and the point is $\left(13, \pi, \cos^{-1}\frac{5}{13}\right)$.

27. $z = \rho\cos\phi = 2\cos 0 = 2$, $\rho^2 = x^2+y^2+z^2 = r^2+z^2 \Rightarrow r = \sqrt{\rho^2 - z^2} = \sqrt{2^2-2^2} = 0$, (or $r = 2\sin 0 = 0$), $\theta = 0$ and the point is $(0, 0, 2)$.

28. $z = 2\sqrt{2}\cos\frac{\pi}{2} = 0$, $r = 2\sqrt{2}\sin\frac{\pi}{2} = 2\sqrt{2}$, $\theta = \frac{3\pi}{2}$ and the point is $\left(2\sqrt{2}, \frac{3\pi}{2}, 0\right)$.

29. $z = 8\cos\frac{\pi}{2} = 0$, $r = 8\sin\frac{\pi}{2} = 8$, $\theta = \frac{\pi}{6}$ and the point is $\left(8, \frac{\pi}{6}, 0\right)$.

30. $z = 4\cos\frac{\pi}{3} = 2$, $r = 4\sin\frac{\pi}{3} = 2\sqrt{3}$, $\theta = \frac{\pi}{4}$ and the point is $\left(2\sqrt{3}, \frac{\pi}{4}, 2\right)$.

31. Since $r = 3$, $x^2+y^2 = 9$ and the surface is a cylinder with radius 3 and axis the z-axis.

32. Since $\rho = 3$, $x^2+y^2+z^2 = 9$ and the surface is a sphere with center the origin and radius 3.

33. Since $\phi = 0$, $x = 0$ and $y = 0$ while $z = \rho \geq 0$. Thus the "surface" is the positive z-axis including the origin.

34. Since $\phi = \frac{\pi}{2}$, $z = 0$ but there are no restrictions on x and y ($x = \rho\cos\theta$, $y = \rho\sin\theta$). Thus the surface is the xy-plane.

35. Since $\phi = \frac{\pi}{3}$, the surface is one frustum of the right circular cone with vertex at the origin and axis the positive z-axis.

36. Whether spherical or cylindrical coordinates, since $\theta = \frac{\pi}{3}$ the surface is a half-plane including the z-axis and intersecting the xy-plane in the half-line $y = \sqrt{3}x$, $x > 0$.

37. $z = r^2 = x^2 + y^2$, so the surface is a circular paraboloid with vertex at the origin and axis the positive z-axis.

38. Since $r = 4\sin\theta$ and $y = r\sin\theta$, $y = 4\sin^2\theta$. Also $r^2 = x^2 + y^2$ so $x^2 + y^2 = 16\sin^2\theta$. Thus $x^2 + y^2 - 4y = 16\sin^2\theta - 16\sin^2\theta = 0$ or $x^2 + (y-2)^2 = 4$, a circular cylinder of radius 2 and with axis parallel to the z-axis.

39. $2 = \rho\cos\phi = z$ is a plane through the point $(0, 0, 2)$ and parallel to the xy-plane.

40. Since $\rho\sin\phi = 2$ and $x = \rho\sin\phi\cos\theta$, $x = 2\cos\theta$. Also $y = \rho\sin\phi\sin\theta$ so $y = 2\sin\theta$. Then $x^2 + y^2 = 4\cos^2\theta + 4\sin^2\theta = 4$, a circular cylinder of radius 2 about the z-axis.

41. $r = 2\cos\theta \quad\Rightarrow\quad r^2 = x^2 + y^2 = 2r\cos\theta = 2x \quad\Leftrightarrow\quad (x-1)^2 + y^2 = 1$, which is the equation of a circular cylinder with radius 1, whose axis is the vertical line $x = 1$, $y = 0$, $z = z$.

42. $\rho = 2\cos\phi \quad\Rightarrow\quad \rho^2 = 2\rho\cos\phi = 2z \quad\Leftrightarrow\quad x^2 + y^2 + z^2 = 2z \quad\Leftrightarrow\quad x^2 + y^2 + (z-1)^2 = 1$. Therefore, the surface is a sphere of radius 1 centered at $(0, 0, 1)$.

43. Since $r^2 + z^2 = 25$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + z^2 = 25$, a sphere with radius 5 and center at the origin.

44. Since $r^2 - 2z^2 = 4$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 - 2z^2 = 4$ or $\frac{1}{4}x^2 + \frac{1}{4}y^2 - \frac{1}{2}z^2 = 1$, a hyperboloid of one sheet with axis the z-axis.

45. Since $x^2 = \rho^2\sin^2\phi\cos^2\theta$ and $z^2 = \rho^2\cos^2\phi$, the equation of the surface in rectangular coordinates is $x^2 + z^2 = 4$. Thus the surface is a circular cylinder of radius 2 about the y-axis.

46. Since $\rho^2\left(\sin^2\phi - 4\cos^2\phi\right) = 1$, $\rho^2\left(\sin^2\phi - 4\cos^2\phi\right) + \rho^2\cos^2\phi - \rho^2\cos^2\phi = 1$ or $\rho^2\left(\sin^2\phi + \cos^2\phi - 5\cos^2\phi\right) = 1$ or $\rho^2\left(1 - 5\cos^2\phi\right) = 1$. But $\rho^2 = x^2 + y^2 + z^2$ and $z^2 = \rho^2\cos^2\phi$, so we can rewrite the equation of the surface as $x^2 + y^2 + z^2 - 5z^2 = 1$ or $x^2 + y^2 - 4z^2 = 1$. Thus the surface is a hyperboloid of one sheet with axis the z-axis.

47. Since $r^2 - r = 0$, $r = 0$ or $r = 1$. But $x^2 + y^2 = r^2$. Thus the surface consists of the right circular cylinder of radius 1 and axis the z-axis along with the surface given by $x^2 + y^2 = 0$, that is, the z-axis.

48. Since $\rho^2 - 6\rho + 8 = 0$, either $\rho = 2$ or $\rho = 4$. Thus the surface consists of two concentric spheres (centered at the origin), one with radius 2 and the other with radius 4.

49. (a) $r^2 = x^2 + y^2$, so $r^2 + z^2 = 16$.

(b) $\rho^2 = x^2 + y^2 + z^2$, so $\rho^2 = 16$ or $\rho = 4$.

50. (a) $r^2 - z^2 = 16$

(b) $x^2 + y^2 - z^2 = x^2 + y^2 + z^2 - 2z^2$, so $\rho^2 - 2\rho^2\cos^2\phi = 16$ or $\rho^2\left(1 - 2\cos^2\phi\right) = 16$.

51. (a) $r\cos\theta + 2r\sin\theta + 3z = 6$

(b) $\rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 6$ or $\rho\left(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi\right) = 6$.

52. (a) $r^2 = 2z$

(b) $\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 2\rho\cos\phi$ or $\rho^2\sin^2\phi = 2\rho\cos\phi$ or $\rho\sin^2\phi = 2\cos\phi$.

53. (a) $r^2\left(\cos^2\theta - \sin^2\theta\right) - 2z^2 = 4$ or $2z^2 = r^2\cos 2\theta - 4$.

(b) $\rho^2\left(\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - 2\cos^2\phi\right) = 4$ or $\rho^2\left(\sin^2\phi\cos 2\theta - 2\cos^2\phi\right) = 4$.

54. (a) $r^2 \sin^2 \theta + z^2 = 1$

(b) $\rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi = 1$ or $\rho^2 \left(\sin^2 \phi \sin^2 \theta + \cos^2 \phi\right) = 1$.

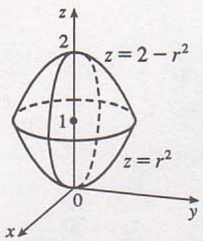
55. (a) $r^2 = 2r \sin \theta$ or $r = 2 \sin \theta$.

(b) $\rho^2 \sin^2 \phi \left(\cos^2 \theta + \sin^2 \theta\right) = 2\rho \sin \phi \sin \theta$ or $\rho \sin^2 \phi = 2 \sin \phi \sin \theta$ or $\rho \sin \phi = 2 \sin \theta$.

56. (a) $z = r^2 \left(\cos^2 \theta - \sin^2 \theta\right)$ or $z = r^2 \cos 2\theta$.

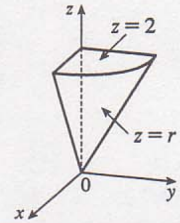
(b) $\rho \cos \phi = \rho^2 \sin^2 \phi \left(\cos^2 \theta - \sin^2 \theta\right)$ or $\cos \phi = \rho \sin^2 \phi \cos 2\theta$.

57.



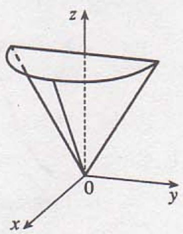
$z = r^2 = x^2 + y^2$ is a circular paraboloid with vertex $(0,0,0)$, opening upward. $z = 2 - r^2 \quad \Rightarrow$ $z - 2 = -\left(x^2 + y^2\right)$ is a circular paraboloid with vertex $(0,0,2)$ opening downward. Thus $r^2 \leq z \leq 2 - r^2$ is the solid region enclosed by these two surfaces.

58.



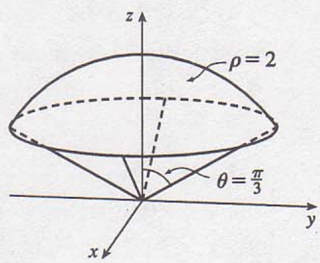
$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward. Thus $r \leq z \leq 2$ is the region above this cone and beneath the horizontal plane $z = 2$. $0 \leq \theta \leq \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

59.



$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ restricts the solid to the 4 octants in which x is positive. $\rho = \sec \phi \quad \Rightarrow$ $\rho \cos \phi = z = 1$, which is the equation of a horizontal plane. $0 \leq \phi \leq \frac{\pi}{6}$ describes a cone, opening upward. So the solid lies above the cone $\phi = \frac{\pi}{6}$ and below the plane $z = 1$.

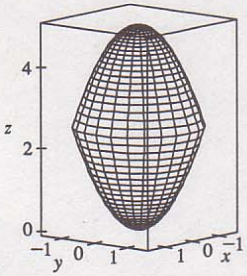
60.



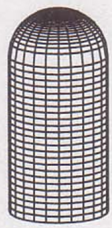
$\rho = 2 \quad \Leftrightarrow \quad x^2 + y^2 + z^2 = 4$, which is a sphere of radius 2, centered at the origin. Hence $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{3}$ restricts the solid to that section of this ball that lies above the cone $\phi = \frac{\pi}{3}$.

61. $z \geq \sqrt{x^2 + y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \geq x^2 + y^2$ $\Rightarrow \quad 2z^2 \geq x^2 + y^2 + z^2 = \rho^2 \quad \Rightarrow \quad z^2 = \rho^2 \cos^2 \phi \geq \frac{1}{2}\rho^2 \quad \Rightarrow \quad \cos^2 \phi \geq \frac{1}{2}$. The cone opens upward so that the inequality is $\cos \phi \geq \frac{1}{\sqrt{2}}$, or equivalently $0 \leq \phi \leq \frac{\pi}{4}$. In spherical coordinates the sphere $z = x^2 + y^2 + z^2$ is $\rho \cos \phi = \rho^2 \quad \Rightarrow \quad \rho = \cos \phi$. $0 \leq \rho \leq \cos \phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \leq \rho \leq \cos \phi$, $0 \leq \phi \leq \frac{\pi}{4}$.

62. In cylindrical coordinates, the equations are $z = r^2$ and $z = 5 - r^2$. The curve of intersection is $r^2 = 5 - r^2$ or $r = \sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0 \le r \le \sqrt{5/2}$. In Maple, we can use either the `coords=cylindrical` option in a regular `plot` command, or the `plots[cylinderplot]` command. In Mathematica, we can use `ParametricPlot3d`.



63. In cylindrical coordinates, the equation of the cylinder is $r = 3$, $0 \le z \le 10$. The hemisphere is the upper part of the sphere radius 3, center $(0, 0, 10)$, equation $r^2 + (z - 10)^2 = 3^2$, $z \ge 10$. In Maple, we can use either the `coords=cylindrical` option in a regular `plot` command, or the `plots[cylinderplot]` command. In Mathematica, we can use `ParametricPlot3d`.



64. We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho = 3960$ mi	$\rho = 3960$ mi
$\theta = 360\,° - 73.60\,° = 286.40\,°$	$\theta = 360\,° - 118.25\,° = 241.75\,°$
$\phi = 90\,° - 45.50\,° = 44.50\,°$	$\phi = 90\,° - 34.06\,° = 55.94\,°$

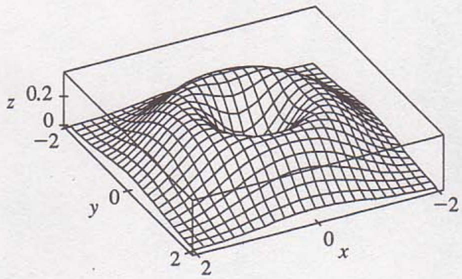
Now we change the above to Cartesian coordinates using $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the Earth). In particular:

Montréal: $\langle 783.67, -2662.67, 2824.47\rangle$ Los Angeles: $\langle -1552.80, -2889.91, 2217.84\rangle$

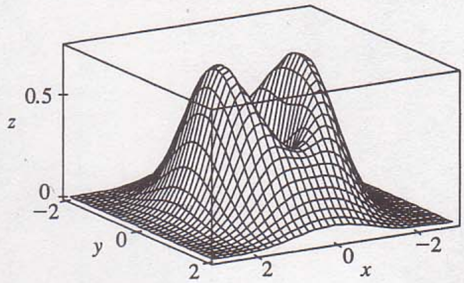
To find the angle α between these two vectors we use the dot product:
$\langle 783.67, -2662.67, 2824.47\rangle \cdot \langle -1552.80, -2889.91, 2217.84\rangle = (3960)^2\cos\alpha \quad\Rightarrow\quad \cos\alpha \approx 0.8126 \quad\Rightarrow$ $\alpha \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\theta \approx 3960\,(0.6223) \approx 2464$ mi.

Laboratory Project □ Families of Surfaces

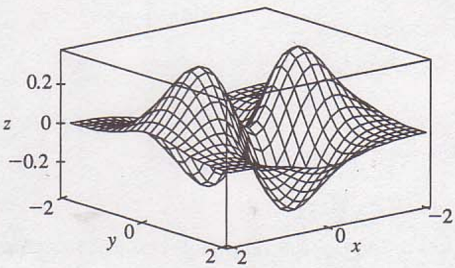
1. $f(x,y) = \left(ax^2 + by^2\right) e^{-x^2-y^2}$. There are only three basic shapes which can be obtained (the fourth and fifth graphs are the reflections of the first and second ones in the xy-plane). Interchanging a and b rotates the graph by $90\,^\circ$ about the z-axis.



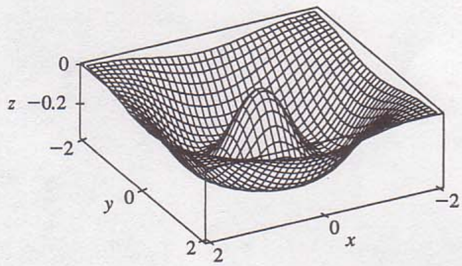
$a = 1, b = 1$



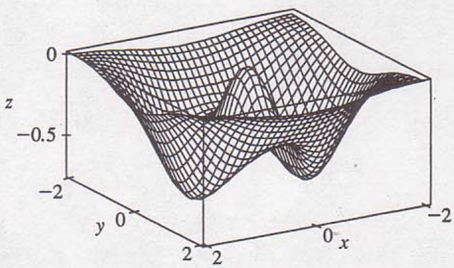
$a = 2, b = 1$



$a = 1, b = -1$



$a = -1, b = -1$

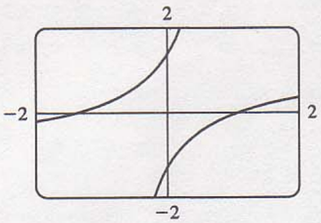


$a = -2, b = -1$

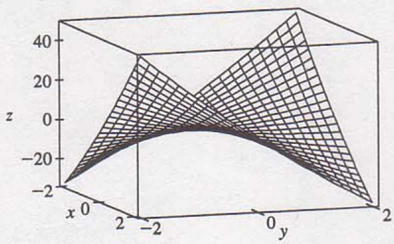
If a and b are both positive ($a \neq b$), we see that the graph has two maximum points whose height increases as a and b increase. If a and b have opposite signs, the graph has two maximum points and two minimum points, and if a and b are both negative, the graph has one maximum point and two minimum points.

2. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See graph below.) It intersects the plane $x = y$ in the parabola $z = (2 + c)\,x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)\,x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

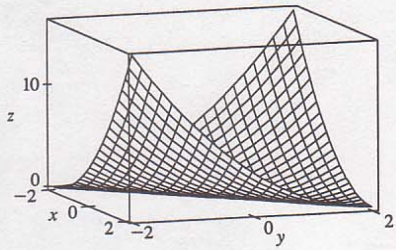
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5$, $z = 2$



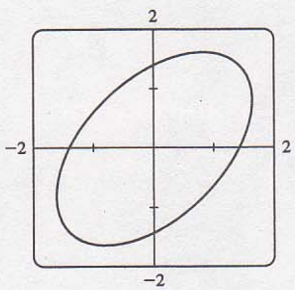
$c = -10$



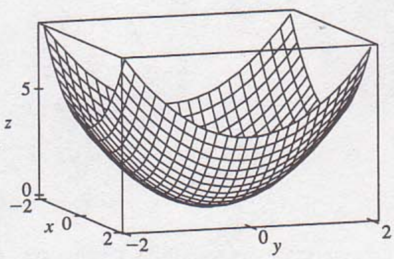
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

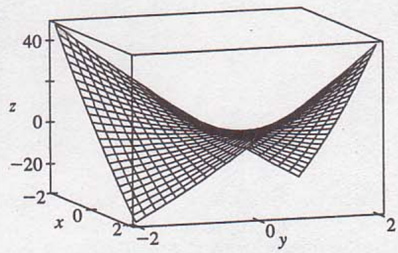
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)\,(-x)\,y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1$, $z = 2$



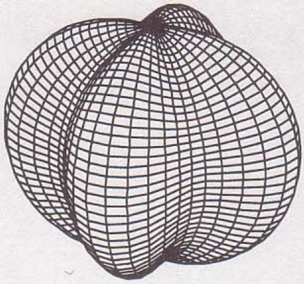
$c = 0$



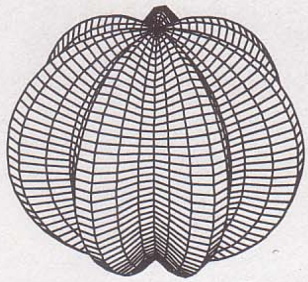
$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

3. $\rho = 1 + 0.2 \sin m\theta \sin n\phi$. If we start with $m = 1, n = 1$ the equation is $\rho = 1 + 0.2 \sin\theta \sin\phi$, whose graph appears spherical or nearly spherical in shape. First we investigate varying just m. Values of $m > 1$ produce vertical ridges in the sphere, the number of ridges corresponding to the value of m. We graph two examples.

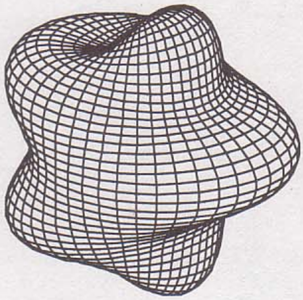


$m = 4, n = 1$

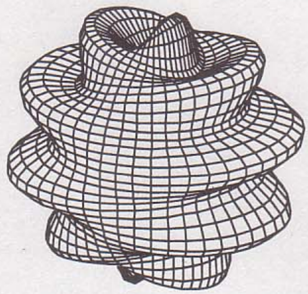


$m = 7, n = 1$

If we leave m fixed at 1 and vary n, we see horizontal ridges that span half the sphere arranged in a staggered fashion. Again, the number of "bumps" coincides with the value of n.

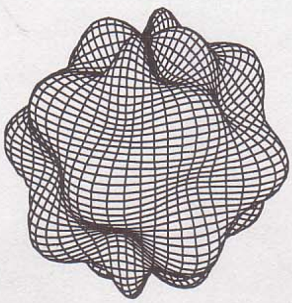


$m = 1, n = 5$

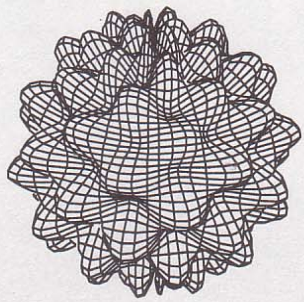


$m = 1, n = 10$

If we allow both m and n to vary, we get combinations of the vertical and horizontal bumps.



$m = 4, n = 5$



$m = 7, n = 10$

The graph on the left shows $m = 4$, $n = 5$. Looking at the top of the bumpy sphere, we can see the 4 vertical ridges which become perturbed horizontally as they progress down the sphere. We can also see the 5 horizontal rows of bumps. (Consequently, there are 20 bumps on the surface.) The graph on the right shows $m = 7$, $n = 10$ which should have 70 bumps.

13 Review

CONCEPT CHECK

1. A scalar is a real number, while a vector is a quantity that has both a real-valued magnitude and a direction.
2. To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law, as illustrated in Figures 4 and 5 in Section 13.2 [ET 12.2]. Algebraically, we add the corresponding components of the vectors.
3. For $c > 0$, $c\mathbf{a}$ is a vector with the same direction as $\mathbf{a}$ and length c times the length of $\mathbf{a}$. If $c < 0$, $c\mathbf{a}$ points in the opposite direction as $\mathbf{a}$ and has length $|c|$ times the length of $\mathbf{a}$. (See Figures 6 and 7 in Section 13.2 [ET 12.2].) Algebraically, to find $c\mathbf{a}$ we multiply each component of $\mathbf{a}$ by c.
4. See (1) in Section 13.2 [ET 12.2].
5. See Theorem 13.3.3 [ET 12.3.3] and Definition 13.3.1 [ET 12.3.1].
6. The dot product can be used to find the angle between two vectors and the scalar projection of one vector onto another. In particular, the dot product can determine if two vectors are orthogonal. Also, the dot product can be used to determine the work done moving an object given the force and displacement vectors.
7. See the boxed equations on page 835 [ET 801] as well as Figures 3 and 4 and the accompanying discussion on page 834 [ET 800].
8. See Theorem 13.4.6 [ET 12.4.6] and the preceding discussion; use either (1) or (4) in Section 13.4 [ET 12.4].
9. The cross product can be used to create a vector orthogonal to two given vectors as well as to determine if two vectors are parallel. The cross product can also be used to find the area of a parallelogram determined by two vectors. In addition, the cross product can be used to determine torque if the force and position vectors are known.
10. (a) The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is the length of the cross product: $|\mathbf{a} \times \mathbf{b}|$.

    (b) The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.
11. If an equation of the plane is known, it can be written as $ax + by + cz + d = 0$. A normal vector, which is perpendicular to the plane, is $\langle a, b, c\rangle$ (or any scalar multiple of $\langle a, b, c\rangle$). If an equation is not known, we can use points on the plane to find two non-parallel vectors which lie in the plane. The cross product of these vectors is a vector perpendicular to the plane.
12. The angle between two intersecting planes is defined as the acute angle between their normal vectors. We can find this angle using Corollary 13.3.6 [ET 12.3.6].
13. See (1), (2), and (3) in Section 13.5 [ET 12.5].
14. See (4), (5), and (6) in Section 13.5 [ET 12.5].
15. (a) Determine the vectors $\overrightarrow{PQ} = \langle a_1, a_2, a_3\rangle$ and $\overrightarrow{PR} = \langle b_1, b_2, b_3\rangle$. If there is a scalar t such that $\langle a_1, a_2, a_3\rangle = t\,\langle b_1, b_2, b_3\rangle$, then the vectors are parallel and the points must all lie on the same line.
    Alternatively, if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P, Q, and R are collinear.
    Thirdly, an algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

    (b) Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P, Q and R, and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. (Or use the reasoning in Example 4 in Section 13.4 [ET 12.4].)
    Alternatively, find an equation for the plane determined by three of the points and check whether or not the fourth point satisfies this equation.

16. (a) See Exercise 13.4.39 [ET 12.4.39].

(b) See Example 8 in Section 13.5 [ET 12.5].

(c) See Example 10 in Section 13.5 [ET 12.5].

17. The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz-plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz-plane) and $z = k$ (parallel to the xy-plane) are found similarly.

18. See Table 1 in Section 13.6 [ET 12.6].

19. (a) See (1) and the discussion accompanying Figure 3 in Section 13.7 [ET 12.7].

(b) See (3) and Figures 6-8, and the accompanying discussion, in Section 13.7 [ET 12.7].

TRUE-FALSE QUIZ

1. True, by Theorem 13.3.2 [ET 12.3.2] #2.

2. False. Theorem 13.4.8 [ET 12.4.8] #1 says that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

3. True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 13.4.6 [ET 12.4.6],
$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = |\mathbf{v}|\,|\mathbf{u}| \sin\theta = |\mathbf{v} \times \mathbf{u}|$.
(Or, by Theorem 13.4.8 [ET 12.4.8], $|\mathbf{u} \times \mathbf{v}| = |-\mathbf{v} \times \mathbf{u}| = |-1|\,|\mathbf{v} \times \mathbf{u}| = |\mathbf{v} \times \mathbf{u}|$.)

4. This is true by Theorem 13.3.2 [ET 12.3.2] #4.

5. Theorem 13.4.8 [ET 12.4.8] #2 tells us that this is true.

6. This is true by Theorem 13.4.8 [ET 12.4.8] #4.

7. This is true by Theorem 13.4.8 [ET 12.4.8] #5.

8. In general, this assertion is false; a counterexample is $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. (See the paragraph preceding Theorem 13.4.8 [ET 12.4.8].)

9. This is true because $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 13.4.5 [ET 12.4.5]), and the dot product of two orthogonal vectors is 0.

10.
$$\begin{aligned}(\mathbf{u}+\mathbf{v}) \times \mathbf{v} &= \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{v} \quad &&\text{(by Theorem 13.4.8 [ET 12.4.8] \#4)}\\ &= \mathbf{u} \times \mathbf{v} + \mathbf{0} \quad &&\text{(by Example 13.4.2 [ET 12.4.2])}\\ &= \mathbf{u} \times \mathbf{v}, \text{ so this is true.}\end{aligned}$$

11. If $|\mathbf{u}| = 1$, $|\mathbf{v}| = 1$ and θ is the angle between these two vectors (so $0 \leq \theta \leq \pi$), then by Theorem 13.4.6 [ET 12.4.6], $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = \sin\theta$, which is equal to 1 if and only if $\theta = \frac{\pi}{2}$ (that is, if and only if the two vectors are orthogonal). Therefore, the assertion that the cross product of two unit vectors is a unit vector is false.

12. This is false, because according to Equation 13.5.7 [ET 12.5.7], $ax + by + cz + d = 0$ is the general equation of a plane.

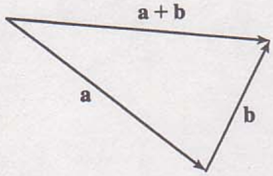
13. This is false. In $\mathbb{R}^2$, $x^2 + y^2 = 1$ represents a circle, but $\{(x, y, z) \mid x^2 + y^2 = 1\}$ represents a *three-dimensional surface*, namely, a circular cylinder with axis the z-axis.

EXERCISES

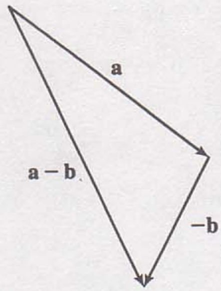
1. (a) By the formula for an equation of a sphere (see Section 13.1 [ET 12.1]), an equation of the sphere with center $(1, -1, 2)$ and radius 3 is $(x-1)^2 + (y+1)^2 + (z-2)^2 = 9$.

(b) Completing squares gives $(x+2)^2 + (y+3)^2 + (z-5)^2 = -2 + 4 + 9 + 25 = 36$. Thus, the sphere is centered at $(-2, -3, 5)$ and has radius 6.

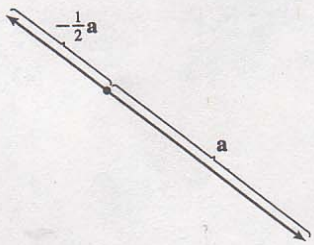
2. (a)



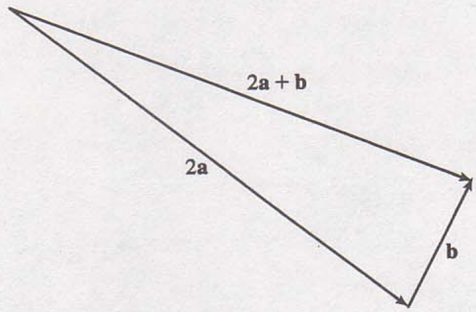
(b)



(c)



(d)



3. $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 45^\circ = (2)(3)\,\frac{\sqrt{2}}{2} = 3\sqrt{2}$. $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin 45^\circ = (2)(3)\,\frac{\sqrt{2}}{2} = 3\sqrt{2}$. By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed out of the page.

4. (a) $2\mathbf{a} + 3\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} = 11\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

(b) $|\mathbf{b}| = \sqrt{9+4+1} = \sqrt{14}$

(c) $\mathbf{a}\cdot\mathbf{b} = (1)(3) + (1)(-2) + (-2)(1) = -1$

(d) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (1-4)\,\mathbf{i} - (1+6)\,\mathbf{j} + (-2-3)\,\mathbf{k} = -3\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$

(e) $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\mathbf{i} + 15\mathbf{j} + 3\mathbf{k}$, $|\mathbf{b}\times\mathbf{c}| = 3\sqrt{9+25+1} = 3\sqrt{35}$

(f) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 9 + 15 - 6 = 18$

(g) $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times(9\mathbf{i}+15\mathbf{j}+3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix} = (3+30)\,\mathbf{i} - (3+18)\,\mathbf{j} + (15-9)\,\mathbf{k}$$

$$= 33\mathbf{i} - 21\mathbf{j} + 6\mathbf{k}.$$

(i) The scalar projection is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = |\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}/\,|\mathbf{a}| = -\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = -\frac{1}{\sqrt{6}}\left(\mathbf{a}/\,|\mathbf{a}|\right) = -\frac{1}{6}\left(\mathbf{i}+\mathbf{j}-2\mathbf{k}\right)$.

(k) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{6}\sqrt{14}} = \dfrac{-1}{2\sqrt{21}}$ and $\theta = \cos^{-1}\dfrac{-1}{2\sqrt{21}} \approx 96^\circ$.

5. For the two vectors to be orthogonal, we need $\langle 3,2,x\rangle\cdot\langle 2x,4,x\rangle = 0 \quad\Leftrightarrow$ $(3)(2x)+(2)(4)+(x)(x)=0 \quad\Leftrightarrow\quad x^2+6x+8=0 \quad\Leftrightarrow\quad (x+2)(x+4)=0 \quad\Leftrightarrow\quad x=-2$ or $x=-4$.

6. We know that the cross product of two vectors is orthogonal to both. So we calculate $(\mathbf{j}+2\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+3\mathbf{k}) = [3-(-4)]\,\mathbf{i}-(0-2)\,\mathbf{j}+(0-1)\,\mathbf{k} = 7\mathbf{i}+2\mathbf{j}-\mathbf{k}$. Then two unit vectors orthogonal to both given vectors are $\pm\dfrac{7\mathbf{i}+2\mathbf{j}-\mathbf{k}}{\sqrt{7^2+2^2+(-1)^2}} = \pm\dfrac{1}{3\sqrt{6}}\,(7\mathbf{i}+2\mathbf{j}-\mathbf{k})$, that is, $\frac{7}{3\sqrt{6}}\mathbf{i}+\frac{2}{3\sqrt{6}}\mathbf{j}-\frac{1}{3\sqrt{6}}\mathbf{k}$ and $-\frac{7}{3\sqrt{6}}\mathbf{i}-\frac{2}{3\sqrt{6}}\mathbf{j}+\frac{1}{3\sqrt{6}}\mathbf{k}$.

7. (a) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 2$

(b) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \mathbf{u}\cdot[-(\mathbf{v}\times\mathbf{w})] = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -2$

(c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -2$

(d) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$

8.
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] &= (\mathbf{a}\times\mathbf{b})\cdot([(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a}) \\ &\qquad \text{(see Exercise 13.4.42 [ET 12.4.42])} \\ &= (\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \\ &= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2\end{aligned}$$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0,0,0)$ to $(1,1,1)$ and $(1,0,0)$ to $(0,1,1)$ are $\langle 1,1,1\rangle$ and $\langle -1,1,1\rangle$. Let θ be the angle between these two vectors. $\langle 1,1,1\rangle\cdot\langle -1,1,1\rangle = -1+1+1 = 1 = |\langle 1,1,1\rangle|\,|\langle -1,1,1\rangle|\cos\theta = 3\cos\theta$ $\Rightarrow\quad \cos\theta = \frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\frac{1}{3} \approx 71^\circ$.

10. $\overrightarrow{AB} = \langle 1,3,-1\rangle$, $\overrightarrow{AC} = \langle -2,1,3\rangle$ and $\overrightarrow{AD} = \langle -1,3,1\rangle$. By Equation 13.4.10 [ET 12.4.10],

$$\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8-3+5 = -6.$$ The volume is $\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right| = 6$ cubic units.

11. $\overrightarrow{AB} = \langle 1, 0, -1 \rangle$, $\overrightarrow{AC} = \langle 0, 4, 3 \rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0 + 4, -(3 + 0), 4 - 0 \rangle = \langle 4, -3, 4 \rangle$.

(b) $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{16 + 9 + 16} = \frac{\sqrt{41}}{2}$.

12. $\mathbf{D} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$, $W = \mathbf{F} \cdot \mathbf{D} = 12 + 15 + 60 = 87$ joules

13. Let F_1 be the magnitude of the force directed $20\,^\circ$ away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives $F_1 \cos 20\,^\circ + F_2 \cos 30\,^\circ = 255$ (1), and $F_1 \sin 20\,^\circ - F_2 \sin 30\,^\circ = 0 \;\Rightarrow$ $F_1 = F_2 \dfrac{\sin 30\,^\circ}{\sin 20\,^\circ}$ (2). Substituting (2) into (1) gives $F_2\left(\sin 30\,^\circ \cot 20\,^\circ + \cos 30\,^\circ\right) = 255 \;\Rightarrow\; F_2 \approx 114$ N. Substituting this into (2) gives $F_1 \approx 166$ N.

14. $|\tau| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.40)(50) \sin(90\,^\circ - 30\,^\circ) \approx 17.3$ joules

15. $x = 1 + 2t$, $y = 2 - t$, $z = 4 + 3t$

16. $\mathbf{v} = \langle 8, -2, 5 \rangle$, so $x = -6 + 8t$, $y = -1 - 2t$ and $z = 5t$.

17. $\mathbf{v} = \langle 4, -3, 5 \rangle$, so $x = 1 + 4t$, $y = -3t$, $z = 1 + 5t$.

18. $2(x - 4) + 6(y + 1) - 3(z + 1) = 0$ or $2x + 6y - 3z = 5$.

19. Since the two planes are parallel, they will have the same normal vectors. So se can take $\mathbf{n} = \langle 1, 2, 5 \rangle$ and an equation of the plane is $1[x - (-4)] + 2(y - 1) + 5(z - 2) = 0$ or $x + 2y + 5z = 8$.

20. Here the vectors $\mathbf{a} = \langle 2 - (-1), 0 - 2, 1 - 0 \rangle = \langle 3, -2, 1 \rangle$ and $\mathbf{b} = \langle -5 - (-1), 3 - 2, 1 - 0 \rangle = \langle -4, 1, 1 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -3, -7, -5 \rangle$ is a normal vector to the plane and an equation of the plane is $-3[x - (-1)] - 7(y - 2) - 5(z - 0) = 0$ or $3x + 7y + 5z = 11$.

21. Substitution of the parametric equations into the equation of the plane gives $2x - y + z = 2(2 - t) - (1 + 3t) + 4t = 2 \;\Rightarrow\; -t + 3 = 2 \;\Rightarrow\; t = 1$. When $t = 1$, the parametric equations give $x = 2 - 1 = 1$, $y = 1 + 3 = 4$ and $z = 4$. Therefore, the point of intersection is $(1, 4, 4)$.

22. Use the formula proven in Exercise 13.4.39 [ET 12.4.39]. In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1, -1, 2 \rangle$. A point on the line is $(1, 2, -1)$ (setting $t = 0$), and therefore $\mathbf{b} = \langle 1 - 0, 2 - 0, -1 - 0 \rangle = \langle 1, 2, -1 \rangle$. Hence

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -1, 2 \rangle \times \langle 1, 2, -1 \rangle|}{\sqrt{1 + 1 + 4}} = \frac{|\langle -3, 3, 3 \rangle|}{\sqrt{6}} = \sqrt{\frac{27}{6}} = \frac{3}{\sqrt{2}}.$$

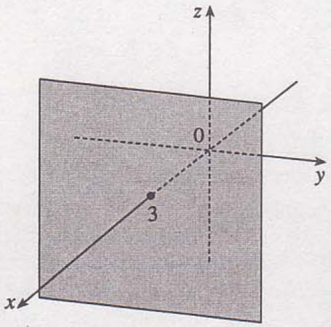
23. Since the direction vectors $\langle 2, 3, 4 \rangle$ and $\langle 6, -1, 2 \rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

24. (a) The normal vectors are $\langle 1, 1, -1 \rangle$ and $\langle 2, -3, 4 \rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

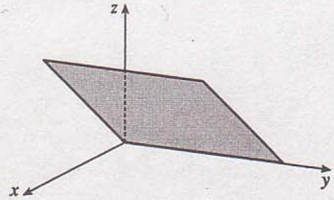
(b) $\cos\theta = \dfrac{\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle}{\sqrt{3}\sqrt{29}} = -\dfrac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122\,^\circ$ (or we can say $\approx 58\,^\circ$).

25. By Exercise 13.6.65 [ET 12.6.65], $D = \dfrac{|2 - 24|}{\sqrt{26}} = \dfrac{22}{\sqrt{26}}$.

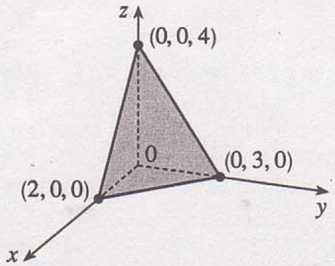
26. The equation $x = 3$ represents a plane parallel to the yz-plane and 3 units in front of it.



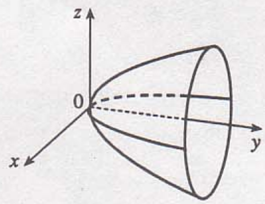
27. The equation $x = z$ represents a plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$.



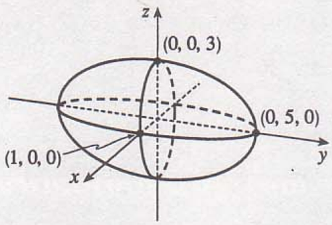
28. An equivalent equation is $(x/2) + (y/3) + (z/4) = 1$ and thus a plane with $x-$, $y-$, z-intercepts 2, 3, and 4.



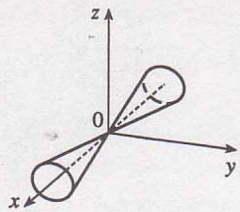
29. A circular paraboloid with vertex the origin and axis the y-axis.



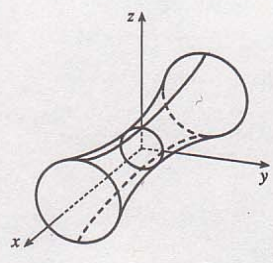
30. An equivalent equation is $x^2 + \dfrac{y^2}{25} + \dfrac{z^2}{9} = 1$, an ellipsoid with center the origin.



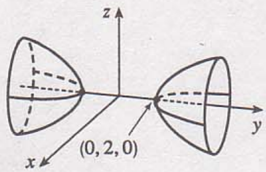
31. A (right elliptical) cone with vertex at the origin and axis the x-axis.



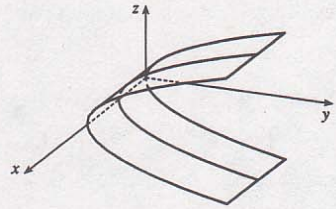
32. A hyperboloid of one sheet with axis the x-axis; traces parallel to the yz-plane are circles.



33. An equivalent equation is $-x^2 + \frac{y^2}{4} - z^2 = 1$, a hyperboloid of two sheets with axis the y-axis. For $|y| > 2$, traces parallel to the xz-plane are circles.



34. A parabolic cylinder whose trace in the xz-plane is the x-axis and which opens to the right.



35. $4x^2 + y^2 = 16 \quad \Leftrightarrow \quad \frac{x^2}{4} + \frac{y^2}{16} = 1$. The equation of the ellipsoid is $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z = 0$ must be the original ellipse. The traces of the ellipsoid in the yz-plane must be circles since the surface is obtained by rotation about the x-axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \quad \Leftrightarrow \quad 4x^2 + y^2 + z^2 = 16$.

36. The distance from a point $P(x, y, z)$ to the plane $y - 1$ is $|y - 1|$, so the given condition becomes
$|y - 1| = 2\sqrt{(x-0)^2 + (y+1)^2 + (z-0)^2} \quad \Rightarrow \quad |y - 1| = 2\sqrt{x^2 + (y+1)^2 + z^2} \quad \Rightarrow$
$(y-1)^2 = 4x^2 + 4(y+1)^2 + 4z^2 \Leftrightarrow -3 = 4x^2 + (3y^2 + 10y) + 4z^2 \quad \Leftrightarrow$
$\frac{16}{3} = 4x^2 + 3\left(y + \frac{5}{3}\right)^2 + 4z^2 \quad \Rightarrow \quad \frac{3}{4}x^2 + \frac{9}{16}\left(y + \frac{5}{3}\right)^2 + \frac{3}{4}z^2 = 1$. This is the equation of an ellipsoid whose center is $\left(0, -\frac{5}{3}, 0\right)$.

37. $x = 2\cos\frac{\pi}{6} = \sqrt{3}$, $y = 2\sin\frac{\pi}{6} = 1$, $z = 2$, so in rectangular coordinates the point is $\left(\sqrt{3}, 1, 2\right)$.
$\rho = \sqrt{3 + 1 + 4} = 2\sqrt{2}$, $\theta = \frac{\pi}{6}$, and $\cos\phi = z/\rho = \frac{1}{\sqrt{2}}$, so $\phi = \frac{\pi}{4}$ and the spherical coordinates are $\left(2\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{4}\right)$.

38. $r = \sqrt{4 + 4} = 2\sqrt{2}$, $z = -1$, $\cos\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$ so $\theta = \frac{\pi}{4}$ and in cylindrical coordinates the point is $\left(2\sqrt{2}, \frac{\pi}{4}, -1\right)$. $\rho = \sqrt{4 + 4 + 1} = 3$, $\cos\phi = -\frac{1}{3}$, so the spherical coordinates are $\left(3, \frac{\pi}{4}, \cos^{-1}\left(-\frac{1}{3}\right)\right)$.

39. $x = 4\sin\frac{\pi}{6}\cos\frac{\pi}{3} = 1$, $y = 4\sin\frac{\pi}{6}\sin\frac{\pi}{3} = \sqrt{3}$, $z = 4\cos\frac{\pi}{6} = 2\sqrt{3}$ so in rectangular coordinates the point is $\left(1, \sqrt{3}, 2\sqrt{3}\right)$. $r^2 = x^2 + y^2 = 4$, $r = 2$, so the cylindrical coordinates are $\left(2, \frac{\pi}{3}, 2\sqrt{3}\right)$.

40. $\phi = \frac{\pi}{4}$. This is one frustum of a circular cone with vertex the origin and axis the positive z-axis.

41. $\theta = \frac{\pi}{4}$. In spherical coordinates, this is a half-plane including the z-axis and intersecting the xy-plane in the half-line $x = y$, $x > 0$.

42. Since $r = \cos\theta$ and $x = r\cos\theta$, $x = \cos^2\theta$. Also $r^2 = x^2 + y^2$ so $x^2 + y^2 = 2\cos^2\theta$. Thus $x^2 + y^2 - 2x = 0$ or $(x-1)^2 + y^2 = 1$. Thus the surface is a circular cylinder with axis the line $x = 1$, $y = 0$, $z = z$.

43. Since $\rho = 3\sec\phi$, $\rho\cos\phi = 3$ or $z = 3$. Thus the surface is a plane parallel to the xy-plane and through the point $(0, 0, 3)$.

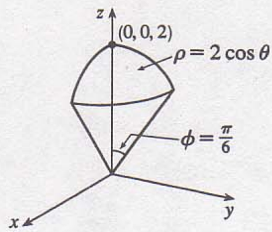
44. $x^2 + y^2 = 4$. In cylindrical coordinates: $r^2 = 4$. In spherical coordinates: $\rho^2 - z^2 = 4$ or $\rho^2 - \rho^2\cos^2\phi = 4$ or $\rho^2\sin^2\phi = 4$ or $\rho\sin\phi = 2$.

45. $x^2 + y^2 + z^2 = 4$. In cylindrical coordinates, this becomes $r^2 + z^2 = 4$. In spherical coordinates, it becomes $\rho^2 = 4$ or $\rho = 2$.

46. In cylindrical coordinates: $r^2 + z^2 = 2r\cos\theta$ or $z^2 = r\,(2\cos\theta - r)$.
In spherical coordinates: $\rho^2 = 2\rho\sin\phi\cos\theta$ or $\rho = 2\sin\phi\cos\theta$.

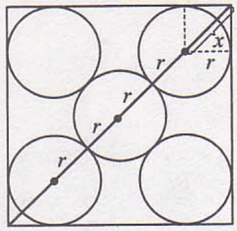
47. The resulting surface is a circular paraboloid with equation $z = 4x^2 + 4y^2$. Changing to cylindrical coordinates we have $z = 4\left(x^2 + y^2\right) = 4r^2$.

48. $\rho = 2\cos\phi \;\Rightarrow\; \rho^2 = 2\rho\cos\phi \;\Rightarrow\; x^2 + y^2 + z^2 = 2z \;\Rightarrow$ $x^2 + y^2 + (z-1)^2 = 1$. This is the equation of a sphere with radius 1, centered at $(0, 0, 1)$. Therefore, $0 \le \rho \le 2\cos\phi$ is the solid ball whose boundary is this sphere. $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{6}$ restrict the solid to the section of this ball that lies above the cone $\phi = \frac{\pi}{6}$ and is in the first octant.



Problems Plus

1. Since three-dimensional situations are often difficult to visualize and work with, let us first try to find an analogous problem in two dimensions. The analogue of a cube is a square and the analogue of a sphere is a circle. Thus a similar problem in two dimensions is the following: if five circles with the same radius r are contained in a square of side 1 m so that the circles touch each other and four of the circles touch two sides of the square, find r.

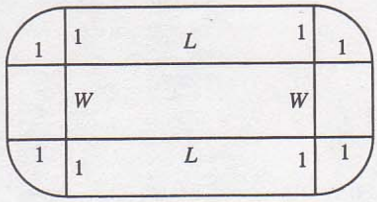


The diagonal of the square is $\sqrt{2}$. The diagonal is also $4r + 2x$. But x is the diagonal of a smaller square of side r. Therefore $x = \sqrt{2}\,r \quad \Rightarrow \quad \sqrt{2} = 4r + 2x = 4r + 2\sqrt{2}\,r = \left(4 + 2\sqrt{2}\right) r \quad \Rightarrow \quad r = \frac{\sqrt{2}}{4+2\sqrt{2}}$.

Let us use these ideas to solve the original three-dimensional problem. The diagonal of the cube is $\sqrt{1^2+1^2+1^2} = \sqrt{3}$. The diagonal of the cube is also $4r + 2x$ where x is the diagonal of a smaller cube with edge r. Therefore $x = \sqrt{r^2+r^2+r^2} = \sqrt{3}\,r \quad \Rightarrow \quad \sqrt{3} = 4r + 2x = 4r + 2\sqrt{3}\,r = \left(4+2\sqrt{3}\right) r$. Thus

$r = \dfrac{\sqrt{3}}{4+2\sqrt{3}} = \dfrac{2\sqrt{3}-3}{2}$. The radius of each ball is $\left(\sqrt{3} - \frac{3}{2}\right)$ m.

2. Try an analogous problem in two dimensions. Consider a rectangle with length L and width W and find the area of S in terms of L and W. Since S contains B, it has area



$$A(S) = LW + \text{the area of two } L \times 1 \text{ rectangles}$$
$$+ \text{ the area of two } 1 \times W \text{ rectangles}$$
$$+ \text{ the area of four quarter-circles of radius 1}$$

as seen in the diagram. So $A(S) = LW + 2L + 2W + \pi \cdot 1^2$.

Now in three dimensions, the volume of S is

$$LWH + 2(L \times W \times 1) + 2(1 \times W \times H) + 2(L \times 1 \times H)$$
$$+ \text{ the volume of 4 quarter-cylinders with radius 1 and height } W$$
$$+ \text{ the volume of 4 quarter-cylinders with radius 1 and height } L$$
$$+ \text{ the volume of 4 quarter-cylinders with radius 1 and height } H$$
$$+ \text{ the volume of 8 eighths of a sphere of radius 1}$$

So

$$\begin{aligned} V(S) &= LWH + 2LW + 2WH + 2LH + \pi \cdot 1^2 \cdot W + \pi \cdot 1^2 \cdot L + \pi \cdot 1^2 \cdot H + \tfrac{4}{3}\pi \cdot 1^3 \\ &= LWH + 2(LW + WH + LH) + \pi(L + W + H) + \tfrac{4}{3}\pi. \end{aligned}$$

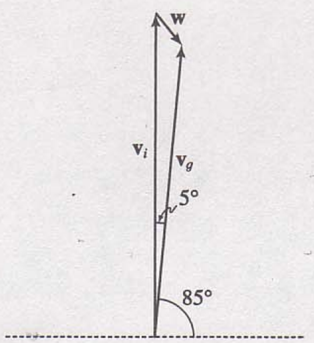
3. (a) We find the line of intersection L as in Example 13.5.7(b) [ET 12.5.7(b)]. Observe that the point $(-1, c, c)$ lies on both planes. Now since L lies in both planes, it is perpendicular to both of the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and thus parallel to their cross product $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & 1 & 1 \\ 1 & -c & c \end{vmatrix} = \langle 2c, -c^2 + 1, -c^2 - 1 \rangle$. So symmetric equations of L can be written as $\dfrac{x+1}{-2c} = \dfrac{y-c}{c^2-1} = \dfrac{z-c}{c^2+1}$, provided that $c \neq 0, \pm 1$.

If $c = 0$, then the two planes are given by $y + z = 0$ and $x = -1$, so symmetric equations of L are $x = -1$, $y = -z$. If $c = -1$, then the two planes are given by $-x + y + z = -1$ and $x + y + z = -1$, and they intersect in the line $x = 0$, $y = -z - 1$. If $c = 1$, then the two planes are given by $x + y + z = 1$ and $x - y + z = 1$, and they intersect in the line $y = 0$, $x = 1 - z$.

(b) If we set $z = t$ in the symmetric equations and solve for x and y separately, we get $x + 1 = \dfrac{(t-c)(-2c)}{c^2+1}$, $y - c = \dfrac{(t-c)(c^2-1)}{c^2+1} \quad \Rightarrow \quad x = \dfrac{-2ct + (c^2-1)}{c^2+1}$, $y = \dfrac{(c^2-1)t + 2c}{c^2+1}$. Eliminating c from these equations, we have $x^2 + y^2 = t^2 + 1$. So the curve traced out by L in the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2+1}$.

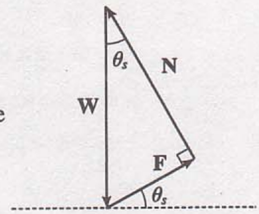
(c) The area of a horizontal cross-section of the solid is $A(z) = \pi(z^2 + 1)$, so $V = \int_0^1 A(z)\,dz = \pi\left[\frac{1}{3}z^3 + z\right]_0^1 = \frac{4\pi}{3}$.

4. (a) We consider velocity vectors for the plane and the wind. Let $\mathbf{v}_i$ be the initial, intended velocity for the plane and $\mathbf{v}_g$ the actual velocity relative to the ground. If $\mathbf{w}$ is the velocity of the wind, $\mathbf{v}_g$ is the resultant, that is, the vector sum $\mathbf{v}_i + \mathbf{w}$ as shown in the figure. We know $\mathbf{v}_i = 180\mathbf{j}$, and since the plane actually flew 80 km in $\frac{1}{2}$ hour, $|\mathbf{v}_g| = 160$. Thus $\mathbf{v}_g = (160\cos 85°)\,\mathbf{i} + (160\sin 85°)\,\mathbf{j} \approx 13.9\mathbf{i} + 159.4\mathbf{j}$. Finally, $\mathbf{v}_i + \mathbf{w} = \mathbf{v}_g$, so $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_i \approx 13.9\mathbf{i} - 20.6\mathbf{j}$. Thus, the wind velocity is about $13.9\mathbf{i} - 20.6\mathbf{j}$, and the wind speed is $|\mathbf{w}| \approx \sqrt{(13.9)^2 + (-20.6)^2} \approx 24.9$ km/h.

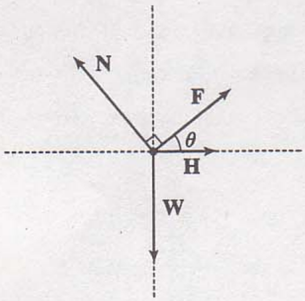


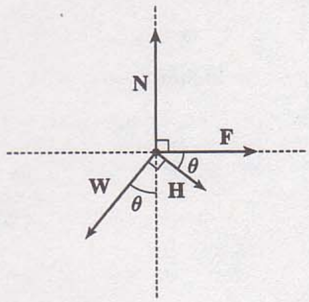
(b) Let $\mathbf{v}$ be the velocity the pilot should take. With the effect of wind, the actual velocity (with respect to the ground) will be $\mathbf{v} + \mathbf{w}$, which we want to be $\mathbf{v}_i$. Thus $\mathbf{v} = \mathbf{v}_i - \mathbf{w} \approx 180\mathbf{j} - (13.9\mathbf{i} - 20.6\mathbf{j}) \approx -13.9\mathbf{i} + 200.6\mathbf{j}$. The angle for this vector can be found by $\tan\theta \approx \frac{200.6}{-13.9} \quad \Rightarrow \quad \theta \approx 94.0°$, or $4.0°$ west of north.

5. (a) When $\theta = \theta_s$, the block is not moving, so the sum of the forces on the block must be $\mathbf{0}$, thus $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This relationship is illustrated geometrically in the figure. Since the vectors form a right triangle, we have $\tan(\theta_s) = \dfrac{|\mathbf{F}|}{|\mathbf{N}|} = \dfrac{\mu_s n}{n} = \mu_s$.



(b) We place the block at the origin and sketch the force vectors acting on the block, including the additional horizontal force $\mathbf{H}$, with initial points at the origin. We then rotate this system so that $\mathbf{F}$ lies along the positive x-axis and the inclined plane is parallel to the x-axis.





$|\mathbf{F}|$ is maximal, so $|\mathbf{F}| = \mu_s n$ for $\theta > \theta_s$. Then the vectors, in terms of components parallel and perpendicular to the inclined plane, are

$$\mathbf{N} = n\,\mathbf{j} \qquad \mathbf{F} = (\mu_s n)\,\mathbf{i}$$

$$\mathbf{W} = (-mg\sin\theta)\,\mathbf{i} + (-mg\cos\theta)\,\mathbf{j}$$

$$\mathbf{H} = (h_{\min}\cos\theta)\,\mathbf{i} + (-h_{\min}\sin\theta)\,\mathbf{j}$$

Equating components, we have

$$\mu_s n - mg\sin\theta + h_{\min}\cos\theta = 0 \quad\Rightarrow\quad h_{\min}\cos\theta + \mu_s n = mg\sin\theta \tag{1}$$

$$n - mg\cos\theta - h_{\min}\sin\theta = 0 \quad\Rightarrow\quad h_{\min}\sin\theta + mg\cos\theta = n \tag{2}$$

(c) Since (2) is solved for n, we substitute into (1):

$$h_{\min}\cos\theta + \mu_s\left(h_{\min}\sin\theta + mg\cos\theta\right) = mg\sin\theta \quad\Rightarrow$$

$$h_{\min}\cos\theta + h_{\min}\mu_s\sin\theta = mg\sin\theta - mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\min} = mg\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)$$

From part (a) we know $\mu_s = \tan\theta_s$, so this becomes $h_{\min} = mg\left(\dfrac{\tan\theta - \tan\theta_s}{1 + \tan\theta_s\tan\theta}\right)$ and using a trigonometric identity, this is $mg\tan(\theta - \theta_s)$ as desired.

Note for $\theta = \theta_s$, $h_{\min} = mg \tan 0 = 0$, which makes sense since the block is at rest for θ_s, thus no additional force $\mathbf{H}$ is necessary to prevent it from moving. As θ increases, the factor $\tan(\theta - \theta_s)$, and hence the value of $h_{\min}$, increases slowly for small values of $\theta - \theta_s$ but much more rapidly as $\theta - \theta_s$ becomes significant. This seems reasonable, as the steeper the inclined plane, the less the horizontal components of the various forces affect the movement of the block, so we would need a much larger magnitude of horizontal force to keep the block motionless. If we allow $\theta \to 90^\circ$, corresponding to the inclined plane being placed vertically, the value of $h_{\min}$ is quite large; this is to be expected, as it takes a great amount of horizontal force to keep an object from moving vertically. In fact, without friction (so $\theta_s = 0$), we would have $\theta \to 90^\circ \quad \Rightarrow \quad h_{\min} \to \infty$, and it would be impossible to keep the block from slipping.

(d) Since $h_{\max}$ is the largest value of h that keeps the block from slipping, the force of friction is keeping the block from moving *up* the inclined plane; thus, $\mathbf{F}$ is directed *down* the plane. Our system of forces is similar to that in part (b), then, except that we have $\mathbf{F} = -(\mu_s n)\,\mathbf{i}$. (Note that $|\mathbf{F}|$ is again maximal.) Following our procedure in parts (b) and (c), we equate components:

$$-\mu_s n - mg\sin\theta + h_{\max}\cos\theta = 0 \quad \Rightarrow \quad h_{\max}\cos\theta - \mu_s n = mg\sin\theta$$

$$n - mg\cos\theta - h_{\max}\sin\theta = 0 \quad \Rightarrow \quad h_{\max}\sin\theta + mg\cos\theta = n$$

Then substituting,

$$h_{\max}\cos\theta - \mu_s\left(h_{\max}\sin\theta + mg\cos\theta\right) = mg\sin\theta \quad \Rightarrow$$

$$h_{\max}\cos\theta - h_{\max}\mu_s\sin\theta = mg\sin\theta + mg\mu_s\cos\theta \quad \Rightarrow$$

$$h_{\max} = mg\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)$$

$$= mg\left(\frac{\tan\theta + \tan\theta_s}{1 - \tan\theta_s\tan\theta}\right) = mg\tan(\theta + \theta_s)$$

We would expect $h_{\max}$ to increase as θ increases, with similar behavior as we established for $h_{\min}$, but with $h_{\max}$ values always larger than $h_{\min}$. We can see that this is the case if we graph $h_{\max}$ as a function of θ, as the curve is the graph of $h_{\min}$ translated $2\theta_s$ to the left, so the equation does seem reasonable. Notice that the equation predicts $h_{\max} \to \infty$ as $\theta \to (90^\circ - \theta_s)$. In fact, as $h_{\max}$ increases, the normal force increases as well. When $(90^\circ - \theta_s) \le \theta \le 90^\circ$, the horizontal force is completely counteracted by the sum of the normal and frictional forces, so no part of the horizontal force contributes to moving the block up the plane no matter how large its magnitude.

14 Vector Functions ET 13

14.1 Vector Functions and Space Curves ET 13.1

1. The component functions t^2, $\sqrt{t-1}$, and $\sqrt{5-t}$ are all defined when $t-1 \geq 0 \quad \Rightarrow \quad t \geq 1$ and $5-t \geq 0 \quad \Rightarrow$ $t \leq 5$, so the domain of $\mathbf{r}(t)$ is $[1,5]$.

2. The component functions $\ln t$, $\dfrac{t}{t-1}$, and e^{-t} are all defined when $t>0$ and $t \neq 1$, so the domain of $\mathbf{r}(t)$ is $(0,1) \cup (1,\infty)$.

3. $\lim\limits_{t\to 0^+} \cos t = \cos 0 = 1$, $\lim\limits_{t\to 0^+} \sin t = \sin 0 = 0$, $\lim\limits_{t\to 0^+} t\ln t = \lim\limits_{t\to 0^+} \dfrac{\ln t}{1/t} = \lim\limits_{t\to 0^+} \dfrac{1/t}{-1/t^2} = \lim\limits_{t\to 0^+} -t = 0$ (by l'Hospital's Rule). Thus $\lim\limits_{t\to 0^+} \langle \cos t, \sin t, t\ln t\rangle = \left\langle \lim\limits_{t\to 0^+} \cos t, \lim\limits_{t\to 0^+} \sin t, \lim\limits_{t\to 0^+} t\ln t \right\rangle = \langle 1,0,0\rangle$.

4. $\lim\limits_{t\to 0} \dfrac{e^t-1}{t} = \lim\limits_{t\to 0} \dfrac{e^t}{1} = 1$ (using l'Hospital's Rule),

$\lim\limits_{t\to 0} \dfrac{\sqrt{1+t}-1}{t} = \lim\limits_{t\to 0} \dfrac{\sqrt{1+t}-1}{t} \cdot \dfrac{\sqrt{1+t}+1}{\sqrt{1+t}+1} = \lim\limits_{t\to 0} \dfrac{1}{\sqrt{1+t}+1} = \dfrac{1}{2}$, $\lim\limits_{t\to 0} \dfrac{3}{1+t} = 3$. Thus, the given limit equals $\left\langle 1, \frac{1}{2}, 3\right\rangle$.

5. $\lim\limits_{t\to 1} \sqrt{t+3} = 2$, $\lim\limits_{t\to 1} \dfrac{t-1}{t^2-1} = \lim\limits_{t\to 1} \dfrac{1}{t+1} = \dfrac{1}{2}$, $\lim\limits_{t\to 1} \left(\dfrac{\tan t}{t}\right) = \tan 1$

Thus the given limit equals $\left\langle 2, \frac{1}{2}, \tan 1\right\rangle$.

6. $\lim\limits_{t\to\infty} e^{-t} = 0$, $\lim\limits_{t\to\infty} \dfrac{t-1}{t+1} = 1$, $\lim\limits_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$, so the given limit equals $\left\langle 0, 1, \frac{\pi}{2}\right\rangle$.

7. $x = \cos 4t$, $y = t$, $z = \sin 4t$. At any point (x,y,z) on the curve, $x^2+z^2 = \cos^2 4t + \sin^2 4t = 1$. So the curve lies on a circular cylinder with axis the y-axis. Since $y=t$, this is a helix. So the graph is V.

8. $x = t^2-2$, $y = t^3$, $z = t^4+1$. Note that $z>0$ for all t, and $x>0$ for $|t|>\sqrt{2}$, so most of the graph must lie where x and z are positive. As $t\to\infty$, $(x,y,z)\to(\infty,\infty,\infty)$, and as $t\to-\infty$, $(x,y,z)\to(\infty,-\infty,\infty)$. So the graph is VI.

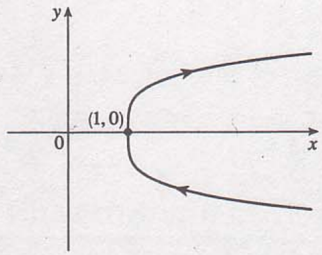
9. $x = t$, $y = 1/\left(1+t^2\right)$, $z = t^2$. Note that y and z are positive for all t. The curve passes through $(0,1,0)$ when $t=0$. As $t\to\infty$, $(x,y,z)\to(\infty,0,\infty)$, and as $t\to-\infty$, $(x,y,z)\to(-\infty,0,\infty)$. So the graph is I.

10. $x = \sin 3t\cos t$, $y = \sin 3t \sin t$, $z = t$. The curve passes through the origin when $t=0$. The values of x and y are periodic, and since $z=t$, the curve is periodic in the z-direction. So the graph is III.

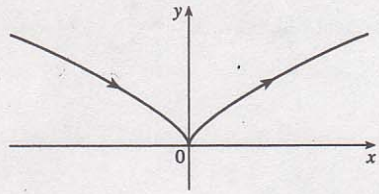
11. $x = \cos t$, $y = \sin t$, $z = \sin 5t$. $\quad x^2+y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z-axis. Each of x, y and z is periodic, and at $t=0$ and $t=2\pi$ the curve passes through the same point, so the curve repeats itself and the graph is IV.

12. $x = \cos t$, $y = \sin t$, $z = \ln t$. $\quad x^2+y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z-axis. As $t\to 0$, $z\to-\infty$, so the graph is II.

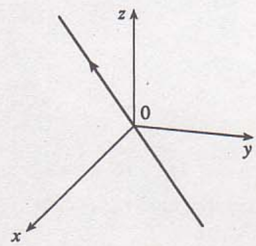
13. The parametric equations for this curve are $x = t^4 + 1$, $y = t$. We can make a table of values, or we can eliminate the parameter: $t = y \Rightarrow x = y^4 + 1$, with $y \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



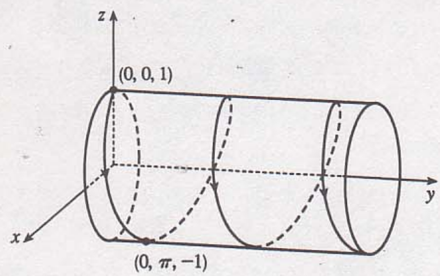
14. The parametric equations for this curve are $x = t^3, y = t^2$. We can make a table of values, or we can eliminate the parameter: $x = t^3 \Rightarrow t = \sqrt[3]{x} \Rightarrow$ $y = t^2 = (\sqrt[3]{x})^2 = x^{2/3}$, with $t \in \mathbb{R} \Rightarrow x \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



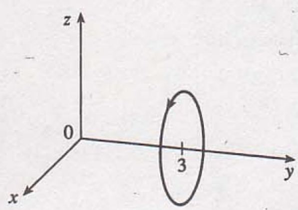
15. The corresponding parametric equations are $x = t, y = -t$, $z = 2t$, which are parametric equations of a line through the origin and with direction vector $\langle 1, -1, 2 \rangle$.



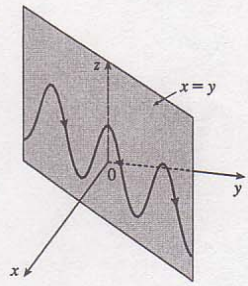
16. The parametric equations give $x^2 + z^2 = \sin^2 t + \cos^2 t = 1, y = t$, so the curve lies on the cylinder $x^2 + z^2 = 1$. Since $y = t$, the curve is a helix.



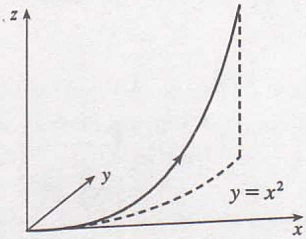
17. The parametric equations give $x^2 + z^2 = \sin^2 t + \cos^2 t = 1, y = 3$, which is a circle of radius 1, center $(0, 3, 0)$ in the plane $y = 3$.



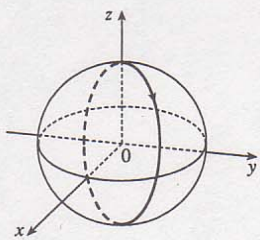
18. The parametric equations are $x = t$, $y = t$, $z = \cos t$. Thus $x = y$, so the curve must lie in the plane $x = y$. Combine this with $z = \cos t$ to determine that the curve traces out the cosine curve in the vertical plane $x = y$.



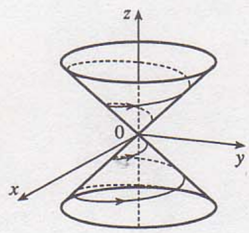
19. The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the xy-plane is $y = x^2$, $y > 0$, a half parabola. On the xz-plane $z = x^3$, $z > 0$, a half cubic, and the yz-plane, $y^3 = z^2$.



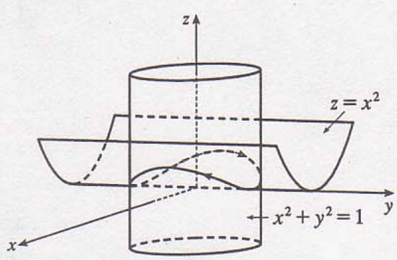
20. The parametric equations give $x^2 + y^2 + z^2 = 2\sin^2 t + 2\cos^2 t = 2$, so the curve lies on the sphere with radius $\sqrt{2}$ and center $(0, 0, 0)$. Furthermore $x = y = \sin t$, so the curve is the intersection of this sphere with the plane $x = y$, that is, the curve is the circle of radius $\sqrt{2}$, center $(0, 0, 0)$ in the plane $x = y$.



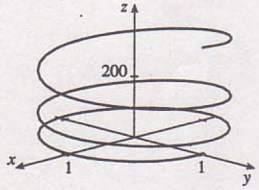
21. If $x = t\cos t$, $y = t\sin t$, and $z = t$, then $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.



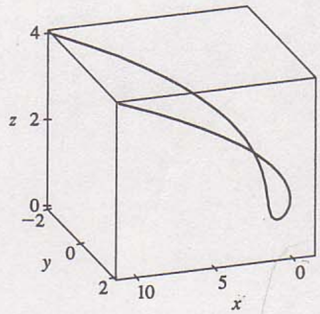
22. Here $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$.



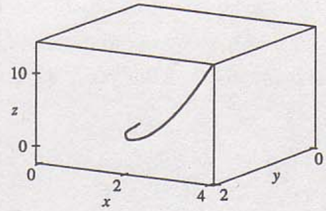
23. $\mathbf{r}(t) = \langle \sin t, \cos t, t^2 \rangle$



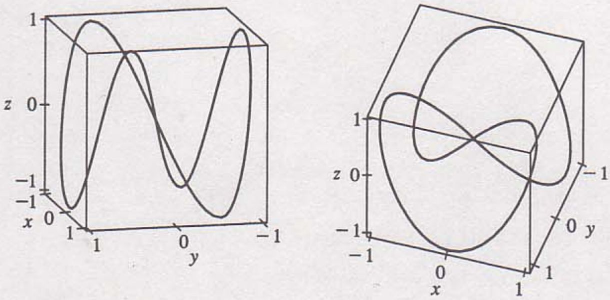
24. $\mathbf{r}(t) = \langle t^4 - t^2 + 1, t, t^2 \rangle$



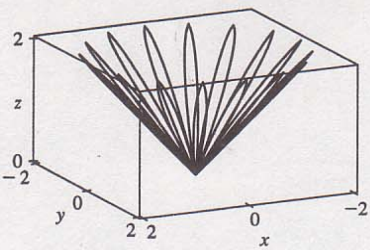
25. $\mathbf{r}(t) = \langle \sqrt{t}, t, t^2 - 2 \rangle$



26. We have the computer plot the parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$, $0 \le t \le 2\pi$. The shape of the curve is not clear from just one viewpoint, so we include a second plot drawn from a different angle.



27.

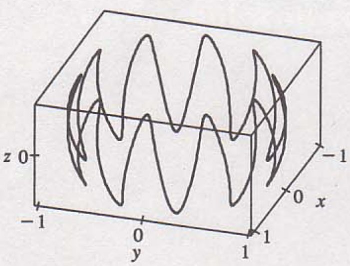


$x = (1 + \cos 16t)\cos t$, $y = (1 + \cos 16t)\sin t$, $z = 1 + \cos 16t$. At any point on the graph,

$$x^2 + y^2 = (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t$$
$$= (1 + \cos 16t)^2 = z^2, \text{ so the graph lies on the cone}$$

$x^2 + y^2 = z^2$. From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

28.



$x = \sqrt{1 - 0.25\cos^2 10t}\,\cos t$, $y = \sqrt{1 - 0.25\cos^2 10t}\,\sin t$, $z = 0.5\cos 10t$. At any point on the graph,

$$x^2 + y^2 + z^2 = (1 - 0.25\cos^2 10t)\cos^2 t$$
$$+ (1 - 0.25\cos^2 10t)\sin^2 t + 0.25\cos^2 t$$
$$= 1 - 0.25\cos^2 10t + 0.25\cos^2 10t = 1,$$

so the graph lies on the sphere $x^2 + y^2 + z^2 = 1$, and since $z = 0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. The graph is generated by $t \in [0, 2\pi]$.

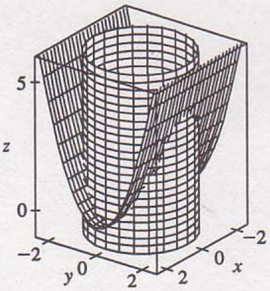
29. If $t=-1$, then $x=1, y=4, z=0$, so the curve passes through the point $(1,4,0)$. If $t=3$, then $x=9, y=-8, z=28$, so the curve passes through the point $(9,-8,28)$. For the point $(4,7,-6)$ to be on the curve, we require $y=1-3t=7 \Rightarrow t=-2$. But then $z=1+(-2)^3=-7\neq-6$, so $(4,7,-6)$ is not on the curve.

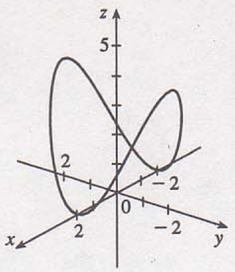
30. The projection of the curve C of intersection onto the xy-plane is the circle $x^2+y^2=4, z=0$. Then we can write $x=2\cos t, y=2\sin t, 0\leq t\leq 2\pi$. Since C also lies on the surface $z=xy$, we have $z=xy=(2\cos t)(2\sin t)=4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for C are $x=2\cos t, y=2\sin t, z=2\sin(2t), 0\leq t\leq 2\pi$, and the corresponding vector function is $\mathbf{r}(t)=2\cos t\,\mathbf{i}+2\sin t\,\mathbf{j}+2\sin(2t)\,\mathbf{k}, 0\leq t\leq 2\pi$.

31. Both equations are solved for z, so we can substitute to eliminate z: $\sqrt{x^2+y^2}=1+y \Rightarrow x^2+y^2=1+2y+y^2 \Rightarrow x^2=1+2y \Rightarrow y=\frac{1}{2}(x^2-1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x=t$, then $y=\frac{1}{2}(t^2-1)$ and $z=1+y=1+\frac{1}{2}(t^2-1)=\frac{1}{2}(t^2+1)$. Thus a vector function representing C is $\mathbf{r}(t)=t\,\mathbf{i}+\frac{1}{2}(t^2-1)\,\mathbf{j}+\frac{1}{2}(t^2+1)\,\mathbf{k}$.

32. The projection of the curve C of intersection onto the xy-plane is the parabola $y=x^2, z=0$. Then we can choose the parameter $x=t \Rightarrow y=t^2$. Since C also lies on the surface $z=4x^2+y^2$, we have $z=4x^2+y^2=4t^2+(t^2)^2$. Then parametric equations for C are $x=t, y=t^2, z=4t^2+t^4$, and the corresponding vector function is $\mathbf{r}(t)=t\,\mathbf{i}+t^2\,\mathbf{j}+(4t^2+t^4)\,\mathbf{k}$.

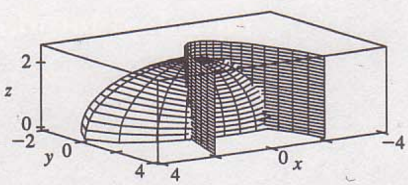
33.

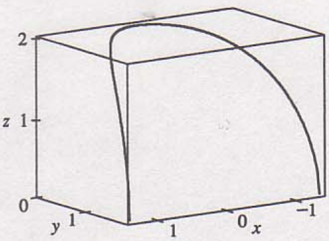




The projection of the curve C of intersection onto the xy-plane is the circle $x^2+y^2=4, z=0$. Then we can write $x=2\cos t, y=2\sin t$, $0\leq t\leq 2\pi$. Since C also lies on the surface $z=x^2$, we have $z=x^2=(2\cos t)^2=4\cos^2 t$. Then parametric equations for C are $x=2\cos t$, $y=2\sin t, z=4\cos^2 t, 0\leq t\leq 2\pi$.

34.





$x=t \Rightarrow y=t^2 \Rightarrow 4z^2=16-x^2-4y^2=16-t^2-4t^4 \Rightarrow z=\sqrt{4-\left(\frac{1}{2}t\right)^2-t^4}$. Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x=t, y=t^2$, $z=\sqrt{4-\frac{1}{4}t^2-t^4}$.

35. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real-valued functions.

(a) $\lim\limits_{t\to a}\mathbf{u}(t) + \lim\limits_{t\to a}\mathbf{v}(t) = \left\langle \lim\limits_{t\to a} u_1(t), \lim\limits_{t\to a} u_2(t), \lim\limits_{t\to a} u_3(t)\right\rangle + \left\langle \lim\limits_{t\to a} v_1(t), \lim\limits_{t\to a} v_2(t), \lim\limits_{t\to a} v_3(t)\right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \to a$. Then adding the two vectors and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned}
\lim_{t\to a}\mathbf{u}(t) + \lim_{t\to a}\mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t) + \lim_{t\to a} v_1(t), \lim_{t\to a} u_2(t) + \lim_{t\to a} v_2(t), \lim_{t\to a} u_3(t) + \lim_{t\to a} v_3(t)\right\rangle \\
&= \left\langle \lim_{t\to a}[u_1(t) + v_1(t)], \lim_{t\to a}[u_2(t) + v_2(t)], \lim_{t\to a}[u_3(t) + v_3(t)]\right\rangle \\
&= \lim_{t\to a}\langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t)\rangle \quad \text{[using (1) backward]} \\
&= \lim_{t\to a}[\mathbf{u}(t) + \mathbf{v}(t)]
\end{aligned}$$

(b) $$\begin{aligned}
\lim_{t\to a} c\mathbf{u}(t) &= \lim_{t\to a}\langle cu_1(t), cu_2(t), cu_3(t)\rangle = \left\langle \lim_{t\to a} cu_1(t), \lim_{t\to a} cu_2(t), \lim_{t\to a} cu_3(t)\right\rangle \\
&= \left\langle c\lim_{t\to a} u_1(t), c\lim_{t\to a} u_2(t), c\lim_{t\to a} u_3(t)\right\rangle = c\left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \\
&= c\lim_{t\to a}\langle u_1(t), u_2(t), u_3(t)\rangle = c\lim_{t\to a}\mathbf{u}(t)
\end{aligned}$$

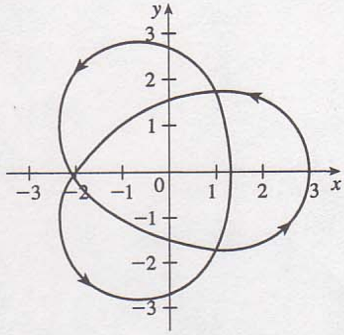
(c) $$\begin{aligned}
\lim_{t\to a}\mathbf{u}(t)\cdot\lim_{t\to a}\mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \cdot \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle \\
&= \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_1(t)\right] + \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_2(t)\right] + \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_3(t)\right] \\
&= \lim_{t\to a} u_1(t)\,v_1(t) + \lim_{t\to a} u_2(t)\,v_2(t) + \lim_{t\to a} u_3(t)\,v_3(t) \\
&= \lim_{t\to a}[u_1(t)\,v_1(t) + u_2(t)\,v_2(t) + u_3(t)\,v_3(t)] = \lim_{t\to a}[\mathbf{u}(t)\cdot\mathbf{v}(t)]
\end{aligned}$$

(d) $$\begin{aligned}
\lim_{t\to a}\mathbf{u}(t)\times\lim_{t\to a}\mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \times \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle \\
&= \Bigg\langle \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_3(t)\right] - \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_2(t)\right], \\
&\qquad \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_1(t)\right] - \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_3(t)\right], \\
&\qquad \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_2(t)\right] - \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_1(t)\right]\Bigg\rangle \\
&= \Big\langle \lim_{t\to a}[u_2(t)\,v_3(t) - u_3(t)\,v_2(t)], \lim_{t\to a}[u_3(t)\,v_1(t) - u_1(t)\,v_3(t)], \\
&\qquad \lim_{t\to a}[u_1(t)\,v_2(t) - u_2(t)\,v_1(t)]\Big\rangle \\
&= \lim_{t\to a}\langle u_2(t)\,v_3(t) - u_3(t)\,v_2(t), u_3(t)\,v_1(t) - u_1(t)\,v_3(t), \\
&\qquad u_1(t)\,v_2(t) - u_2(t)\,v_1(t)\rangle \\
&= \lim_{t\to a}[\mathbf{u}(t)\times\mathbf{v}(t)]
\end{aligned}$$

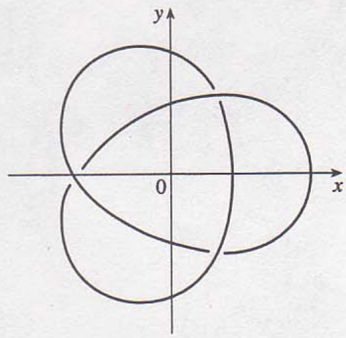
36. The projection of the curve onto the xy-plane is given by the parametric equations $x = (2 + \cos 1.5t)\cos t$, $y = (2 + \cos 1.5t)\sin t$. If we convert to polar coordinates, we have

$$\begin{aligned} r^2 = x^2 + y^2 &= [(2 + \cos 1.5t)\cos t]^2 + [(2 + \cos 1.5t)\sin t]^2 = (2 + \cos 1.5t)^2 \left(\cos^2 t + \sin^2 t\right) \\ &= (2 + \cos 1.5t)^2 \end{aligned}$$

$\Rightarrow \quad r = 2 + \cos 1.5t$. Also, $\tan\theta = \dfrac{y}{x} = \dfrac{(2 + \cos 1.5t)\sin t}{(2 + \cos 1.5t)\cos t} = \tan t \quad \Rightarrow \quad \theta = t$.

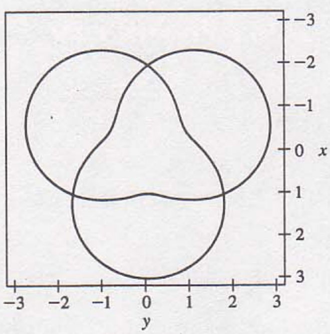


Thus the polar equation of the curve is $r = 2 + \cos 1.5\theta$. At $\theta = 0$, we have $r = 3$, and r decreases to 1 as θ increases to $\frac{2\pi}{3}$. For $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$, r increases to 3; r decreases to 1 again at $\theta = 2\pi$, increases to 3 at $\theta = \frac{8\pi}{3}$, decreases to 1 at $\theta = \frac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta = 4\pi$. We sketch an approximate graph as shown in the figure.

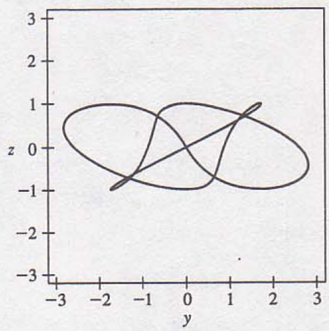


We can determine how the curve passes over itself by investigating the maximum and minimum values of z for $t = \theta \in [0, 4\pi]$. Since $z = \sin 1.5t$, z is maximized where $\sin 1.5t = 1 \quad \Rightarrow \quad 1.5t = \frac{\pi}{2}, \frac{5\pi}{2}$, or $\frac{9\pi}{2} \quad \Rightarrow$ $t = \frac{\pi}{3}, \frac{5\pi}{3}$, or 3π. z is minimized where $\sin 1.5t = -1 \quad \Rightarrow$ $1.5t = \frac{3\pi}{2}, \frac{7\pi}{2}$, or $\frac{11\pi}{2} \quad \Rightarrow \quad t = \pi, \frac{7\pi}{3}$, or $\frac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t = 0 \quad \Rightarrow$ $r = 2$, and on the graph of the projection, these six points appear to be at the three self-intersections we see. Comparing the maximum and minimum values of z at these intersections, we can determine where the curve passes over itself, as indicated in the figure.

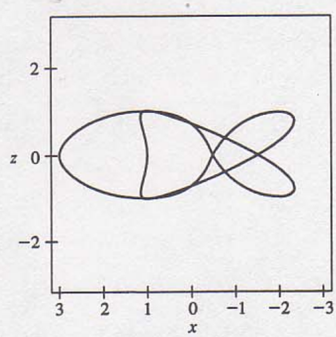
We show a computer-drawn graph of the curve from above, as well as views from the front and from the right side.



Top view

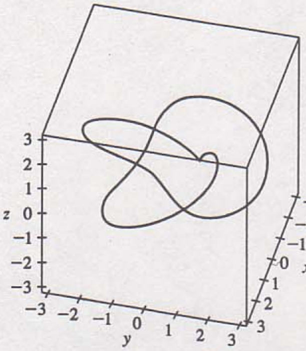


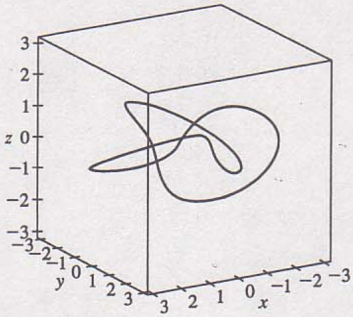
Front view

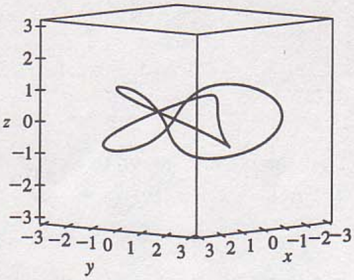


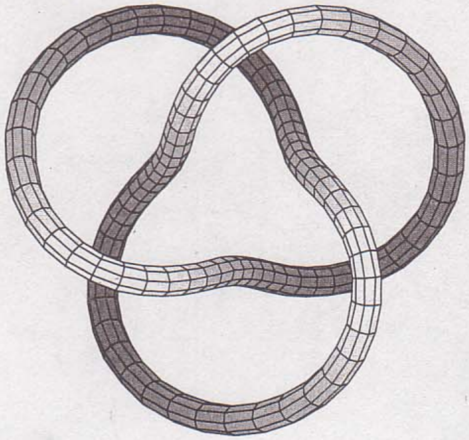
Side view

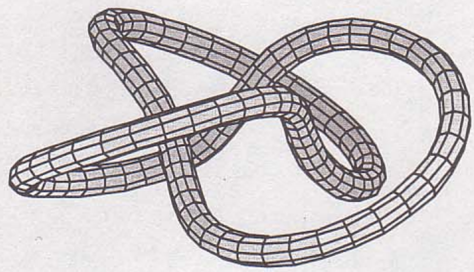
The top view graph shows a more accurate representation of the projection of the trefoil knot on the xy-plane (the axes are rotated 90 °). Notice the indentations the graph exhibits at the points corresponding to $r = 1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.











37. Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. If $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{b}$, then $\lim\limits_{t\to a} \mathbf{r}(t)$ exists, so by (1), $\mathbf{b} = \lim\limits_{t\to a} \mathbf{r}(t) = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t)\right\rangle$. By the definition of equal vectors we have $\lim\limits_{t\to a} f(t) = b_1$, $\lim\limits_{t\to a} g(t) = b_2$ and $\lim\limits_{t\to a} h(t) = b_3$. But these are limits of real-valued functions, so by the definition of limits, for every $\epsilon > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ so $|f(t) - b_1| < \epsilon/3$ whenever $0 < |t - a| < \delta_1$, $|g(t) - b_2| < \epsilon/3$ whenever $0 < |t - a| < \delta_2$, and $|h(t) - b_3| < \epsilon/3$ whenever $0 < |t - a| < \delta_3$. Letting $\delta =$ minimum of $\{\delta_1, \delta_2, \delta_3\}$, we have $|f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ whenever $0 < |t - a| < \delta$. But $|\mathbf{r}(t) - \mathbf{b}| = |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3\rangle|$
$= \sqrt{(f(t) - b_1)^2 + (g(t) - b_2)^2 + (h(t) - b_3)^2} \le \sqrt{[f(t) - b_1]^2} + \sqrt{[g(t) - b_2]^2} + \sqrt{[h(t) - b_3]^2}$
$= |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3|$. Thus for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{r}(t) - b| \le |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. Conversely, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathbf{r}(t) - \mathbf{b}| < \epsilon$ whenever $0 < |t - a| < \delta$, then

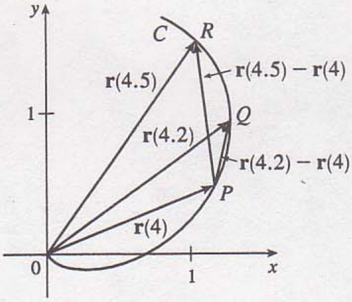
$$|\langle f(t) - b_1, g(t) - b_2, h(t) - b_3\rangle| < \epsilon \quad\Leftrightarrow\quad \sqrt{[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2} < \epsilon \quad\Leftrightarrow$$

$[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$. But each term on the left side of this inequality is positive so $[f(t) - b_1]^2 < \epsilon^2$, $[g(t) - b_2]^2 < \epsilon^2$ and $[h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$, or

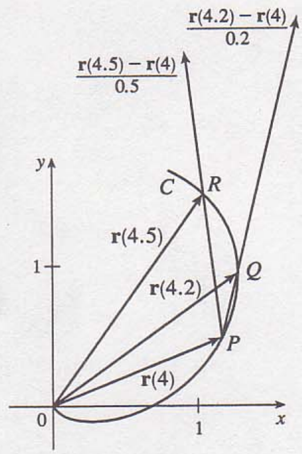
taking the square root of both sides in each of the above we have $|f(t) - b_1| < \epsilon$, $|g(t) - b_2| < \epsilon$ and $|h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. And by definition of limits of real-valued functions we have $\lim_{t \to a} f(t) = b_1$, $\lim_{t \to a} g(t) = b_2$ and $\lim_{t \to a} h(t) = b_3$. But by (1), $\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$, so $\lim_{t \to a} \mathbf{r}(t) = \langle b_1, b_2, b_3 \rangle = \mathbf{b}$.

14.2 Derivatives and Integrals of Vector Functions ET 13.2

1. (a)



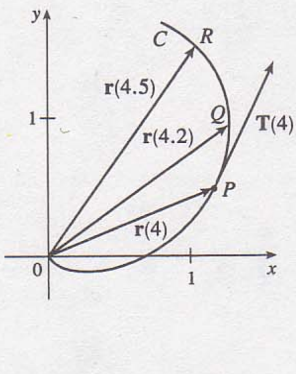
(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$. $\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.



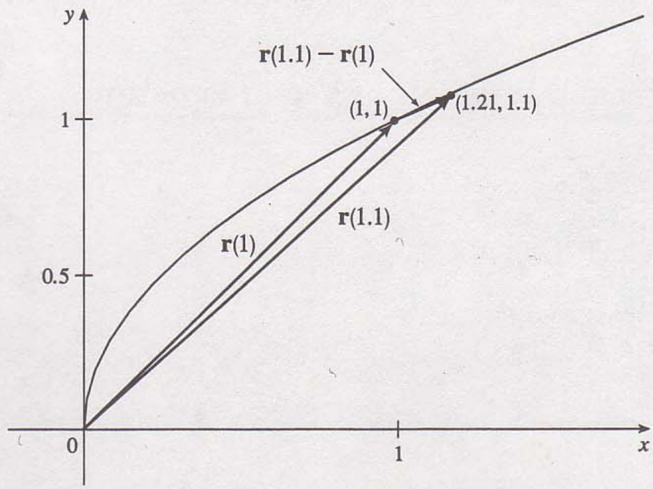
(c) By Definition 1, $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$.

$\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.

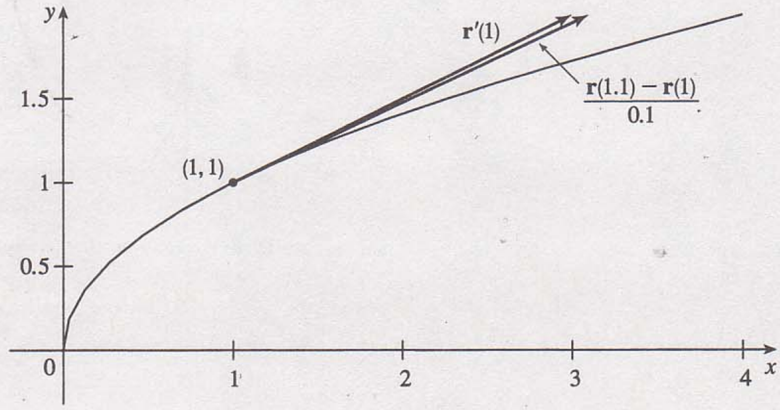


2. (a) The curve can be represented by the parametric equations $x = t^2$, $y = t$, $0 \leq t \leq 2$. Eliminating the parameter, we have $x = y^2$, $0 \leq y \leq 2$, a portion of which we graph here, along with the vectors $\mathbf{r}\,(1)$, $\mathbf{r}\,(1.1)$, and $\mathbf{r}\,(1.1) - \mathbf{r}\,(1)$.



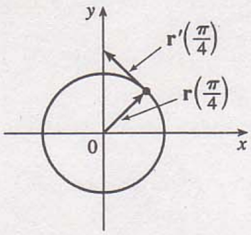
(b) Since $\mathbf{r}\,(t) = \langle t^2, t\rangle$, we differentiate components, giving $\mathbf{r}'\,(t) = \langle 2t, 1\rangle$, so $\mathbf{r}'\,(1) = \langle 2, 1\rangle$.

$$\frac{\mathbf{r}\,(1.1) - \mathbf{r}\,(1)}{0.1} = \frac{\langle 1.21, 1.1\rangle - \langle 1, 1\rangle}{0.1} = 10\,\langle 0.21, 0.1\rangle = \langle 2.1, 1\rangle.$$



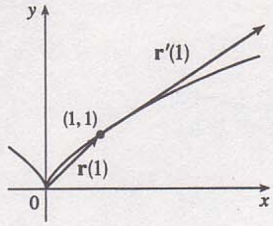
As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'\,(1)$ is defined to be $\lim\limits_{h\to 0} \dfrac{\mathbf{r}\,(1+h) - \mathbf{r}\,(1)}{h}$, and we recognize $\dfrac{\mathbf{r}\,(1.1) - \mathbf{r}\,(1)}{0.1}$ as the expression after the limit sign with $h = 0.1$. Since h is close to 0, we would expect $\dfrac{\mathbf{r}\,(1.1) - \mathbf{r}\,(1)}{0.1}$ to be a vector close to $\mathbf{r}'\,(1)$.

3. (a), (c)



(b) $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$

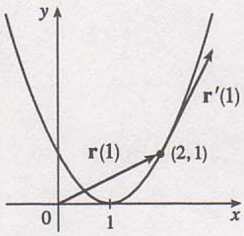
4. (a), (c)



(b) $\mathbf{r}'(t) = \langle 3t^2, 2t \rangle$

5. Since $(x-1)^2 = t^2 = y$, the curve is a parabola.

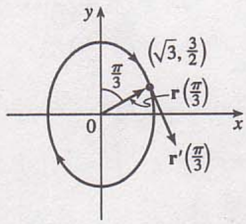
(a), (c)



(b) $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

6. $x = 2\sin t$, $y = 3\cos t$, so $(x/2)^2 + (y/3)^2 = \sin^2 t + \cos^2 t = 1$ and the curve is an ellipse.

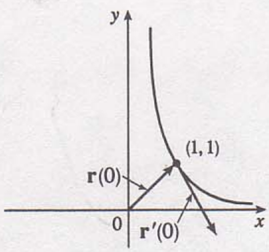
(a), (c)



(b) $\mathbf{r}'(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$

7. $x^{-2} = e^{-2t} = y$, so $y = 1/x^2$, $x > 0$.

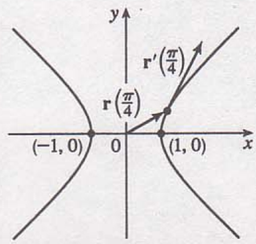
(a), (c)



(b) $\mathbf{r}'(t) = e^t\mathbf{i} - 2e^{-2t}\mathbf{j}$

8. $x^2 - y^2 = \sec^2 t - \tan^2 t = 1$, so the curve is a hyperbola.

(a), (c)



(b) $\mathbf{r}'(t) = \sec t \tan t\,\mathbf{i} + \sec^2 t\,\mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \frac{d}{dt}\left[t^2\right], \frac{d}{dt}\left[1-t\right], \frac{d}{dt}\left[\sqrt{t}\right] \right\rangle = \left\langle 2t, -1, \frac{1}{2\sqrt{t}} \right\rangle$

10. $\mathbf{r}(t) = \langle \cos 3t, t, \sin 3t \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle -3\sin 3t, 1, 3\cos 3t \rangle$

11. $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + e^{4t}\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = 0\,\mathbf{i} + 0\,\mathbf{j} + 4e^{4t}\mathbf{k} = 4e^{4t}\mathbf{k}$

12. $\mathbf{r}(t) = \sin^{-1} t\,\mathbf{i} + \sqrt{1-t^2}\,\mathbf{j} + \mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = \frac{1}{\sqrt{1-t^2}}\mathbf{i} - \frac{t}{\sqrt{1-t^2}}\mathbf{j}$

13. $\mathbf{r}'(t) = -\frac{2t}{4-t^2}\,\mathbf{i} + \frac{1}{2\sqrt{1+t}}\,\mathbf{j} - 12e^{3t}\,\mathbf{k}$

14. $\mathbf{r}'(t) = -e^{-t}(\cos t + \sin t)\,\mathbf{i} + e^{-t}(\cos t - \sin t)\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}$

15. $\mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2\,\mathbf{c}\,t = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

16. To find $\mathbf{r}'(t)$, we first expand $\mathbf{r}(t) = t\,\mathbf{a} \times (\mathbf{b} + t\,\mathbf{c}) = t\,(\mathbf{a} \times \mathbf{b}) + t^2\,(\mathbf{a} \times \mathbf{c})$, so $\mathbf{r}'(t) = \mathbf{a} \times \mathbf{b} + 2t\,(\mathbf{a} \times \mathbf{c})$.

17. $\mathbf{r}'(t) = \langle 30t^4, 12t^2, 2\rangle \quad\Rightarrow\quad \mathbf{r}'(1) = \langle 30, 12, 2\rangle$. So $|\mathbf{r}'(1)| = \sqrt{30^2 + 12^2 + 2^2} = \sqrt{1048} = 2\sqrt{262}$ and

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{2\sqrt{262}}\langle 30, 12, 2\rangle = \left\langle \frac{15}{\sqrt{262}}, \frac{6}{\sqrt{262}}, \frac{1}{\sqrt{262}} \right\rangle.$$

18. $\mathbf{r}'(t) = \left\langle \dfrac{1}{2\sqrt{t}}, 1 - 2t, \dfrac{1}{1+t^2} \right\rangle \quad\Rightarrow\quad \mathbf{r}'(1) = \left\langle \frac{1}{2}, -1, \frac{1}{2} \right\rangle$. Thus $|\mathbf{r}'(1)| = \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{2}}$

and $\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \dfrac{1}{\sqrt{3/2}}\left\langle \frac{1}{2}, -1, \frac{1}{2} \right\rangle = \left\langle \frac{1}{2}\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \frac{1}{2}\sqrt{\frac{2}{3}} \right\rangle = \left\langle \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}} \right\rangle$.

19. $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\,\mathbf{j} - 3\sin t\,\mathbf{k}$, $\mathbf{r}'\left(\frac{\pi}{6}\right) = \mathbf{i} + \sqrt{3}\,\mathbf{j} - \frac{3}{2}\,\mathbf{k}$. Thus

$$\mathbf{T}\left(\tfrac{\pi}{6}\right) = \frac{1}{\sqrt{1^2 + \left(\sqrt{3}\right)^2 + (-3/2)^2}}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j} - \tfrac{3}{2}\,\mathbf{k}\right) = \frac{1}{5/2}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j} - \tfrac{3}{2}\,\mathbf{k}\right) = \tfrac{2}{5}\,\mathbf{i} + \tfrac{2\sqrt{3}}{5}\,\mathbf{j} - \tfrac{3}{5}\,\mathbf{k}.$$

20. $\mathbf{r}'(t) = 2e^{2t}\,(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}) + e^{2t}\,(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}) = e^{2t}\,[(2\cos t - \sin t)\,\mathbf{i} + (2\sin t + \cos t)\,\mathbf{j} + 2\,\mathbf{k}]$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = e^{\pi}\,(-\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k})$

Thus $\mathbf{T}\left(\frac{\pi}{2}\right) = \dfrac{e^{\pi}}{e^{\pi}\sqrt{9}}\,(-\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}) = -\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$.

21. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. Then $\mathbf{r}'(1) = \langle 1, 2, 3\rangle$ and $|\mathbf{r}'(1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \dfrac{1}{\sqrt{14}}\langle 1, 2, 3\rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$. $\mathbf{r}''(t) = \langle 0, 2, 6t\rangle$, so

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}\mathbf{k}$$

$$= \left(12t^2 - 6t^2\right)\mathbf{i} - (6t - 0)\,\mathbf{j} + (2 - 0)\,\mathbf{k} = \left\langle 6t^2, -6t, 2\right\rangle.$$

22. $\mathbf{r}(t) = \left\langle e^{2t}, e^{-2t}, te^{2t}\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle 2e^{2t}, -2e^{-2t}, (2t+1)\,e^{2t}\right\rangle \quad\Rightarrow$

$\mathbf{r}'(0) = \left\langle 2e^0, -2e^0, (0+1)\,e^0\right\rangle = \langle 2, -2, 1\rangle$ and $|\mathbf{r}'(0)| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$. Then

$\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{3}\langle 2, -2, 1\rangle = \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle$. $\mathbf{r}''(t) = \left\langle 4e^{2t}, 4e^{-2t}, (4t+4)\,e^{2t}\right\rangle \quad\Rightarrow$

$\mathbf{r}''(0) = \left\langle 4e^0, 4e^0, (0+4)\,e^0\right\rangle = \langle 4, 4, 4\rangle$.

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \left\langle 2e^{2t}, -2e^{-2t}, (2t+1)\,e^{2t}\right\rangle \cdot \left\langle 4e^{2t}, 4e^{-2t}, (4t+4)\,e^{2t}\right\rangle$$

$$= \left(2e^{2t}\right)\left(4e^{2t}\right) + \left(-2e^{-2t}\right)\left(4e^{-2t}\right) + \left((2t+1)\,e^{2t}\right)\left((4t+4)\,e^{2t}\right)$$

$$= 8e^{4t} - 8e^{-4t} + \left(8t^2 + 12t + 4\right)e^{4t} = \left(8t^2 + 12t + 12\right)e^{4t} - 8e^{-4t}$$

23. The vector equation for the curve is $\mathbf{r}(t) = \left\langle t^5, t^4, t^3\right\rangle$, so $\mathbf{r}'(t) = \left\langle 5t^4, 4t^3, 3t^2\right\rangle$. The point $(1, 1, 1)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 5, 4, 3\rangle$. Thus, the tangent line goes through the point $(1, 1, 1)$ and is parallel to the vector $\langle 5, 4, 3\rangle$. Parametric equations are $x = 1 + 5t$, $y = 1 + 4t$, $z = 1 + 3t$.

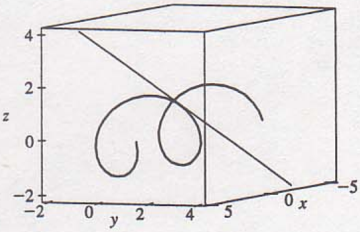
24. The vector equation for the curve is $\mathbf{r}(t) = \left\langle t^2 - 1, t^2 + 1, t + 1\right\rangle$, so $\mathbf{r}'(t) = \langle 2t, 2t, 1\rangle$. The point $(-1, 1, 1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 0, 0, 1\rangle$. Thus, the tangent line is parallel to the vector $\langle 0, 0, 1\rangle$ and parametric equations are $x = -1 + 0t = -1$, $y = 1 + 0t = 1$, $z = 1 + 1 \cdot t = 1 + t$.

25. $\mathbf{r}(t) = \langle t\cos 2\pi t, t\sin 2\pi t, 4t\rangle$, $\mathbf{r}'(t) = \langle \cos 2\pi t - 2\pi t \sin 2\pi t, \sin 2\pi t + 2\pi t\cos 2\pi t, 4\rangle$. At $\left(0, \frac{1}{4}, 1\right)$, $t = \frac{1}{4}$ and $\mathbf{r}'\left(\frac{1}{4}\right) = \left\langle 0 - \frac{\pi}{2}, 1 + 0, 4\right\rangle = \left\langle -\frac{\pi}{2}, 1, 4\right\rangle$. Thus, parametric equations of the tangent line are $x = -\frac{\pi}{2}t$, $y = \frac{1}{4} + t$, $z = 1 + 4t$.

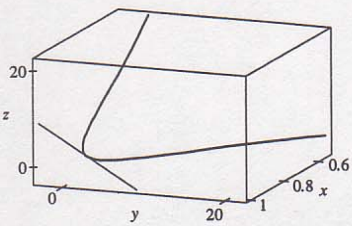
26. $\mathbf{r}(t) = \left\langle \sin \pi t, \sqrt{t}, \cos \pi t\right\rangle$, $\mathbf{r}'(t) = \left\langle \pi\cos \pi t, 1/\left(2\sqrt{t}\right), -\pi \sin \pi t\right\rangle$. At $(0, 1, -1)$, $t = 1$ and $\mathbf{r}'(1) = \left\langle -\pi, \frac{1}{2}, 0\right\rangle$. Thus, parametric equations of the tangent line are $x = -\pi t$, $y = 1 + \frac{1}{2}t$, $z = -1$.

27. $\mathbf{r}(t) = \left\langle t, \sqrt{2}\cos t, \sqrt{2}\sin t\right\rangle \Rightarrow$

$\mathbf{r}'(t) = \left\langle 1, -\sqrt{2}\sin t, \sqrt{2}\cos t\right\rangle$. At $\left(\frac{\pi}{4}, 1, 1\right)$, $t = \frac{\pi}{4}$ and $\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle 1, -1, 1\rangle$. Thus, parametric equations of the tangent line are $x = \frac{\pi}{4} + t$, $y = 1 - t$, $z = 1 + t$.



28. $\mathbf{r}(t) = \left\langle \cos t, 3e^{2t}, 3e^{-2t}\right\rangle$, $\mathbf{r}'(t) = \left\langle -\sin t, 6e^{2t}, -6e^{-2t}\right\rangle$. At $(1, 3, 3)$, $t = 0$ and $\mathbf{r}'(0) = \langle 0, 6, -6\rangle$. Thus, parametric equations of the tangent line are $x = 1$, $y = 3 + 6t$, $z = 3 - 6t$.



29. (a) $\mathbf{r}(t) = \left\langle t^3, t^4, t^5\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 3t^2, 4t^3, 5t^4\right\rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0\rangle = \mathbf{0}$, the curve is not smooth.

(b) $\mathbf{r}(t) = \left\langle t^3 + t, t^4, t^5\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 3t^2 + 1, 4t^3, 5t^4\right\rangle$. $\mathbf{r}'(t)$ is continuous since its component functions are continuous. Also, $\mathbf{r}'(t) \neq \mathbf{0}$, as the y- and z-components are 0 only for $t = 0$, but $\mathbf{r}'(0) = \langle 1, 0, 0\rangle \neq \mathbf{0}$. Thus, the curve is smooth.

(c) $\mathbf{r}(t) = \left\langle \cos^3 t, \sin^3 t\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle -3\cos^2 t \sin t, 3\sin^2 t\cos t\right\rangle$. Since $\mathbf{r}'(0) = \left\langle -3\cos^2 0 \sin 0, 3\sin^2 0\cos 0\right\rangle = \langle 0, 0\rangle = \mathbf{0}$, the curve is not smooth.

30. (a) The tangent line at $t = 0$ is the line through the point with position vector $\mathbf{r}(0) = \langle \sin 0, 2\sin 0, \cos 0\rangle = \langle 0, 0, 1\rangle$, and in the direction of the tangent vector,

$\mathbf{r}'(0) = \langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0\rangle = \langle \pi, 2\pi, 0\rangle$.

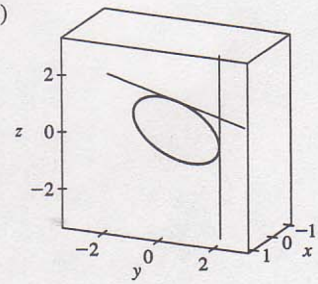
So an equation of the line is

$\langle x, y, z\rangle = \mathbf{r}(0) + u\mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1\rangle = \langle \pi u, 2\pi u, 1\rangle$

$\mathbf{r}\left(\frac{1}{2}\right) = \left\langle \sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right\rangle = \langle 1, 2, 0\rangle$, $\mathbf{r}'\left(\frac{1}{2}\right) = \left\langle \pi\cos\frac{\pi}{2}, 2\pi\cos\frac{\pi}{2}, -\pi\sin\frac{\pi}{2}\right\rangle = \langle 0, 0, -\pi\rangle$.

So the equation of the second line is $\langle x, y, z\rangle = \langle 1, 2, 0\rangle + v\langle 0, 0, -\pi\rangle = \langle 1, 2, -\pi v\rangle$. The lines intersect where $\langle \pi u, 2\pi u, 1\rangle = \langle 1, 2, -\pi v\rangle$, so the point of intersection is $(1, 2, 1)$.

(b)



31. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \left\langle 1, 2t, 3t^2\right\rangle$ and $t = 0$ at $(0, 0, 0)$, $\mathbf{r}_1'(0) = \langle 1, 0, 0\rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0, 0, 0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1\rangle$ and since $\mathbf{r}_2(0) = \langle 0, 0, 0\rangle$, $\mathbf{r}_2'(0) = \langle 1, 2, 1\rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0, 0, 0)$. If θ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}}\langle 1, 0, 0\rangle \cdot \langle 1, 2, 1\rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\frac{1}{\sqrt{6}} \approx 66^\circ$.

32. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1, 0, 4)$. To find the angle θ of intersection, we proceed as in Exercise 31. The tangent vectors to the respective curves at $(1, 0, 4)$ are $\mathbf{r}_1'(1) = \langle 1, -1, 2\rangle$ and $\mathbf{r}_2'(2) = \langle -1, 1, 4\rangle$. So $\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1 - 1 + 8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\frac{1}{\sqrt{3}} \approx 55^\circ$.
Note: In Exercise 31, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

33. $\int_0^1 (t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k})\,dt = \left(\int_0^1 t\,dt\right)\mathbf{i} + \left(\int_0^1 t^2\,dt\right)\mathbf{j} + \left(\int_0^1 t^3\,dt\right)\mathbf{k} = \left[\frac{t^2}{2}\right]_0^1\mathbf{i} + \left[\frac{t^3}{3}\right]_0^1\mathbf{j} + \left[\frac{t^4}{4}\right]_0^1\mathbf{k}$

$= \frac{1}{2}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} + \frac{1}{4}\,\mathbf{k}$

34. $\int_1^2 \left[(1 + t^2)\,\mathbf{i} - 4t^4\,\mathbf{j} - (t^2 - 1)\,\mathbf{k}\right]dt = \left[\left(t + \frac{1}{3}t^3\right)\mathbf{i} - \frac{4}{5}t^5\,\mathbf{j} - \left(\frac{1}{3}t^3 - t\right)\mathbf{k}\right]_1^2$

$= \left[\left(2 + \frac{8}{3}\right)\mathbf{i} - \frac{128}{5}\,\mathbf{j} - \left(\frac{8}{3} - 2\right)\mathbf{k}\right] - \left[\left(1 + \frac{1}{3}\right)\mathbf{i} - \frac{4}{5}\,\mathbf{j} - \left(\frac{1}{3} - 1\right)\mathbf{k}\right] = \frac{10}{3}\,\mathbf{i} - \frac{124}{5}\,\mathbf{j} - \frac{4}{3}\,\mathbf{k}$

35. $\int_0^{\pi/4} (\cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + t\sin t\,\mathbf{k})\,dt = \left[\frac{1}{2}\sin 2t\,\mathbf{i} - \frac{1}{2}\cos 2t\,\mathbf{j}\right]_0^{\pi/4} + \left[\left[-t\cos t\right]_0^{\pi/4} + \int_0^{\pi/4}\cos t\,dt\right]\mathbf{k}$

$= \frac{1}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \left[-\frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right]\mathbf{k} = \frac{1}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)\mathbf{k} = \frac{1}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \frac{4 - \pi}{4\sqrt{2}}\,\mathbf{k}$

36. $\int_1^4 \left(\sqrt{t}\,\mathbf{i} + te^{-t}\,\mathbf{j} + t^{-2}\,\mathbf{k}\right)dt = \left[\frac{2}{3}t^{3/2}\,\mathbf{i} - t^{-1}\,\mathbf{k}\right]_1^4 + \left(\left[-te^{-t}\right]_1^4 + \int_1^4 e^{-t}\,dt\right)\mathbf{j}$

$= \left(\frac{16}{3} - \frac{2}{3}\right)\mathbf{i} - \left(\frac{1}{4} - 1\right)\mathbf{k} + \left(-4e^{-4} + e^{-1} - e^{-4} + e^{-1}\right)\mathbf{j} = \frac{14}{3}\,\mathbf{i} + e^{-1}\left(2 - 5e^{-3}\right)\mathbf{j} + \frac{3}{4}\,\mathbf{k}$

37. $\int \left(e^t\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}\right)dt = \left(\int e^t\,dt\right)\mathbf{i} + \left(\int 2t\,dt\right)\mathbf{j} + \left(\int \ln t\,dt\right)\mathbf{k} = e^t\,\mathbf{i} + t^2\,\mathbf{j} + (t\ln t - t)\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a vector constant of integration.

38. $\int (\cos\pi t\,\mathbf{i} + \sin\pi t\,\mathbf{j} + t\,\mathbf{k})\,dt = \left(\int\cos\pi t\,dt\right)\mathbf{i} + \left(\int\sin\pi t\,dt\right)\mathbf{j} + \left(\int t\,dt\right)\mathbf{k} = \frac{1}{\pi}\sin\pi t\,\mathbf{i} - \frac{1}{\pi}\cos\pi t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k} + \mathbf{C}$

39. $\mathbf{r}'(t) = t^2\,\mathbf{i} + 4t^3\,\mathbf{j} - t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}(t) = \frac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector. But $\mathbf{j} = \mathbf{r}(0) = (0)\,\mathbf{i} + (0)\,\mathbf{j} - (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = \mathbf{j}$ and
$\mathbf{r}(t) = \frac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k} + \mathbf{j} = \frac{1}{3}t^3\,\mathbf{i} + (t^4 + 1)\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k}$.

40. $\mathbf{r}'(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}(t) = (-\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C}$. But $\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \mathbf{r}(0) = -\mathbf{i} + (0)\,\mathbf{j} + (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = 2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$ and
$\mathbf{r}(t) = (2 - \cos t)\,\mathbf{i} + (1 - \sin t)\,\mathbf{j} + (2 + t^2)\,\mathbf{k}$.

For Exercises 41–44, let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

41. $\frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] = \frac{d}{dt}\langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t)\rangle$

$= \left\langle \frac{d}{dt}\left[u_1(t) + v_1(t)\right], \frac{d}{dt}\left[u_2(t) + v_2(t)\right], \frac{d}{dt}\left[u_3(t) + v_3(t)\right]\right\rangle$

$= \langle u_1'(t) + v_1'(t), u_2'(t) + v_2'(t), u_3'(t) + v_3'(t)\rangle$

$= \langle u_1'(t), u_2'(t), u_3'(t)\rangle + \langle v_1'(t), v_2'(t), v_3'(t)\rangle = \mathbf{u}'(t) + \mathbf{v}'(t)$.

42.
$$\begin{aligned}\frac{d}{dt}[f(t)\,\mathbf{u}(t)] &= \frac{d}{dt}\langle f(t)\,u_1(t), f(t)\,u_2(t), f(t)\,u_3(t)\rangle \\ &= \left\langle \frac{d}{dt}[f(t)\,u_1(t)], \frac{d}{dt}[f(t)\,u_2(t)], \frac{d}{dt}[f(t)\,u_3(t)]\right\rangle \\ &= \langle f'(t)\,u_1(t) + f(t)\,u_1'(t), f'(t)\,u_2(t) + f(t)\,u_2'(t), f'(t)\,u_3(t) + f(t)\,u_3'(t)\rangle \\ &= f'(t)\,\langle u_1(t), u_2(t), u_3(t)\rangle + f(t)\,\langle u_1'(t), u_2'(t), u_3'(t)\rangle \\ &= f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)\end{aligned}$$

43.
$$\begin{aligned}&\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] \\ &= \frac{d}{dt}\langle u_2(t)\,v_3(t) - u_3(t)\,v_2(t), u_3(t)\,v_1(t) - u_1(t)\,v_3(t), u_1(t)\,v_2(t) - u_2(t)\,v_1(t)\rangle \\ &= \langle u_2'v_3(t) + u_2(t)\,v_3'(t) - u_3'(t)\,v_2(t) - u_3(t)\,v_2'(t), \\ &\qquad u_3'(t)\,v_1(t) + u_3(t)\,v_1'(t) - u_1'(t)\,v_3(t) - u_1(t)\,v_3'(t), \\ &\qquad\quad u_1'(t)\,v_2(t) + u_1(t)\,v_2'(t) - u_2'(t)\,v_1(t) - u_2(t)\,v_1'(t)\rangle \\ &= \langle u_2'(t)\,v_3(t) - u_3'(t)\,v_2(t), u_3'(t)\,v_1(t) - u_1'(t)\,v_3(t), u_1'(t)\,v_2(t) - u_2'(t)\,v_1(t)\rangle \\ &\qquad + \langle u_2(t)\,v_3'(t) - u_3(t)\,v_2'(t), u_3(t)\,v_1'(t) - u_1(t)\,v_3'(t), u_1(t)\,v_2'(t) - u_2(t)\,v_1'(t)\rangle \\ &= \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)\end{aligned}$$

Alternate Solution: Let $\mathbf{r}(t) = \mathbf{u}(t)\times\mathbf{v}(t)$. Then

$$\begin{aligned}\mathbf{r}(t+h) - \mathbf{r}(t) &= [\mathbf{u}(t+h)\times\mathbf{v}(t+h)] - [\mathbf{u}(t)\times\mathbf{v}(t)] \\ &= [\mathbf{u}(t+h)\times\mathbf{v}(t+h)] - [\mathbf{u}(t)\times\mathbf{v}(t)] + [\mathbf{u}(t+h)\times\mathbf{v}(t)] - [\mathbf{u}(t+h)\times\mathbf{v}(t)] \\ &= \mathbf{u}(t+h)\times[\mathbf{v}(t+h) - \mathbf{v}(t)] + [\mathbf{u}(t+h) - \mathbf{u}(t)]\times\mathbf{v}(t)\end{aligned}$$

(Be careful of the order of the cross product.)
Dividing through by h and taking the limit as $h \to 0$ we have

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times[\mathbf{v}(t+h) - \mathbf{v}(t)]}{h} + \lim_{h\to 0}\frac{[\mathbf{u}(t+h) - \mathbf{u}(t)]\times\mathbf{v}(t)}{h} \\ &= \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)\end{aligned}$$

by Exercise 14.1.35(a) [ET 13.1.35(a)] and Definition 1.

44.
$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(f(t))] &= \frac{d}{dt}\langle u_1(f(t)), u_2(f(t)), u_3(f(t))\rangle \\ &= \left\langle \frac{d}{dt}[u_1(f(t))], \frac{d}{dt}[u_2(f(t))], \frac{d}{dt}[u_3(f(t))]\right\rangle \\ &= \langle f'(t)\,u_1'(f(t)), f'(t)\,u_2'(f(t)), f'(t)\,u_3'(f(t))\rangle \\ &= f'(t)\,\mathbf{u}'(t)\end{aligned}$$

45. $D_t\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$ by Formula 4 of Theorem 3

$$=\left(-4t\,\mathbf{j}+9t^2\,\mathbf{k}\right)\cdot\left(t\,\mathbf{i}+\cos t\,\mathbf{j}+\sin t\,\mathbf{k}\right)+\left(\mathbf{i}-2t^2\,\mathbf{j}+3t^3\,\mathbf{k}\right)\cdot\left(\mathbf{i}-\sin t\,\mathbf{j}+\cos t\,\mathbf{k}\right)$$

$$=-4t\cos t+9t^2\sin t+1+2t^2\sin t+3t^3\cos t$$

$$=1-4t\cos t+11t^2\sin t+3t^3\cos t$$

46. $D_t\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$ by Formula 5 of Theorem 3

$$=\left(-4t\,\mathbf{j}+9t^2\,\mathbf{k}\right)\times\left(t\,\mathbf{i}+\cos t\,\mathbf{j}+\sin t\,\mathbf{k}\right)+\left(\mathbf{i}-2t^2\,\mathbf{j}+3t^3\,\mathbf{k}\right)\times\left(\mathbf{i}-\sin t\,\mathbf{j}+\cos t\,\mathbf{k}\right)$$

$$=\left(-4t\sin t-9t^2\cos t\right)\mathbf{i}+\left(9t^3-0\right)\mathbf{j}+\left(0+4t^2\right)\mathbf{k}$$

$$+\left(-2t^2\cos t+3t^3\sin t\right)\mathbf{i}+\left(3t^3-\cos t\right)\mathbf{j}+\left(-\sin t+2t^2\right)\mathbf{k}$$

$$=\left[(\sin t)\left(3t^3-4t\right)-11t^2\cos t\right]\mathbf{i}+\left(12t^3-\cos t\right)\mathbf{j}+\left(6t^2-\sin t\right)\mathbf{k}$$

47. $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right]=\mathbf{r}'(t)\times\mathbf{r}'(t)+\mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t)=\mathbf{0}$ (see Example 13.4.2 [ET 12.4.2]). Thus, $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right]=\mathbf{r}(t)\times\mathbf{r}''(t)$.

48. $\dfrac{d}{dt}\left(\mathbf{u}(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]\right)=\mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]+\mathbf{u}(t)\cdot\dfrac{d}{dt}\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]$

$$=\mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]+\mathbf{u}(t)\cdot\left[\mathbf{v}'(t)\times\mathbf{w}(t)+\mathbf{v}(t)\times\mathbf{w}'(t)\right]$$

$$=\mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]+\mathbf{u}(t)\cdot\left[\mathbf{v}'(t)\times\mathbf{w}(t)\right]+\mathbf{u}(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}'(t)\right]$$

$$=\mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]-\mathbf{v}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{w}(t)\right]+\mathbf{w}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]$$

49. $\dfrac{d}{dt}\left|\mathbf{r}(t)\right|=\dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{1/2}=\frac{1}{2}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{-1/2}\left[2\mathbf{r}(t)\cdot\mathbf{r}'(t)\right]=\dfrac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\left|\mathbf{r}(t)\right|}$

50. Since $\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$, we have $0=2\mathbf{r}(t)\cdot\mathbf{r}'(t)=\dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]=\dfrac{d}{dt}\left|\mathbf{r}(t)\right|^2$. Thus $\left|\mathbf{r}(t)\right|^2$, and so $\left|\mathbf{r}(t)\right|$, is a constant, and hence the curve lies on a sphere with center the origin.

51. Since $\mathbf{u}(t)=\mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]$,

$$\mathbf{u}'(t)=\mathbf{r}'(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]+\mathbf{r}(t)\cdot\frac{d}{dt}\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]$$

$$=0+\mathbf{r}(t)\cdot\left[\mathbf{r}''(t)\times\mathbf{r}''(t)+\mathbf{r}'(t)\times\mathbf{r}'''(t)\right]\qquad\left[\text{since }\mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)\right]$$

$$=\mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}'''(t)\right]\qquad\left[\text{since }\mathbf{r}''(t)\times\mathbf{r}''(t)=\mathbf{0}\right]$$

14.3 Arc Length and Curvature ET 13.3

1. $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2} = \sqrt{29}$. Then using Formula 3, we have $L = \int_{-10}^{10} |\mathbf{r}'(t)|\,dt = \int_{-10}^{10} \sqrt{29}\,dt = \sqrt{29}\,t\big]_{-10}^{10} = 20\sqrt{29}$.

2. $\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5}\,|t| = \sqrt{5}\,t$ for $0 \le t \le \pi$. Then using Formula 3, we have $L = \int_0^{\pi} |\mathbf{r}'(t)|\,dt = \int_0^{\pi} \sqrt{5}\,t\,dt = \sqrt{5}\,\dfrac{t^2}{2}\Big]_0^{\pi} = \dfrac{\sqrt{5}}{2}\pi^2$.

3. $\mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ (since $e^t + e^{-t} > 0$).

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 \left(e^t + e^{-t}\right)dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

4. $\mathbf{r}'(t) = \langle 2t, 2, 1/t\rangle$, $|\mathbf{r}'(t)| = \sqrt{4t^2 + 4 + (1/t)^2} = \dfrac{1 + 2t^2}{|t|} = \dfrac{1 + 2t^2}{t}$ for $1 \le t \le e$.

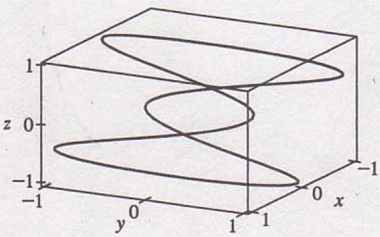
$$L = \int_1^e \frac{1 + 2t^2}{t}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right)dt = \left[\ln t + t^2\right]_1^e = e^2$$

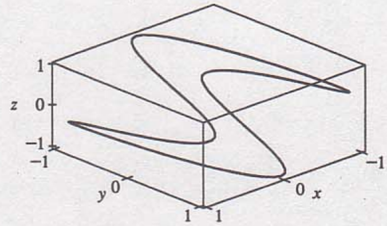
5. The point $(2, 4, 8)$ corresponds to $t = 2$, so by Equation 2, $L = \int_0^2 \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}\,dt$.

If $f(t) = \sqrt{1 + 4t^2 + 9t^4}$, then Simpson's Rule gives

$L \approx \dfrac{2 - 0}{10 \cdot 3}\left[f(0) + 4f(0.2) + 2f(0.4) + \cdots + 4f(1.8) + f(2)\right] \approx 9.5706$.

6. Here are two views of the curve with parametric equations $x = \cos t$, $y = \sin 3t$, $z = \sin t$:





The complete curve is given by the parameter interval $[0, 2\pi]$, so

$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (3\cos 3t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} \sqrt{1 + 9\cos^2 3t}\,dt \approx 13.9744$.

7. $\mathbf{r}'(t) = e^t(\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j}$,

$ds/dt = |\mathbf{r}'(t)| = e^t\sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2} = e^t\sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}e^t$.

$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}e^u\,du = \sqrt{2}\left(e^t - 1\right) \quad \Rightarrow \quad \frac{1}{\sqrt{2}}s + 1 = e^t \quad \Rightarrow \quad t(s) = \ln\left(\frac{1}{\sqrt{2}}s + 1\right)$.

Therefore, $\mathbf{r}(t(s)) = \left(\frac{1}{\sqrt{2}}s + 1\right)\left[\sin\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right)\mathbf{i} + \cos\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right)\mathbf{j}\right]$.

8. $\mathbf{r}'(t) = 2\,\mathbf{i} + \mathbf{j} - 5\,\mathbf{k}$, $ds/dt = |\mathbf{r}'(t)| = \sqrt{4 + 1 + 25} = \sqrt{30}$ and $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{30}\,du = \sqrt{30}\,t$ $\Rightarrow \quad t(s) = \frac{1}{\sqrt{30}}s$. Therefore, $\mathbf{r}(t(s)) = \left(1 + \frac{2}{\sqrt{30}}s\right)\mathbf{i} + \left(3 + \frac{1}{\sqrt{30}}s\right)\mathbf{j} - \frac{5}{\sqrt{30}}s\,\mathbf{k}$.

9. $|\mathbf{r}'(t)| = \sqrt{(3\cos t)^2 + 16 + (-3\sin t)^2} = \sqrt{9+16} = 5$ and $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t \Rightarrow$ $t(s) = \frac{1}{5}s$. Therefore, $\mathbf{r}(t(s)) = 3\sin\left(\frac{1}{5}s\right)\mathbf{i} + \frac{4}{5}s\,\mathbf{j} + 3\cos\left(\frac{1}{5}s\right)\mathbf{k}$.

10. $\mathbf{r}'(t) = \dfrac{-4t}{(t^2+1)^2}\,\mathbf{i} + \dfrac{-2t^2+2}{(t^2+1)^2}\,\mathbf{j}$,

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}$$
$$= \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1}$$

Since the initial point $(1,0)$ corresponds to $t=0$, the arc length function $s(t) = \displaystyle\int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \frac{2}{u^2+1}\,du = 2\arctan t$. Then $\arctan t = \frac{1}{2}s \Rightarrow t = \tan\frac{1}{2}s$. Substituting, we have

$$\mathbf{r}(t(s)) = \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1} - 1\right]\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\,\mathbf{j} = \frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{j}$$
$$= \frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + 2\tan\left(\tfrac{1}{2}s\right)\cos^2\left(\tfrac{1}{2}s\right)\mathbf{j}$$
$$= \left[\cos^2\left(\tfrac{1}{2}s\right) - \sin^2\left(\tfrac{1}{2}s\right)\right]\mathbf{i} + 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right)\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1,0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ (k an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

11. (a) $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}$. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t\rangle$ or $\left\langle \frac{2}{\sqrt{29}}\cos t, \frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\sin t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle \Rightarrow |\mathbf{T}'(t)| = \frac{1}{\sqrt{29}}\sqrt{4\sin^2 t + 0 + 4\cos^2 t} = \frac{2}{\sqrt{29}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{29}}{2/\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/\sqrt{29}}{\sqrt{29}} = \dfrac{2}{29}$.

12. (a) $\mathbf{r}'(t) = \langle 2t, t\sin t, t\cos t\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{5t^2} = \sqrt{5}\,t$ (since $t > 0$). Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \Rightarrow |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$.

13. (a) $\mathbf{r}'(t) = \langle t^2, 2t, 2\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{t^2+2}\langle t^2, 2t, 2\rangle.$$

$$\mathbf{T}'(t) = \frac{-2t}{(t^2+2)^2}\langle t^2, 2t, 2\rangle + \frac{1}{t^2+2}\langle 2t, 2, 0\rangle \qquad \text{(by Theorem 14.2.3 [ET 13.2.3] \#3)}$$

$$= \frac{1}{(t^2+2)^2}\langle -2t^3, -4t^2, -4t\rangle + \frac{1}{(t^2+2)^2}\langle 2t^3 + 4t, 2t^2 + 4, 0\rangle = \frac{1}{(t^2+2)^2}\langle 4t, 4 - 2t^2, -4t\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(t^2+2)^2}\sqrt{16t^2 + (16 - 16t^2 + 4t^4) + 16t^2} = \frac{1}{(t^2+2)^2}\sqrt{4t^4 + 16t^2 + 16}$$

$$= \frac{1}{(t^2+2)^2}\sqrt{4(t^2+2)^2} = \frac{2(t^2+2)}{(t^2+2)^2} = \frac{2}{t^2+2}$$

Thus $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/(t^2+2)^2}{2/(t^2+2)}\langle 4t, 4 - 2t^2, -4t\rangle = \dfrac{1}{t^2+2}\langle 2t, 2 - t^2, -2t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/(t^2+2)}{t^2+2} = \dfrac{2}{(t^2+2)^2}$

14. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4t^2 + 4 + (1/t)^2}}\langle 2t, 2, 1/t\rangle = \dfrac{|t|}{2t^2+1}\langle 2t, 2, 1/t\rangle$. But since the

$\mathbf{k}$-component is $\ln t$, t is positive, $|t| = t$ and $\mathbf{T}(t) = \dfrac{1}{2t^2+1}\langle 2t^2, 2t, 1\rangle$. Then

$\mathbf{T}'(t) = \dfrac{1}{2t^2+1}\langle 4t, 2, 0\rangle - (2t^2+1)^{-2}(4t)\langle 2t^2, 2t, 1\rangle = \dfrac{1}{(2t^2+1)^2}\langle 4t, 2 - 4t^2, -4t\rangle$, so

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle 4t, 2 - 4t^2, -4t\rangle}{\sqrt{(4t)^2 + (2 - 4t^2)^2 + (-4t)^2}} = \frac{1}{2t^2+1}\langle 2t, 1 - 2t^2, -2t\rangle.$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2}{2t^2+1}\left(\dfrac{t}{2t^2+1}\right) = \dfrac{2t}{(2t^2+2)^2}$

15. $\mathbf{r}'(t) = 2t\,\mathbf{i} + \mathbf{k}$, $\mathbf{r}''(t) = 2\mathbf{i}$, $|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 0^2 + 1^2} = \sqrt{4t^2+1}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\mathbf{j}$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2}{\left(\sqrt{4t^2+1}\right)^3} = \dfrac{2}{(4t^2+1)^{3/2}}$.

16. $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{r}''(t) = 2\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + 1^2 + (2t)^2} = \sqrt{4t^2+2}$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\mathbf{i} - 2\mathbf{j}$, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{2^2 + 2^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$. Then

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{2}}{\left(\sqrt{4t^2+2}\right)^3} = \frac{2\sqrt{2}}{\left(\sqrt{2}\sqrt{2t^2+1}\right)^3} = \frac{1}{(2t^2+1)^{3/2}}.$$

17. $\mathbf{r}'(t) = \langle \cos t, -\sin t, \cos t\rangle$, $\mathbf{r}''(t) = \langle -\sin t, -\cos t, -\sin t\rangle$, $|\mathbf{r}'(t)|^3 = \left(\sqrt{\cos^2 t + 1}\right)^3$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 1, 0, -1\rangle| = \sqrt{2}$, $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{2}}{(1 + \cos^2 t)^{3/2}}$

18. $\mathbf{r}'(t) = \langle e^t \cos t - e^t \sin t, e^t \cos t + e^t \sin t, 1 \rangle$. The point $(1, 0, 0)$ corresponds to $t = 0$, and

$\mathbf{r}'(0) = \langle 1, 1, 1 \rangle \quad \Rightarrow \quad |\mathbf{r}'(0)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

$\mathbf{r}''(t) = \langle e^t \cos t - e^t \sin t - e^t \cos t - e^t \sin t, e^t \cos t - e^t \sin t + e^t \cos t + e^t \sin t, 0 \rangle$

$= \langle -2e^t \sin t, 2e^t \cos t, 0 \rangle \quad \Rightarrow \quad \mathbf{r}''(0) = \langle 0, 2, 0 \rangle$. $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -2, 0, 2 \rangle$.

$|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$. Then $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{(\sqrt{3})^3} = \dfrac{2\sqrt{2}}{3\sqrt{3}}$

or $\dfrac{2\sqrt{6}}{9}$.

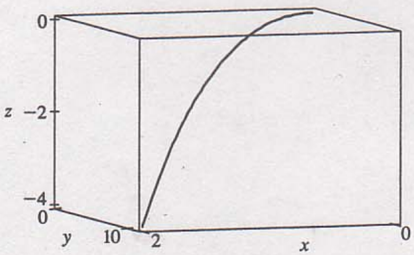
19. $\mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$. The point $(0, 1, 1)$ corresponds to $t = 0$, and

$\mathbf{r}'(0) = \langle \sqrt{2}, 1, -1 \rangle \quad \Rightarrow \quad |\mathbf{r}'(0)| = \sqrt{(\sqrt{2})^2 + 1^2 + (-1)^2} = 2$.

$\mathbf{r}''(t) = \langle 0, e^t, e^{-t} \rangle \quad \Rightarrow \quad \mathbf{r}''(0) = \langle 0, 1, 1 \rangle$. $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 2, -\sqrt{2}, \sqrt{2} \rangle$,

$|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{2^2 + (-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$. Then $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{2^3} = \dfrac{\sqrt{2}}{4}$.

20.



$\mathbf{r}(t) = \left\langle t, 4t^{3/2}, -t^2 \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 1, 6t^{1/2}, -2t \right\rangle$,

$\mathbf{r}''(t) = \left\langle 0, 3t^{-1/2}, -2 \right\rangle$, $|\mathbf{r}'(t)|^3 = (1 + 36t + 4t^2)^{3/2}$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -12t^{1/2} + 6t^{1/2}, 2, 3t^{-1/2} \right\rangle \quad \Rightarrow$

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t + 4 + 9t^{-1}} = \left[\dfrac{36t^2 + 4t + 9}{t}\right]^{1/2}$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \left(\frac{36t^2 + 4t + 9}{t}\right)^{1/2} \frac{1}{(1 + 36t + 4t^2)^{3/2}} = \frac{\sqrt{36t^2 + 4t + 9}}{t^{1/2}(1 + 36t + 4t^2)^{3/2}}.$$

The point $(1, 4, -1)$ corresponds to $t = 1$, so the curvature at this point is $\kappa(1) = \dfrac{\sqrt{36 + 4 + 9}}{(1 + 36 + 4)^{3/2}} = \dfrac{7}{41\sqrt{41}}$.

21. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $\kappa(x) = \dfrac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \dfrac{6|x|}{(1 + 9x^4)^{3/2}}$

22. $y' = \dfrac{1}{2\sqrt{x}}$, $y'' = -\dfrac{1}{4(x)^{3/2}}$, $\kappa(x) = \dfrac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \dfrac{1}{4|x^{3/2}|} \dfrac{1}{[1 + 1/(4x)]^{3/2}} = \dfrac{2}{(4x + 1)^{3/2}}$

23. $y' = \cos x$, $y'' = -\sin x$, $\kappa(x) = \dfrac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \dfrac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$

24. $y' = \dfrac{1}{x}$, $y'' = -\dfrac{1}{x^2}$,

$$\kappa(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right| \frac{1}{(1 + 1/x^2)^{3/2}} = \frac{1}{x^2} \frac{(x^2)^{3/2}}{(x^2 + 1)^{3/2}} = \frac{|x|}{(x^2 + 1)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}$$

(since $x > 0$). To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2 + 1)^{3/2} - x\left(\frac{3}{2}\right)(x^2 + 1)^{1/2}(2x)}{\left[(x^2 + 1)^{3/2}\right]^2} = \frac{(x^2 + 1)^{1/2}\left[(x^2 + 1) - 3x^2\right]}{(x^2 + 1)^3} = \frac{1 - 2x^2}{(x^2 + 1)^{5/2}};$$

$\kappa'(x) = 0 \Rightarrow 1 - 2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$. Since $\lim\limits_{x\to\infty} \dfrac{x}{(x^2+1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

25. Since $y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{\left[1+(y'(x))^2\right]^{3/2}} = \dfrac{e^x}{(1+e^{2x})^{3/2}} = e^x\left(1+e^{2x}\right)^{-3/2}$. To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$\kappa'(x) = e^x\left(1+e^{2x}\right)^{-3/2} + e^x\left(-\frac{3}{2}\right)\left(1+e^{2x}\right)^{-5/2}\left(2e^{2x}\right) = e^x\dfrac{1+e^{2x}-3e^{2x}}{(1+e^{2x})^{5/2}} = e^x\dfrac{1-2e^{2x}}{(1+e^{2x})^{5/2}}$.

$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and $1 - 2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point

$\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$. Since $\lim\limits_{x\to\infty} e^x\left(1+e^{2x}\right)^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

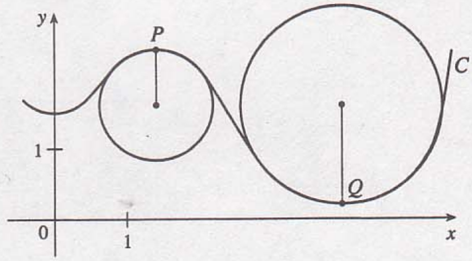
26. We can take the parabola as having its vertex at the origin and opening upward, so the equation is

$f(x) = ax^2, a > 0$. Then by Equation 11, $\kappa(x) = \dfrac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}} = \dfrac{|2a|}{\left[1+(2ax)^2\right]^{3/2}} = \dfrac{2a}{(1+4a^2x^2)^{3/2}}$,

thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

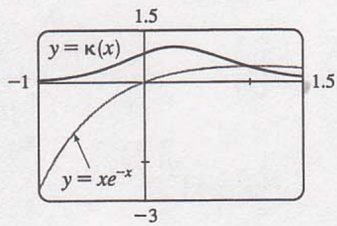
27. (a) C appears to be changing direction more quickly at P than Q, so we would expect the curvature to be greater at P.

(b) First we sketch approximate osculating circles at P and Q. Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \Rightarrow \kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at Q to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.

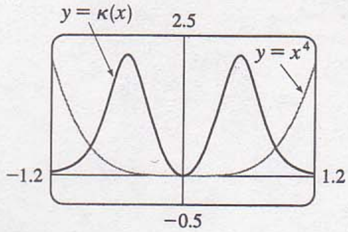


28. $y = xe^{-x} \Rightarrow y' = e^{-x}(1-x), y'' = e^{-x}(x-2)$, and

$\kappa(x) = \dfrac{|y''|}{\left[1+(y')^2\right]^{3/2}} = \dfrac{e^{-x}|x-2|}{\left[1+e^{-2x}(1-x)^2\right]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of xe^{-x} is bending most sharply slightly to the right of the origin. As $x \to \infty$, the graph of xe^{-x} is asymptotic to the x-axis, and so the curvature approaches zero.



29. $y = x^4 \Rightarrow y' = 4x^3$, $y'' = 12x^2$, and $\kappa(x) = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \dfrac{12x^2}{(1 + 16x^6)^{3/2}}$. The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^4$ is very flat around the origin, and so here the curvature is zero.



30. Here $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, $\mathbf{r}'(t) = \langle f'(t), g'(t)\rangle$, $\mathbf{r}''(t) = \langle f''(t), g''(t)\rangle$,

$|\mathbf{r}'(t)|^3 = \left[\sqrt{(f'(t))^2 + (g'(t))^2}\right]^3 = \left[(f'(t))^2 + (g'(t))^2\right]^{3/2} = (\dot{x}^2 + \dot{y}^2)^{3/2}$, and

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, f'(t)\,g''(t) - f''(t)\,g'(t)\rangle| = \left[(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2\right]^{1/2} = |\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.

Thus $\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

31. $$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(3t^2)(2) - (6t)(2t)|}{(9t^4 + 4t^2)^{3/2}} = \frac{6t^2}{(t^2)^{3/2}(9t^2 + 4)^{3/2}} = \frac{6t^2}{|t|^3(9t^2 + 4)^{3/2}}$$
$$= \frac{6}{|t|(9t^2 + 4)^{3/2}}$$

32. $$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(\sin t + t\cos t)(-2\sin t - t\cos t) - (2\cos t - t\sin t)(\cos t - t\sin t)|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$
$$= \frac{\left|\left(-2\sin^2 t - 3t\sin t\cos t - t^2\cos^2 t\right) - \left(2\cos^2 t - 3t\cos t\sin t + t^2\sin^2 t\right)\right|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$
$$= \frac{\left|-\left(\sin^2 t + \cos^2 t\right)\left(2 + t^2\right)\right|}{\left(\sin^2 t + 2t\cos t\sin t + \cos^2 t + t^2\cos^2 t - 2t\sin t\cos t + t^2\sin t\right)^{3/2}} = \frac{2 + t^2}{(1 + t^2)^{3/2}}$$

33. $\left(1, \frac{2}{3}, 1\right)$ corresponds to $t = 1$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\langle 2t, 2t^2, 1\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

$$\mathbf{T}'(t) = -4t\left(2t^2 + 1\right)^{-2}\left\langle 2t, 2t^2, 1\right\rangle + \left(2t^2 + 1\right)^{-1}\langle 2, 4t, 0\rangle \text{ (By Theorem 14.2.3 [ET 13.2.3] \#3)}$$
$$= \left(2t^2 + 1\right)^{-2}\left\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\right\rangle = 2\left(2t^2 + 1\right)^{-2}\left\langle 1 - 2t^2, 2t, -2t\right\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2\left(2t^2 + 1\right)^{-2}\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{2(2t^2 + 1)^{-2}\sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}}$$
$$= \frac{\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{1 + 2t^2}$$

$\mathbf{N}(1) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left\langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\right\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle$.

34. $(1, 0, 1)$ corresponds to $t = 0$. $\mathbf{r}(t) = e^t \langle 1, \sin t, \cos t \rangle$, so

$\mathbf{r}'(t) = e^t \langle 1, \sin t, \cos t \rangle + e^t \langle 0, \cos t, -\sin t \rangle = e^t \langle 1, \sin t + \cos t, \cos t - \sin t \rangle$ and

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{e^t \langle 1, \sin t + \cos t, \cos t - \sin t \rangle}{e^t \sqrt{1 + \sin^2 t + 2 \sin t \cos t + \cos^2 t + \cos^2 t - 2 \sin t \cos t + \sin^2 t}}$$

$$= \frac{\langle 1, \sin t + \cos t, \cos t - \sin t \rangle}{\sqrt{3}},$$

$\mathbf{T}(0) = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$. $\mathbf{T}'(t) = \frac{1}{\sqrt{3}} \langle 0, \cos t - \sin t, -\sin t - \cos t \rangle$, so

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{1}{\sqrt{3}} \langle 0, \cos t - \sin t, -\sin t - \cos t \rangle}{\frac{1}{\sqrt{3}} \sqrt{0^2 + \cos^2 t - 2 \cos t \sin t + \sin^2 t + \sin^2 t + 2 \sin t \cos t + \cos^2 t}}$$

$$= \frac{1}{\sqrt{2}} \langle 0, \cos t - \sin t, -\sin t - \cos t \rangle.$$

$\mathbf{N}(0) = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ and $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$.

35. $(0, \pi, -2)$ corresponds to $t = \pi$. $\mathbf{r}(t) = \langle 2 \sin 3t, t, 2 \cos 3t \rangle \quad \Rightarrow$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 6 \cos 3t, 1, -6 \sin 3t \rangle}{\sqrt{36 \cos^2 3t + 1 + 36 \sin^2 3t}} = \frac{1}{\sqrt{37}} \langle 6 \cos 3t, 1, -6 \sin 3t \rangle.\ \mathbf{T}(\pi) = \frac{1}{\sqrt{37}} \langle -6, 1, 0 \rangle$$

is a normal vector for the normal plane, and so $\langle -6, 1, 0 \rangle$ is also normal. Thus an equation for the plane is $-6(x - 0) + 1(y - \pi) + 0(z + 2) = 0$ or $y - 6x = \pi$. $\mathbf{T}'(t) = \frac{1}{\sqrt{37}} \langle -18 \sin 3t, 0, -18 \cos 3t \rangle \quad \Rightarrow$

$$|\mathbf{T}'(t)| = \frac{\sqrt{18^2 \sin^2 3t + 18^2 \cos^2 3t}}{\sqrt{37}} = \frac{18}{\sqrt{37}} \quad \Rightarrow \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin 3t, 0, -\cos 3t \rangle.$$ So

$\mathbf{N}(\pi) = \langle 0, 0, 1 \rangle$ and $\mathbf{B}(\pi) = \frac{1}{\sqrt{37}} \langle -6, 1, 0 \rangle \times \langle 0, 0, 1 \rangle = \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$. Since $\mathbf{B}(\pi)$ is a normal to the osculating plane, so is $\langle 1, 6, 0 \rangle$ and an equation for the plane is $1(x - 0) + 6(y - \pi) + 0(z + 2) = 0$ or $x + 6y = 6\pi$.

36. $t = 1$ at $(1, 1, 1)$. $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle$ is normal to the normal plane, so an equation for this plane is $1(x - 1) + 2(y - 1) + 3(z - 1) = 0$, or $x + 2y + 3z = 6$.

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1 + 4t^2 + 9t^4}} \langle 1, 2t, 3t^2 \rangle$. Using the product rule on each term of $\mathbf{T}(t)$ gives

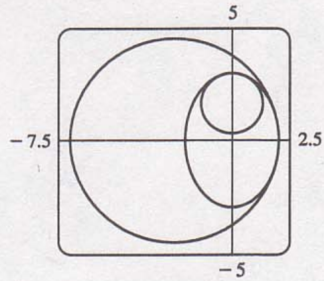
$$\mathbf{T}'(t) = \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \Big\langle -\tfrac{1}{2}(8t + 36t^3), 2(1 + 4t^2 + 9t^4) - \tfrac{1}{2}(8t + 36t^3) 2t,$$
$$6t(1 + 4t^2 + 9t^4) - \tfrac{1}{2}(8t + 36t^3) 3t^2 \Big\rangle$$
$$= \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \langle -4t - 18t^3, 2 - 18t^4, 6t + 12t^3 \rangle = \frac{-2}{(14)^{3/2}} \langle 11, 8, -9 \rangle \text{ when } t = 1.$$

$\mathbf{N}(1) \parallel \mathbf{T}'(1) \parallel \langle 11, 8, -9 \rangle$ and $\mathbf{T}(1) \parallel \mathbf{r}'(1) = \langle 1, 2, 3 \rangle \quad \Rightarrow \quad$ a normal vector to the osculating plane is $\langle 11, 8, -9 \rangle \times \langle 1, 2, 3 \rangle = \langle 42, -42, 14 \rangle$ or equivalently $\langle 3, -3, 1 \rangle$. An equation for the plane is $3(x - 1) - 3(y - 1) + (z - 1) = 0$ or $3x - 3y + z = 1$.

37. The ellipse is given by the parametric equations $x = 2\cos t$, $y = 3\sin t$, so using the result from Exercise 30,

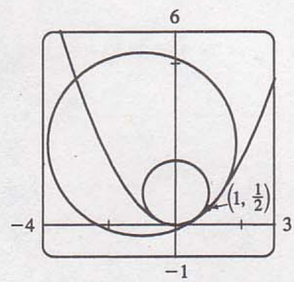
$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(-2\sin t)(-3\sin t) - (3\cos t)(-2\cos t)|}{(4\sin^2 t + 9\cos^2 t)^{3/2}} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}.$$

At $(2, 0)$, $t = 0$. Now $\kappa(0) = \frac{6}{27} = \frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0) = \frac{9}{2}$ and its center is $\left(-\frac{5}{2}, 0\right)$. Its equation is therefore $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$. At $(0, 3)$, $t = \frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right) = \frac{6}{8} = \frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0, \frac{5}{3}\right)$. Hence its equation is $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$.



38. $y = \frac{1}{2}x^2 \quad\Rightarrow\quad y' = x$ and $y'' = 1$, so Formula 11 gives $\kappa(x) = \dfrac{1}{(1 + x^2)^{3/2}}$. So the curvature at $(0, 0)$ is $\kappa(0) = 1$ and the osculating circle has radius 1 and center $(0, 1)$, and hence equation $x^2 + (y - 1)^2 = 1$. The curvature at $\left(1, \frac{1}{2}\right)$ is $\kappa(1) = \dfrac{1}{(1 + 1^2)^{3/2}} = \dfrac{1}{2\sqrt{2}}$.

The tangent line to the parabola at $\left(1, \frac{1}{2}\right)$ has slope 1, so the normal line has slope -1. Thus the center of the osculating circle lies in the direction of the unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2} \right\rangle + 2\sqrt{2}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -1, \frac{5}{2} \right\rangle$. So the equation of the circle is $(x + 1)^2 + \left(y - \frac{5}{2}\right)^2 = 8$.



39. The tangent vector is normal to the normal plane, and the vector $\langle 6, 6, -8\rangle$ is normal to the given plane. But $\mathbf{T}(t) \parallel \mathbf{r}'(t)$ and $\langle 6, 6, -8\rangle \parallel \langle 3, 3, -4\rangle$, so we need to find t such that $\mathbf{r}'(t) \parallel \langle 3, 3, -4\rangle$. $\mathbf{r}(t) = \left\langle t^3, 3t, t^4\right\rangle \quad\Rightarrow$ $\mathbf{r}'(t) = \left\langle 3t^2, 3, 4t^3\right\rangle \parallel \langle 3, 3, -4\rangle$ when $t = -1$. So the planes are parallel at the point $\mathbf{r}(-1) = (-1, -3, 1)$.

40. To find the osculating plane, we first calculate the tangent and normal vectors.

In Maple, we set `x:=t^3;` `y:=3*t;` and `z:=t^4;` and then calculate the components of the tangent vector $\mathbf{T}(t)$ using the `diff` command. We find that $\mathbf{T}(t) = \dfrac{\left\langle 3t^2, 3, 4t^3\right\rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. Differentiating the components of $\mathbf{T}(t)$, we find that

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\left\langle -6t\left(8t^6 - 9\right), 3\left(48t^5 + 18t^3\right), 36t^2\left(t^4 + 3\right)\right\rangle}{\sqrt{144t^2\left(8t^6 - 9\right)^2 + 9\left(96t^5 + 36t^3\right)^2 + 5{,}184t^{12} + 31{,}104t^8 + 46{,}656t^4}}.$$

In Maple, we can calculate $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ using the `linalg` package. First we define $\mathbf{T}$ and $\mathbf{N}$ using `T:=array([f,g,h]);` and `N:=array([F,G,H]);` where `f`, `g`, `h`, `F`, `G`, and `H` are the components of $\mathbf{T}$ and $\mathbf{N}$. Then we use the command `B:=crossprod(T,N);`. After normalization and simplification, we find that $\mathbf{B}(t) = b\left\langle 6t, -2t^3, -3\right\rangle$, where

$$b = \frac{t\sqrt{16t^6 + 9t^4 + 9}}{\sqrt{16t^2\left(8t^6 - 9\right)^2 + \left(96t^5 + 36t^3\right)^2 + 576t^{12} + 3456t^8 + 5184t^4}}.$$

In Mathematica, we use the command `Dt` to differentiate the components of $\mathbf{r}(t)$ and subsequently $\mathbf{T}(t)$, and then

load the vector analysis package with the command `<<Calculus'VectorAnalysis'`. After setting `T={f,g,h}` and `N={F,G,H}`, we use `CrossProduct[T,N]` to find $\mathbf{B}$ (before normalization).
Now $\mathbf{B}(t)$ is parallel to $\langle 6t, -2t^3, -3\rangle$, so if $\mathbf{B}(t)$ is parallel to $\langle 1,1,1\rangle$ for some t, then $6t = 1 \Rightarrow t = \frac{1}{6}$, but $-2\left(\frac{1}{6}\right)^3 \neq 1$. So there is no such osculating plane.

41. $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right| = \dfrac{|d\mathbf{T}/dt|}{ds/dt}$ and $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, so $\kappa\mathbf{N} = \dfrac{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{d\mathbf{T}}{dt}}{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{ds}{dt}} = \dfrac{d\mathbf{T}/dt}{ds/dt} = \dfrac{d\mathbf{T}}{ds}$ by the Chain Rule.

42. For a plane curve, $\mathbf{T} = |\mathbf{T}|\cos\phi\,\mathbf{i} + |\mathbf{T}|\sin\phi\,\mathbf{j} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$. Then
$\dfrac{d\mathbf{T}}{ds} = \left(\dfrac{d\mathbf{T}}{d\phi}\right)\left(\dfrac{d\phi}{ds}\right) = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\left(\dfrac{d\phi}{ds}\right)$ and $\left|\dfrac{d\mathbf{T}}{ds}\right| = |-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}|\left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa = |d\phi/ds|$.

43. (a) $|\mathbf{B}| = 1 \Rightarrow \mathbf{B}\cdot\mathbf{B} = 1 \Rightarrow \dfrac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 0 \Rightarrow 2\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0 \Rightarrow \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \Rightarrow$

$$\begin{aligned}\frac{d\mathbf{B}}{ds} &= \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{ds/dt} = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{|\mathbf{r}'(t)|}\\ &= [(\mathbf{T}'\times\mathbf{N}) + (\mathbf{T}\times\mathbf{N}')]\frac{1}{|\mathbf{r}'(t)|} = \left[\left(\mathbf{T}'\times\frac{\mathbf{T}'}{|\mathbf{T}'|}\right) + (\mathbf{T}\times\mathbf{N}')\right]\frac{1}{|\mathbf{r}'(t)|} = \frac{\mathbf{T}\times\mathbf{N}'}{|\mathbf{r}'(t)|}\end{aligned}$$

$\Rightarrow \dfrac{d\mathbf{B}}{ds} \perp \mathbf{T}$

(c) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \Rightarrow \mathbf{T}\perp\mathbf{N}$, $\mathbf{B}\perp\mathbf{T}$ and $\mathbf{B}\perp\mathbf{N}$. So $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Therefore, $d\mathbf{B}/ds = -\tau(s)\,\mathbf{N}$, where $\tau(s)$ is a scalar.

(d) Since $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, $\mathbf{T}\perp\mathbf{N}$ and both $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is a unit vector mutually perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. For a plane curve, $\mathbf{T}$ and $\mathbf{N}$ always lie in the plane of the curve, so that $\mathbf{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\mathbf{B}/ds = \mathbf{0}$, but $d\mathbf{B}/ds = -\tau(s)\,\mathbf{N}$ and $\mathbf{N} \neq \mathbf{0}$, so $\tau(s) = 0$.

44. $\mathbf{N} = \mathbf{B}\times\mathbf{T} \Rightarrow$

$$\begin{aligned}\frac{d\mathbf{N}}{ds} &= \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = \frac{d\mathbf{B}}{ds}\times\mathbf{T} + \mathbf{B}\times\frac{d\mathbf{T}}{ds} \quad \text{(by Theorem 14.2.3 [ET 13.2.3] \#5)}\\ &= -\tau\mathbf{N}\times\mathbf{T} + \mathbf{B}\times\kappa\mathbf{N} \quad \text{(by Formulas 3 and 1)}\\ &= -\tau(\mathbf{N}\times\mathbf{T}) + \kappa(\mathbf{B}\times\mathbf{N}) \quad \text{(by Theorem 13.4.8 [ET 12.4.8] \#2)}\end{aligned}$$

But $\mathbf{B}\times\mathbf{N} = \mathbf{B}\times(\mathbf{B}\times\mathbf{T}) = (\mathbf{B}\cdot\mathbf{T})\,\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\,\mathbf{T}$ (by Theorem 13.4.8 [12.4.8] #6) $= -\mathbf{T} \Rightarrow$
$d\mathbf{N}/ds = \tau(\mathbf{T}\times\mathbf{N}) - \kappa\mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

45. (a) $\mathbf{r}' = s'\mathbf{T} \Rightarrow \mathbf{r}'' = s''\mathbf{T} + s'\mathbf{T}' = s''\mathbf{T} + s'\dfrac{d\mathbf{T}}{ds}s' = s''\mathbf{T} + \kappa(s')^2\,\mathbf{N}$ by the first Serret-Frenet formula.

(b) Using part (a), we have

$$\begin{aligned}\mathbf{r}'\times\mathbf{r}'' &= (s'\mathbf{T})\times\left[s''\mathbf{T} + \kappa(s')^2\,\mathbf{N}\right]\\ &= [(s'\mathbf{T})\times(s''\mathbf{T})] + \left[(s'\mathbf{T})\times\left(\kappa(s')^2\,\mathbf{N}\right)\right] \quad \text{(By Theorem 13.4.8 [ET 12.4.8] \#3)}\\ &= (s's'')(\mathbf{T}\times\mathbf{T}) + \kappa(s')^3(\mathbf{T}\times\mathbf{N}) = \mathbf{0} + \kappa(s')^3\,\mathbf{B} = \kappa(s')^3\,\mathbf{B}\end{aligned}$$

(c) Using part (a), we have

$$\begin{aligned}
\mathbf{r}''' &= \left[s''\mathbf{T} + \kappa\,(s')^2\,\mathbf{N}\right]' = s'''\mathbf{T} + s''\mathbf{T}' + \kappa'\,(s')^2\,\mathbf{N} + 2\kappa s' s''\mathbf{N} + \kappa\,(s')^2\,\mathbf{N}' \\
&= s'''\mathbf{T} + s''\frac{d\mathbf{T}}{ds}s' + \kappa'\,(s')^2\,\mathbf{N} + 2\kappa s' s''\mathbf{N} + \kappa\,(s')^2\,\frac{d\mathbf{N}}{ds}s' \\
&= s'''\mathbf{T} + s''s'\kappa\mathbf{N} + \kappa'\,(s')^2\,\mathbf{N} + 2\kappa s' s''\mathbf{N} + \kappa\,(s')^3\,(-\kappa\mathbf{T} + \tau\mathbf{B}) \quad \text{(by the second formula)} \\
&= \left[s''' - \kappa^2\,(s')^3\right]\mathbf{T} + \left[3\kappa s' s'' + \kappa'\,(s')^2\right]\mathbf{N} + \kappa\tau\,(s')^3\,\mathbf{B}
\end{aligned}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B}\cdot\mathbf{T} = 0$, $\mathbf{B}\cdot\mathbf{N} = 0$, and $\mathbf{B}\cdot\mathbf{B} = 1$, we get

$$\begin{aligned}
\frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} &= \frac{\kappa\,(s')^3\,\mathbf{B}\cdot\left\{\left[s''' - \kappa^2\,(s')^3\right]\mathbf{T} + \left[3\kappa s' s'' + \kappa'\,(s')^2\right]\mathbf{N} + \kappa\tau\,(s')^3\,\mathbf{B}\right\}}{\left|\kappa\,(s')^3\,\mathbf{B}\right|^2} \\
&= \frac{\kappa\,(s')^3\,\kappa\tau\,(s')^3}{\left[\kappa\,(s')^3\right]^2} = \tau
\end{aligned}$$

46. First we find the quantities required to compute κ:

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \Rightarrow \mathbf{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle \Rightarrow \mathbf{r}'''(t) = \langle a\sin t, -a\cos t, 0\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t\,\mathbf{i} - ab\cos t\,\mathbf{j} + a^2\,\mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4}$$

$$(\mathbf{r}'(t) \times \mathbf{r}''(t))\cdot\mathbf{r}'''(t) = (ab\sin t)\,(a\sin t) + (-ab\cos t)\,(-a\cos t) + (a^2)\,(0) = a^2b$$

Then by Theorem 10,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{a^2b^2 + a^4}}{\left(\sqrt{a^2 + b^2}\right)^3} = \frac{a\sqrt{a^2 + b^2}}{\left(\sqrt{a^2 + b^2}\right)^3} = \frac{a}{a^2 + b^2}$$

which is a constant.

From Exercise 45(d), the torsion τ is given by

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{a^2b}{\left(\sqrt{a^2b^2 + a^4}\right)^2} = \frac{b}{a^2 + b^2}$$

which is also a constant.

47. $\mathbf{r} = \left\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3\right\rangle \quad \Rightarrow \quad \mathbf{r}' = \left\langle 1, t, t^2\right\rangle, \mathbf{r}'' = \langle 0, 1, 2t\rangle, \mathbf{r}''' = \langle 0, 0, 2\rangle \quad \Rightarrow \quad \mathbf{r}' \times \mathbf{r}'' = \left\langle t^2, -2t, 1\right\rangle \quad \Rightarrow$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\left\langle t^2, -2t, 1\right\rangle\cdot\langle 0, 0, 2\rangle}{t^4 + 4t^2 + 1} = \frac{2}{t^4 + 4t^2 + 1}$$

48. $\mathbf{r} = \langle \sinh t, \cosh t, t \rangle \quad \Rightarrow \quad \mathbf{r}' = \langle \cosh t, \sinh t, 1 \rangle$, $\mathbf{r}'' = \langle \sinh t, \cosh t, 0 \rangle$, $\mathbf{r}''' = \langle \cosh t, \sinh t, 0 \rangle \quad \Rightarrow$
$\mathbf{r}' \times \mathbf{r}'' = \langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t \rangle = \langle -\cosh t, \sinh t, 1 \rangle \quad \Rightarrow$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle -\cosh t, \sinh t, 1 \rangle|}{|\langle \cosh t, \sinh t, 1 \rangle|^3} = \frac{\sqrt{\cosh^2 t + \sinh^2 t + 1}}{\left(\cosh^2 t + \sinh^2 t + 1\right)^{3/2}} = \frac{1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2\cosh^2 t},$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle -\cosh t, \sinh t, 1 \rangle \cdot \langle \cosh t, \sinh t, 0 \rangle}{\cosh^2 t + \sinh^2 t + 1} = \frac{-\cosh^2 t + \sinh^2 t}{2\cosh^2 t} = \frac{-1}{2\cosh^2 t}$$

So at the point $(0, 1, 0)$, $t = 0$, and $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{2}$.

49. For one helix, the vector equation is $\mathbf{r}(t) = \langle 10\cos t, 10\sin t, 34t/(2\pi) \rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t. Using the arc length formula, letting t go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$\begin{aligned} L &= \int_0^{2.9\times10^8\times2\pi} \left|\mathbf{r}'(t)\right| dt \\ &= \int_0^{2.9\times10^8\times2\pi} \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\, dt \\ &= \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\, t\,\Big]_0^{2.9\times10^8\times2\pi} \\ &= 2.9 \times 10^8 \times 2\pi\sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2} \\ &\approx 2.07 \times 10^{10}\ \text{Å} \text{ — more than two meters!} \end{aligned}$$

50. (a) For the function $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ to be continuous, we must have $P(0) = 0$ and $P(1) = 1$. For F' to be continuous, we must have $P'(0) = P'(1) = 0$. The curvature of the curve $y = F(x)$ at the point $(x, F(x))$ is $\kappa(x) = \dfrac{|F''(x)|}{\left(1 + [F'(x)]^2\right)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0) = P''(1) = 0$. Write $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Then $P'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ and $P''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$. Our six conditions are:

$$P(0) = 0 \quad \Rightarrow \quad f = 0 \tag{1}$$

$$P(1) = 1 \quad \Rightarrow \quad a + b + c + d + e + f = 1 \tag{2}$$

$$P'(0) = 0 \quad \Rightarrow \quad e = 0 \tag{3}$$

$$P'(1) = 0 \quad \Rightarrow \quad 5a + 4b + 3c + 2d + e = 0 \tag{4}$$

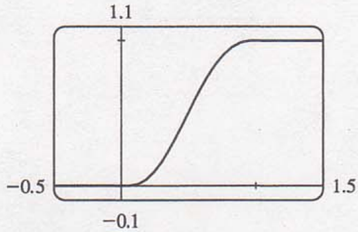
$$P''(0) = 0 \quad \Rightarrow \quad d = 0 \tag{5}$$

$$P''(1) = 0 \quad \Rightarrow \quad 20a + 12b + 6c + 2d = 0 \tag{6}$$

From (1), (3), and (5), we have $d = e = f = 0$. Thus (2), (4) and (6) become (7) $a + b + c = 1$, (8) $5a + 4b + 3c = 0$, and (9) $10a + 6b + 3c = 0$. Subtracting (8) from (9) gives (10) $5a + 2b = 0$.

Multiplying (7) by 3 and subtracting from (8) gives (11) $2a + b = -3$. Multiplying (11) by 2 and subtracting from (10) gives $a = 6$. By (10), $b = -15$. By (7), $c = 10$. Thus, $P(x) = 6x^5 - 15x^4 + 10x^3$.

(b)



14.4 Motion in Space: Velocity and Acceleration ET 13.4

1. (a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time t, then the average velocity over the time interval $[0, 1]$ is

$\mathbf{v}_{\text{ave}} = \dfrac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \dfrac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$. Similarly, over the other intervals we have

$$[0.5, 1]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5} = 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$$

$$[1, 2]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{1} = 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$$

$$[1, 1.5]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5} = 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$$

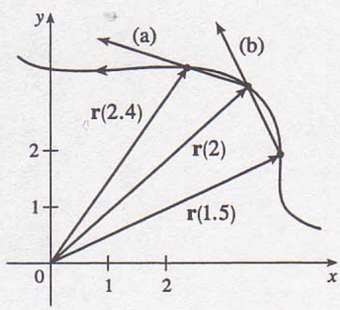
(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals $[0.5, 1]$ and $[1, 1.5]$: $\mathbf{v}(1) \approx \frac{1}{2}\left[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})\right] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$. Then the speed is $|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$.

2. (a) The average velocity over $2 \le t \le 2.4$ is

$\dfrac{\mathbf{r}(2.4) - \mathbf{r}(2)}{2.4 - 2} = 2.5\left[\mathbf{r}(2.4) - \mathbf{r}(2)\right]$, so we sketch a vector in the same direction but 2.5 times the length of $[\mathbf{r}(2.4) - \mathbf{r}(2)]$.

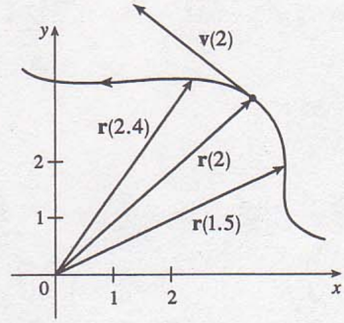
(b) The average velocity over $1.5 \le t \le 2$ is

$\dfrac{\mathbf{r}(2) - \mathbf{r}(1.5)}{2 - 1.5} = 2\left[\mathbf{r}(2) - \mathbf{r}(1.5)\right]$, so we sketch a vector in the same direction but twice the length of $[\mathbf{r}(2) - \mathbf{r}(1.5)]$.



(c) Using Equation 2 we have $\mathbf{v}(2) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(2+h) - \mathbf{r}(2)}{h}$.

(d) $\mathbf{v}(2)$ is tangent to the curve at $\mathbf{r}(2)$ and points in the direction of increasing t. Its length is the speed of the particle at $t = 2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2 \le t \le 2.4$ and $1.5 \le t \le 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7. Thus, we estimate the speed at $t = 2$ to be $|\mathbf{v}(2)| \approx \frac{1}{2}(2.8 + 2.7) = 2.75$ and we draw the velocity vector $\mathbf{v}(2)$ with this length.

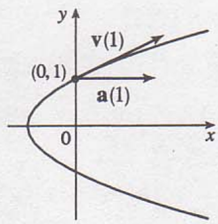


3. $\mathbf{r}(t) = \langle t^2 - 1, t \rangle \;\Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 1 \rangle$, $\mathbf{v}(1) = \langle 2, 1 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2, 0 \rangle$, $\mathbf{a}(1) = \langle 2, 0 \rangle$

$|\mathbf{v}(t)| = \sqrt{4t^2 + 1}$



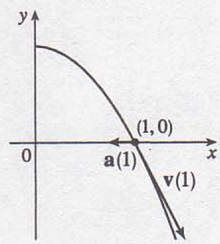
4. $\mathbf{r}(t) = \langle \sqrt{t}, 1 - t \rangle \;\Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \left\langle \frac{1}{2}t^{-1/2}, -1 \right\rangle$, $\mathbf{v}(1) = \langle \frac{1}{2}, -1 \rangle$

$\mathbf{a}(t) = \left\langle -\frac{1}{4}t^{-3/2}, 0 \right\rangle$, $\mathbf{a}(1) = \langle -\frac{1}{4}, 0 \rangle$

$|\mathbf{v}(t)| = \sqrt{\frac{1}{4}t^{-1} + 1}$

Since $x^2 = t$, $y = 1 - t = 1 - x^2$, but $x = \sqrt{t}$, so $x \ge 0$.



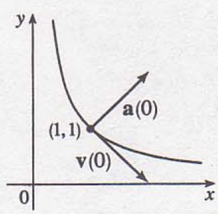
5. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j} \;\Rightarrow$ At $t = 0$:

$\mathbf{v}(t) = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$, $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$,

$\mathbf{a}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$ $\mathbf{a}(0) = \mathbf{i} + \mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{e^{2t} + e^{-2t}} = e^{-t}\sqrt{e^{4t} + 1}$

Since $x = e^t$, $t = \ln x$ and $y = e^{-t} = e^{-\ln x} = 1/x$, and $x > 0, y > 0$.



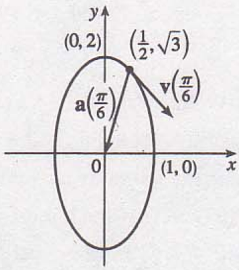
6. $\mathbf{r}(t) = \langle \sin t, 2\cos t\rangle \quad \Rightarrow$

$\mathbf{v}(t) = \langle \cos t, -2\sin t\rangle$, $\mathbf{v}\left(\frac{\pi}{6}\right) = \left\langle \frac{\sqrt{3}}{2}, -1\right\rangle$

$\mathbf{a}(t) = \langle -\sin t, -2\cos t\rangle$, $\mathbf{a}\left(\frac{\pi}{6}\right) = \left\langle -\frac{1}{2}, -\sqrt{3}\right\rangle$

$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 4\sin^2 t} = \sqrt{1 + 3\sin^2 t}$

And $x^2 + y^2/4 = \sin^2 t + \cos^2 t = 1$, an ellipse.



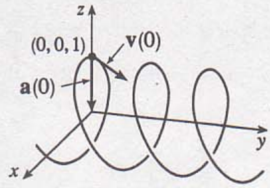
7. $\mathbf{r}(t) = \langle \sin t, t, \cos t\rangle \quad \Rightarrow$

$\mathbf{v}(t) = \langle \cos t, 1, -\sin t\rangle$, $\mathbf{v}(0) = \langle 1, 1, 0\rangle$

$\mathbf{a}(t) = \langle -\sin t, 0, -\cos t\rangle$, $\mathbf{a}(0) = \langle 0, 0, -1\rangle$

$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 1 + \sin^2 t} = \sqrt{2}$

Since $x^2 + z^2 = 1$, $y = t$, the path of the particle is a helix about the y-axis.



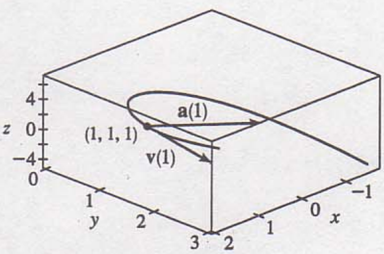
8. $\mathbf{r}(t) = \left\langle t, t^2, t^3\right\rangle \quad \Rightarrow$

$\mathbf{v}(t) = \left\langle 1, 2t, 3t^2\right\rangle$, $\mathbf{v}(1) = \langle 1, 2, 3\rangle$

$\mathbf{a}(t) = \langle 0, 2, 6t\rangle$, $\mathbf{a}(1) = \langle 0, 2, 6\rangle$

$|\mathbf{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$

The path is a "twisted cubic" (see Example 14.1.6 [ET 13.1.6]).



9. $\mathbf{r}(t) = \left\langle t, t^2, t^3\right\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2, 6t\rangle$,

$|\mathbf{v}(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$.

10. $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3, 2\cos t\rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle -2\cos t, 0, -2\sin t\rangle$,

$|\mathbf{v}(t)| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}$.

11. $\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$,

$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$.

12. $\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + t^{-1}\,\mathbf{j} + \mathbf{k}$, $\mathbf{a}(t) = \mathbf{v}'(t) = 2\mathbf{i} - t^{-2}\,\mathbf{j}$,

$|\mathbf{v}(t)| = \sqrt{4t^2 + t^{-2} + 1}$.

13. $\mathbf{r}(t) = e^t\langle \cos t, \sin t, t\rangle \quad \Rightarrow$

$\mathbf{v}(t) = \mathbf{r}'(t) = e^t\langle \cos t, \sin t, t\rangle + e^t\langle -\sin t, \cos t, 1\rangle = e^t\langle \cos t - \sin t, \sin t + \cos t, t + 1\rangle$

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= e^t\langle \cos t - \sin t - \sin t - \cos t, \sin t + \cos t + \cos t - \sin t, t + 1 + 1\rangle \\ &= e^t\langle -2\sin t, 2\cos t, t + 2\rangle\end{aligned}$$

$$\begin{aligned}|\mathbf{v}(t)| &= e^t\sqrt{\cos^2 t + \sin^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1} \\ &= e^t\sqrt{t^2 + 2t + 3}\end{aligned}$$

14. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j} + t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = (\sin t + t\cos t)\,\mathbf{i} + (\cos t - t\sin t)\,\mathbf{j} + 2t\,\mathbf{k}$,
$\mathbf{a}(t) = \mathbf{v}'(t) = (2\cos t - t\sin t)\,\mathbf{i} + (-2\sin t - t\cos t)\,\mathbf{j} + 2\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{(\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + 4t^2} = \sqrt{5t^2+1}$.

15. $\mathbf{a}(t) = \mathbf{k} \quad\Rightarrow\quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \mathbf{k}\,dt = t\,\mathbf{k} + \mathbf{c}_1$ and $\mathbf{i} - \mathbf{j} = \mathbf{v}(0) = 0\,\mathbf{k} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + t\,\mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} - \mathbf{j} + t\,\mathbf{k})\,dt = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k} + \mathbf{c}_2$. But $\mathbf{0} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k}$.

16. $\mathbf{a}(t) = -10\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v}(t) = \int (-10\mathbf{k})\,dt = -10t\,\mathbf{k} + \mathbf{c}_1$, and $\mathbf{i} + \mathbf{j} - \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - (10t+1)\,\mathbf{k}$.
$\mathbf{r}(t) = \int [\mathbf{i} + \mathbf{j} - (10t+1)\,\mathbf{k}]\,dt = t\,\mathbf{i} + t\,\mathbf{j} - (5t^2 + t)\,\mathbf{k} + \mathbf{c}_2$. But $2\,\mathbf{i} + 3\,\mathbf{j} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = 2\,\mathbf{i} + 3\,\mathbf{j}$ and $\mathbf{r}(t) = (t+2)\,\mathbf{i} + (t+3)\,\mathbf{j} - (5t^2+t)\,\mathbf{k}$.

17. (a) $\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j} + 2t\,\mathbf{k} \quad\Rightarrow$

$\mathbf{v}(t) = \int (\mathbf{i} + 2\mathbf{j} + 2t\,\mathbf{k})\,dt = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{c}_1$, and

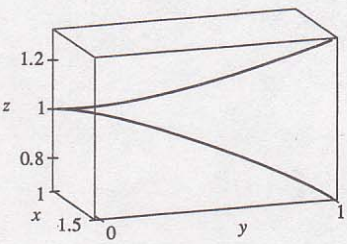
$\mathbf{0} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{0}$ and $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$.

$\mathbf{r}(t) = \int (t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k})\,dt = \frac{1}{2}t^2\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{c}_2$.

But $\mathbf{i} + \mathbf{k} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{i} + \mathbf{k}$ and

$\mathbf{r}(t) = \left(1 + \frac{1}{2}t^2\right)\mathbf{i} + t^2\,\mathbf{j} + \left(1 + \frac{1}{3}t^3\right)\mathbf{k}$.

(b)



18. (a) $\mathbf{a}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k} \quad\Rightarrow$

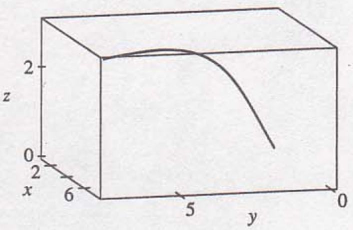
$$\mathbf{v}(t) = \int (t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k})\,dt = \frac{t^2}{2}\,\mathbf{i} + \frac{t^3}{3}\,\mathbf{j} + \frac{\sin 2t}{2}\,\mathbf{k} + \mathbf{c}_1$$

and $\mathbf{i} + \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{k}$ and
$\mathbf{v}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + \frac{1}{3}t^3\,\mathbf{j} + \left(1 + \frac{1}{2}\sin 2t\right)\mathbf{k}$.

$$\mathbf{r}(t) = \int \left[\left(\tfrac{1}{2}t^2 + 1\right)\mathbf{i} + \tfrac{1}{3}t^3\,\mathbf{j} + \left(1 + \tfrac{1}{2}\sin 2t\right)\mathbf{k}\right]dt = \left(\tfrac{1}{6}t^3 + t\right)\mathbf{i} + \tfrac{1}{12}t^4\,\mathbf{j} + \left(t - \tfrac{1}{4}\cos 2t\right)\mathbf{k} + \mathbf{c}_2$$

But $\mathbf{j} = \mathbf{r}(0) = -\frac{1}{4}\,\mathbf{k} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{j} + \frac{1}{4}\,\mathbf{k}$ and
$\mathbf{r}(t) = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \left(1 + \frac{1}{12}t^4\right)\mathbf{j} + \left(\frac{1}{4} + t - \frac{1}{4}\cos 2t\right)\mathbf{k}$.

(b)



19. $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle \quad\Rightarrow\quad \mathbf{v}(t) = \langle 2t, 5, 2t - 16\rangle$,
$|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$ and
$\dfrac{d}{dt}|\mathbf{v}(t)| = \frac{1}{2}\left(8t^2 - 64t + 281\right)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is, $16t - 64 = 0$ or $t = 4$. Since $\dfrac{d}{dt}|\mathbf{v}(t)| < 0$ for $t < 4$ and $\dfrac{d}{dt}|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained at $t = 4$ units of time.

20. Since $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$. By Newton's Second Law, $\mathbf{F}(t) = m\,\mathbf{a}(t) = 6mt\,\mathbf{i} + 2m\,\mathbf{j} + 6mt\,\mathbf{k}$ is the required force.

21. $|\mathbf{F}(t)| = 20$ N in the direction of the positive z-axis, so $\mathbf{F}(t) = 20\,\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Since $20\mathbf{k} = \mathbf{F}(t) = 4\,\mathbf{a}(t)$, $\mathbf{a}(t) = 5\,\mathbf{k}$. Then $\mathbf{v}(t) = 5t\,\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\,\mathbf{k}$ and the speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$.

22. The argument here is the same as that in Example 14.2.5 [ET 13.2.5] with $\mathbf{r}(t)$ replaced by $\mathbf{v}(t)$ and $\mathbf{r}'(t)$ replaced by $\mathbf{a}(t)$.

23. $|\mathbf{v}(0)| = 500$ m/s and since the angle of elevation is 30°, the direction of the velocity is $\frac{1}{2}\left(\sqrt{3}\,\mathbf{i} + \mathbf{j}\right)$. Thus $\mathbf{v}(0) = 250\left(\sqrt{3}\,\mathbf{i} + \mathbf{j}\right)$ and if we set up the axes so the projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the only force is that due to gravity, so $\mathbf{F}(t) = -mg\,\mathbf{j}$ where $g \approx 9.8$ m/s^2. Thus $\mathbf{a}(t) = -g\,\mathbf{j}$ and $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{c}_1$. But $250\left(\sqrt{3}\,\mathbf{i} + \mathbf{j}\right) = \mathbf{v}(0) = \mathbf{c}_1$, so $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}t\,\mathbf{i} + \left(250t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{c}_2$ where $\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2$. Thus $\mathbf{r}(t) = 250\sqrt{3}t\,\mathbf{i} + \left(250t - \frac{1}{2}gt^2\right)\mathbf{j}$.

(a) Setting $250t - \frac{1}{2}gt^2 = 0$ gives $t = 0$ or $t = \frac{500}{g} \approx 51.0$ s. So the range is $250\sqrt{3} \cdot \frac{500}{g} \approx 22$ km.

(b) $0 = \dfrac{d}{dt}\left(250t - \frac{1}{2}gt^2\right) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s. Thus, the maximum height is $(250)\,(250/g) - g\,(250/g)^2\,\frac{1}{2} = (250)^2/\,(2g) \approx 3.2$ km.

(c) From part (a), impact occurs at $t = 500/g \approx 51.0$. Thus, the velocity at impact is $\mathbf{v}(500/g) = 250\sqrt{3}\,\mathbf{i} + [250 - g\,(500/g)]\,\mathbf{j} = 250\sqrt{3}\,\mathbf{i} - 250\,\mathbf{j}$ and the speed is $|\mathbf{v}(500/g)| = 250\sqrt{3 + 1} = 500$ m/s.

24. As in Exercise 23, $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}t\,\mathbf{i} + \left(250t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{c}_2$. But $\mathbf{r}(0) = 200\,\mathbf{j}$, so $\mathbf{c}_2 = 200\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}t\,\mathbf{i} + \left(200 + 250t - \frac{1}{2}gt^2\right)\mathbf{j}$.

(a) $200 + 250t - \frac{1}{2}gt^2 = 0$ implies that $gt^2 - 500t - 400 = 0$ or $t = \dfrac{500 \pm \sqrt{500^2 + 1600g}}{2g}$. Taking the positive ty-value gives $t = \dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g} \approx 51.8$ s. Thus the range is

$$\left(250\sqrt{3}\right)\frac{500 + \sqrt{250{,}000 + 1600g}}{2g} \approx 22.4 \text{ km}.$$

(b) $0 = \dfrac{d}{dt}\left(200 + 250t - \frac{1}{2}gt^2\right) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s and thus the maximum height is

$$\left[200 + (250)\left(\frac{250}{g}\right) - \frac{g}{2}\left(\frac{250}{g}\right)^2\right] = 200 + \frac{(250)^2}{2g} \approx 3.4 \text{ km}.$$

Alternate Solution: Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 200 m higher, the maximum height reached is 200 m higher than that found in Exercise 23, that is, 3.2 km + 200 m = 3.4 km.

(c) From part (a), impact occurs at $t = \dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g}$. Thus the velocity at impact is

$$250\sqrt{3}\,\mathbf{i} + \left[250 - g\frac{500 + \sqrt{250{,}000 + 1600g}}{2g}\right]\mathbf{j}, \text{ so } |\mathbf{v}| \approx \sqrt{(250)^2\,(3) + (250 - 51.8g)^2} \approx 504 \text{ m/s}.$$

25. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45^\circ)\, t\,\mathbf{i} + \left[(v_0 \sin 45^\circ)\, t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{2}\,t\,\mathbf{i} + \left(v_0\sqrt{2}\,t - gt^2\right)\mathbf{j}\right]$. Then the ball lands at $t = \dfrac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = \frac{1}{2}v_0\sqrt{2}\dfrac{v_0\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity is $v_0 = \sqrt{90g} \approx 30$ m/s.

26. Here the initial speed $v_0 = 120$ m/s; let α be the angle of elevation. Assuming the object is lying flat on the ground, the object will be hit at time $t = \dfrac{240 \sin\alpha}{g}$ s (again refer to Example 5). Then $\dfrac{(120)^2 \sin 2\alpha}{g} = 500$ or $\sin 2\alpha = \dfrac{500g}{(120)^2} = \dfrac{5g}{144}$ and $2\alpha = \sin^{-1}\dfrac{5g}{144} \approx 19.9^\circ$ so $\alpha \approx 9.9^\circ$.

27. From (4), $x = (v_0 \cos\alpha)\, t$ or $t = \dfrac{x}{v_0 \cos\alpha}$. Thus

$$y = (v_0 \sin\alpha)\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)\, x - \frac{g}{2v_0^2\cos^2\alpha}x^2$$. Thus

the trajectory is a parabola. Continuing by completing the square, we see that

$$y - \frac{(\tan^2\alpha)\, v_0^2 \cos^2\alpha}{2g} = -\frac{g}{2v_0^2\cos^2\alpha}\left[x - \frac{(\tan\alpha)\, v_0^2\,(\cos^2\alpha)}{g}\right]^2 \quad\text{or}$$

$$y - \frac{v_0^2\sin^2\alpha}{2g} = -\frac{g}{2v_0^2\cos^2\alpha}\left(x - \frac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2$$. Thus the vertex of the parabola lies at $\left(\dfrac{v_0^2\sin\alpha\,\cos\alpha}{g}, \dfrac{v_0^2\sin^2\alpha}{2g}\right)$, so the maximum height is $y = \dfrac{v_0^2\sin^2\alpha}{2g}$.

28. $\mathbf{r}(t) = (1+t)\,\mathbf{i} + \left(t^2 - 2t\right)\mathbf{j} \quad\Rightarrow\quad \mathbf{r}'(t) = \mathbf{i} + (2t-2)\,\mathbf{j}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t-2)^2} = \sqrt{4t^2 - 8t + 5}$, $\mathbf{r}''(t) = 2\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = 2\mathbf{k}$. Then Equation 9 gives $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{2\,(2t-2)}{\sqrt{4t^2-8t+5}}$ and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2}{\sqrt{4t^2-8t+5}}$.

29. $\mathbf{r}(t) = \left(3t - t^3\right)\mathbf{i} + 3t^2\,\mathbf{j} \quad\Rightarrow\quad \mathbf{r}'(t) = \left(3 - 3t^2\right)\mathbf{i} + 6t\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{(3-3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3+3t^2)^2} = 3 + 3t^2$,

$\mathbf{r}''(t) = -6t\,\mathbf{i} + 6\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = \left(18 + 18t^2\right)\mathbf{k}$. Then Equation 9 gives

$a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{(3-3t^2)(-6t) + (6t)(6)}{3+3t^2} = \dfrac{18t + 18t^3}{3+3t^2} = \dfrac{18t\,(1+t^2)}{3\,(1+t^2)} = 6t$ (or by Equation 8,

$a_T = v' = \dfrac{d}{dt}\left[3 + 3t^2\right] = 6t$) and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{18+18t^2}{3+3t^2} = \dfrac{18\,(1+t^2)}{3\,(1+t^2)} = 6$.

30. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + 3^2} = \sqrt{4t^2+10}$, $\mathbf{r}''(t) = 2\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = -6\mathbf{i} + 2\mathbf{k}$. Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2+10}}$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2\sqrt{10}}{\sqrt{4t^2+10}}$.

31. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$,

$\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}$.

Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{\sin t\cos t - \sin t\cos t}{\sqrt{2}} = 0$ and

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

32. $\mathbf{r}(t) = t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \sin^2 t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = \mathbf{i} - 2\cos t \sin t\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k} = \mathbf{i} - \sin 2t\,\mathbf{j} + \sin 2t\,\mathbf{k}$,
$|\mathbf{r}'(t)| = \sqrt{1 + 2\sin^2 2t}$, $\mathbf{r}''(t) = 2\left(\sin^2 t - \cos^2 t\right)\mathbf{j} + 2\left(\cos^2 t - \sin^2 t\right)\mathbf{k} = -2\cos 2t\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$. So
$a_T = \dfrac{2\sin 2t\cos 2t + 2\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}} = \dfrac{4\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}}$ and $a_N = \dfrac{|-2\cos 2t\,\mathbf{j} - 2\cos 2t\,\mathbf{k}|}{\sqrt{1+2\sin^2 2t}} = \dfrac{2\sqrt{2}\,|\cos 2t|}{\sqrt{1+2\sin^2 t}}$.

33. $\mathbf{r}(t) = e^t\,\mathbf{i} + \sqrt{2}t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = e^t\,\mathbf{i} + \sqrt{2}\,\mathbf{j} - e^{-t}\,\mathbf{k}$,
$|\mathbf{r}(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$, $\mathbf{r}''(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{k}$.
Then $a_T = \dfrac{e^{2t} - e^{-2t}}{e^t + e^{-t}} = \dfrac{\left(e^t + e^{-t}\right)\left(e^t - e^{-t}\right)}{e^t + e^{-t}} = e^t - e^{-t} = 2\sinh t$ and
$a_N = \dfrac{|\sqrt{2}e^{-t}\mathbf{i} - 2\mathbf{j} - \sqrt{2}e^t\mathbf{k}|}{e^t + e^{-t}} = \dfrac{\sqrt{2\left(e^{-2t} + 2 + e^{2t}\right)}}{e^t + e^{-t}} = \sqrt{2}\,\dfrac{e^t + e^{-t}}{e^t + e^{-t}} = \sqrt{2}$.

34. $\mathbf{L}(t) = m\,\mathbf{r}(t) \times \mathbf{v}(t) \quad \Rightarrow$

$\mathbf{L}'(t) = m\left[\mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)\right]$ (by Theorem 14.2.3 [ET 13.2.3] #5)

$= m\left[\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)\right] = m\left[\mathbf{0} + \mathbf{r}(t) \times \mathbf{a}(t)\right] = \boldsymbol{\tau}(t)$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{0}$ for all t, and so $\mathbf{L}(t)$ is constant.

35. If the engines are turned off at time t, then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need a t such that for some scalar $s > 0$, $\mathbf{r}(t) + s\,\mathbf{v}(t) = \langle 6, 4, 9\rangle$. $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{t}\mathbf{j} + \dfrac{8t}{(t^2+1)^2}\mathbf{k} \quad \Rightarrow$
$\mathbf{r}(t) + s\,\mathbf{v}(t) = \left\langle 3 + t + s, 2 + \ln t + \dfrac{s}{t}, 7 - \dfrac{4}{t^2+1} + \dfrac{8st}{(t^2+1)^2}\right\rangle \quad \Rightarrow \quad 3 + t + s = 6 \quad \Rightarrow \quad s = 3 - t$, so
$7 - \dfrac{4}{t^2+1} + \dfrac{8(3-t)t}{(t^2+1)^2} = 9 \quad \Leftrightarrow \quad \dfrac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \quad \Leftrightarrow \quad t^4 + 8t^2 - 12t + 3 = 0$. It is easily seen that
$t = 1$ is a root of this polynomial. Also $2 + \ln 1 + \dfrac{3-1}{1} = 4$, so $t = 1$ is the desired solution.

36. (a) $m\dfrac{d\mathbf{v}}{dt} = \dfrac{dm}{dt}\mathbf{v}_e \quad \Leftrightarrow \quad \dfrac{d\mathbf{v}}{dt} = \dfrac{1}{m}\dfrac{dm}{dt}\mathbf{v}_e$. Integrating both sides of this equation with respect to t gives
$\displaystyle\int_0^t \frac{d\mathbf{v}}{du}\,du = \mathbf{v}_e \int_0^t \frac{1}{m}\frac{dm}{du}\,du \quad \Rightarrow \quad \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}_e \int_{m(0)}^{m(t)} \frac{dm}{m}$ (Substitution Rule) $\Rightarrow$
$\mathbf{v}(t) - \mathbf{v}(0) = \ln\left(\dfrac{m(t)}{m(0)}\right)\mathbf{v}_e \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\dfrac{m(0)}{m(t)}\right)\mathbf{v}_e$.

(b) $|\mathbf{v}(t)| = 2\,|\mathbf{v}_e|$, and $|\mathbf{v}(0)| = 0$. Therefore, by part (a), $2\,|\mathbf{v}_e| = \left|-\ln\left(\dfrac{m(0)}{m(t)}\right)\mathbf{v}_e\right| \quad \Rightarrow$

$2\,|\mathbf{v}_e| = \ln\left(\dfrac{m(0)}{m(t)}\right)|\mathbf{v}_e|$. [*Note:* $m(0) > m(t)$ so that $\ln\left(m(0)/m(t)\right) > 0$] $\quad \Rightarrow \quad m(t) = e^{-2}m(0)$.

Thus $\dfrac{m(0) - e^{-2}m(0)}{m(0)} = 1 - e^{-2}$ is the fraction of the initial mass that is burned as fuel.

Applied Project □ Kepler's Laws

1. With $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$ and $\mathbf{h} = \alpha\,\mathbf{k}$ where $\alpha > 0$,

(a) $\mathbf{h} = \mathbf{r} \times \mathbf{r}' = [(r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}] \times \left[\left(r'\cos\theta - r\sin\theta\dfrac{d\theta}{dt}\right)\mathbf{i} + \left(r'\sin\theta + r\cos\theta\dfrac{d\theta}{dt}\right)\mathbf{j}\right]$

$$= \left[rr'\cos\theta\sin\theta + r^2\cos^2\theta\frac{d\theta}{dt} - rr'\cos\theta\sin\theta + r^2\sin^2\theta\frac{d\theta}{dt}\right]\mathbf{k} = r^2\frac{d\theta}{dt}\,\mathbf{k}$$

(b) Since $\mathbf{h} = \alpha\,\mathbf{k}$, $\alpha > 0$, $\alpha = |\mathbf{h}|$. But by part (a), $\alpha = |\mathbf{h}| = r^2\,(d\theta/dt)$.

(c) $A(t) = \frac{1}{2}\int_{\theta_0}^{\theta} |\mathbf{r}|^2\,d\theta = \frac{1}{2}\int_{t_0}^{t} r^2\,(d\theta/dt)\,dt$ in polar coordinates. Thus, by the Fundamental Theorem of Calculus, $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt}$.

(d) $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt} = \dfrac{h}{2} =$ constant since $\mathbf{h}$ is a constant vector and $h = |\mathbf{h}|$.

2. (a) Since $dA/dt = \frac{1}{2}h$, a constant, $A(t) = \frac{1}{2}ht + c_1$. But $A(0) = 0$, so $A(t) = \frac{1}{2}ht$. But $A(T) =$ area of the ellipse $= \pi ab$ and $A(T) = \frac{1}{2}hT$, so $T = 2\pi ab/h$.

(b) $h^2/(GM) = ed$ where e is the eccentricity of the ellipse. But $a = ed/\left(1 - e^2\right)$ or $ed = a\left(1 - e^2\right)$ and $1 - e^2 = b^2/a^2$. Hence $h^2/(GM) = ed = b^2/a$.

(c) $T^2 = \dfrac{4\pi a^2 b^2}{h^2} = 4\pi^2 a^2 b^2 \dfrac{a}{GMb^2} = \dfrac{4\pi^2}{GM}a^3$.

3. From Problem 2, $T^2 = \dfrac{4\pi^2}{GM}a^3$. $T \approx 365.25$ days $\times\ 24 \cdot 60^2\ \dfrac{\text{seconds}}{\text{day}} \approx 3.1558 \times 10^7$ seconds. Therefore

$a^3 = \dfrac{GMT^2}{4\pi^2} \approx \dfrac{\left(6.67 \times 10^{-11}\right)\left(1.99 \times 10^{30}\right)\left(3.1558 \times 10^7\right)^2}{4\pi^2} \approx 3.348 \times 10^{33}\ \text{m}^3 \quad \Rightarrow$

$a \approx 1.496 \times 10^{11}$ m. Thus, the length of the major axis of the earth's orbit (that is, $2a$) is approximately $2.99 \times 10^{11}\ \text{m} = 2.99 \times 10^8$ km.

4. We can adapt the equation $T^2 = \dfrac{4\pi^2}{GM}a^3$ from Problem 2(c) with Earth at the center of the system, so T is the period of the satellite's orbit about Earth, M is the mass of Earth, and a is the length of the semimajor axis of the satellite's orbit (measured from Earth's center). Since we want the satellite to remain fixed above a particular point on Earth's equator, T must coincide with the period of Earth's own rotation, so $T = 24\ \text{h} = 86{,}400$ s. The mass of Earth is $M = 5.98 \times 10^{24}$ kg (see Exercise 6.4.26 [ET 6.4.26]), so

$a = \left(\dfrac{T^2 GM}{4\pi^2}\right)^{1/3} \approx \left[\dfrac{(86,400)^2\left(6.67 \times 10^{-11}\right)\left(5.98 \times 10^{24}\right)}{4\pi^2}\right]^{1/3} \approx 4.23 \times 10^7$ m. If we assume a circular orbit, the radius of the orbit is a, and since the radius of Earth is approximately 6.37×10^6 m (again see Exercise 6.4.26 [ET 6.4.26]), the required altitude above Earth's surface for the satellite is $4.23 \times 10^7 - 6.37 \times 10^6 \approx 3.59 \times 10^7$ m, or 35,900 km.

14 Review ET 13

CONCEPT CHECK

1. A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component of the vector function.
2. The tip of the moving vector $\mathbf{r}(t)$ of a continuous vector function traces out a space curve.
3. (a) A curve represented by the vector function $\mathbf{r}(t)$ is smooth if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on its parametric domain (except possibly at the endpoints).
   (b) The tangent vector to a smooth curve at a point P with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at P is the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.
4. (a)–(f) See Theorem 14.2.3 [ET 13.2.3].
5. Use Formula 14.3.2 [ET 13.3.2], or equivalently 14.3.3 [ET 13.3.3].
6. (a) The curvature of a curve is $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.
   (b) $\kappa(t) = \left|\dfrac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$ (c) $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ (d) $\kappa(x) = \dfrac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$
7. (a) The unit normal vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. The binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.
   (b) See the discussion preceding Example 7 in Section 14.3 [ET 13.3].
8. (a) If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.
   (b) $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where $a_T = v'$ and $a_N = \kappa v^2$.
9. See the statement of Kepler's Laws on page 897 [ET 863].

TRUE-FALSE QUIZ

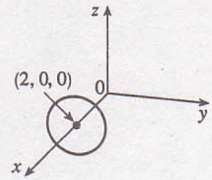
1. True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j} + 3u\,\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$.
2. True. $\mathbf{r}'(t) = \langle 1, 3t^2, 5t^4\rangle$ is continuous for all t (since its component functions are each continuous) and since $x'(t) = 1$, we have $\mathbf{r}'(t) \neq \mathbf{0}$ for all t.
3. False. $\mathbf{r}'(t) = \langle -\sin t, 2t, 4t^3\rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0\rangle = \mathbf{0}$, the curve is not smooth.
4. True. See Theorem 14.2.2 [ET 13.2.2].
5. False. By Formula 5 of Theorem 14.2.3 [ET 13.2.3], $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.
6. False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \Rightarrow \dfrac{d}{dt}|\mathbf{r}(t)| = 0$, but $|\mathbf{r}'(t)| = |\langle -\sin t, \cos t\rangle| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.
7. False. κ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s, not with respect to t.
8. False. The binormal vector, by the definition given in Section 14.3 [ET 13.3], is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -[\mathbf{N}(t) \times \mathbf{T}(t)]$.

9. True. See the discussion preceding Example 7 in Section 14.3 [ET 13.3].

10. False. For example, $\mathbf{r}_1(t) = \langle t, t \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t \rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}_1'(t) = \langle 1, 1 \rangle$ for all t, while $\mathbf{r}_2'(t) = \langle 2, 2 \rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

EXERCISES

1. (a) Since $x = 2$ and $y^2 + z^2 = 1$, the curve is a circle in the plane $x = 2$ with center $(2, 0, 0)$ and radius 1.



(b) $\mathbf{r}'(t) = \cos t\,\mathbf{j} - \sin t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}''(t) = -\sin t\,\mathbf{j} - \cos t\,\mathbf{k}$

2. $\mathbf{r}(t) = \langle t^3, \sqrt{t+2}, (\sin t)/t \rangle$

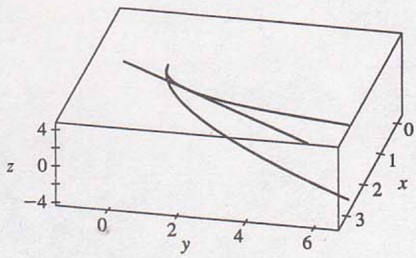
(a) The expressions t^3, $\sqrt{t+2}$, and $(\sin t)/t$ are all defined when $t + 2 \geq 0 \;\Rightarrow\; t \geq -2$ and $t \neq 0$, so the domain is $[-2, 0) \cup (0, \infty)$.

(b) $\lim\limits_{t \to 0} \mathbf{r}(t) = \left\langle \lim\limits_{t \to 0} t^3, \lim\limits_{t \to 0} \sqrt{t+2}, \lim\limits_{t \to 0} \dfrac{\sin t}{t} \right\rangle = \langle 0, \sqrt{2}, 1 \rangle$ (using Equation 3.5.2 [ET 3.4.2] for the z-component).

(c) $\mathbf{r}'(t) = \left\langle \dfrac{d}{dt} t^3, \dfrac{d}{dt}\sqrt{t+2}, \dfrac{d}{dt}\dfrac{\sin t}{t} \right\rangle = \left\langle 3t^2, \dfrac{1}{2\sqrt{t+2}}, \dfrac{t\cos t - \sin t}{t^2} \right\rangle$

3. The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 16$, $z = 0$. So we can write $x = 4\cos t$, $y = 4\sin t$, $0 \leq t \leq 2\pi$. From the equation of the plane, we have $z = 5 - x = 5 - 4\cos t$, so parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $z = 5 - 4\cos t$, $0 \leq t \leq 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (5 - 4\cos t)\,\mathbf{k}$, $0 \leq t \leq 2\pi$.

4. The curve is given by $\mathbf{r}(t) = \langle t^2, t^4, t^3 \rangle$, so $\mathbf{r}'(t) = \langle 2t, 4t^3, 3t^2 \rangle$. The point $(1, 1, 1)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 2, 4, 3 \rangle$. Then the tangent line has direction vector $\langle 2, 4, 3 \rangle$ and includes the point $(1, 1, 1)$, so parametric equations are $x = 1 + 2t$, $y = 1 + 4t$, $z = 1 + 3t$.



5. $\int_0^1 \left[(t + t^2)\,\mathbf{i} + (2 + t^3)\,\mathbf{j} + t^4\,\mathbf{k}\right] dt = \left[\left(\tfrac{1}{2}t^2 + \tfrac{1}{3}t^3\right)\mathbf{i} + \left(2t + \tfrac{1}{4}t^4\right)\mathbf{j} + \left(\tfrac{1}{5}t^5\right)\mathbf{k}\right]_0^1 = \tfrac{5}{6}\,\mathbf{i} + \tfrac{9}{4}\,\mathbf{j} + \tfrac{1}{5}\,\mathbf{k}$

6. (a) C intersects the xz-plane where $y = 0 \;\Rightarrow\; 2t - 1 = 0 \;\Rightarrow\; t = \tfrac{1}{2}$, so the point is $\left(2 - \left(\tfrac{1}{2}\right)^3, 0, \ln\tfrac{1}{2}\right) = \left(\tfrac{15}{8}, 0, -\ln 2\right)$.

(b) The curve is given by $\mathbf{r}(t) = \langle 2 - t^3, 2t - 1, \ln t \rangle$, so $\mathbf{r}'(t) = \langle -3t^2, 2, 1/t \rangle$. The point $(1, 1, 0)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$. Then the tangent line has direction vector $\langle -3, 2, 1 \rangle$ and includes the point $(1, 1, 0)$, so parametric equations are $x = 1 - 3t$, $y = 1 + 2t$, $z = t$.

(c) The normal plane has normal vector $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$ and equation $-3(x - 1) + 2(y - 1) + z = 0$ or $3x - 2y - z = 1$.

7. $t = 1$ at $(1, 4, 2)$ and $t = 4$ at $(2, 1, 17)$, so

$$L = \int_1^4 \sqrt{\frac{1}{4t} + \frac{16}{t^4} + 4t^2}\, dt$$
$$\approx \frac{4-1}{3 \cdot 4} \left[\sqrt{\tfrac{1}{4} + 16 + 4} + 4 \cdot \sqrt{\frac{1}{4 \cdot \frac{7}{4}} + \frac{16}{\left(\frac{7}{4}\right)^4} + 4\left(\tfrac{7}{4}\right)^2} + 2 \cdot \sqrt{\frac{1}{4 \cdot \frac{10}{4}} + \frac{16}{\left(\frac{10}{4}\right)^4} + 4\left(\tfrac{10}{4}\right)^2} \right.$$
$$\left. + 4 \cdot \sqrt{\frac{1}{4 \cdot \frac{13}{4}} + \frac{16}{\left(\frac{13}{4}\right)^4} + 4\left(\tfrac{13}{4}\right)^2} + \sqrt{\frac{1}{4 \cdot 4} + \frac{16}{4^4} + 4 \cdot 4^2} \right]$$
$$\approx 15.9241$$

8. $\mathbf{r}'(t) = \left\langle 3t^{1/2}, -2\sin 2t, 2\cos 2t \right\rangle$, $|\mathbf{r}'(t)| = \sqrt{9t + 4\left(\sin^2 2t + \cos^2 2t\right)} = \sqrt{9t+4}$. Thus
$L = \int_0^1 \sqrt{9t+4}\, dt = \int_4^{13} \frac{1}{9} u^{1/2}\, du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big]_4^{13} = \frac{2}{27}\left(13^{3/2} - 8\right)$.

9. The angle of intersection of the two curves, θ, is the angle between their respective tangents at the point of intersection. For both curves the point $(1, 0, 0)$ occurs when $t = 0$. $\mathbf{r}_1'(t) = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_1'(0) = \mathbf{j} + \mathbf{k}$ and $\mathbf{r}_2'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \Rightarrow \mathbf{r}_2'(0) = \mathbf{i}$. $\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0) = (\mathbf{j} + \mathbf{k}) \cdot \mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = \frac{\pi}{2}$.

10. The parametric value corresponding to the point $(1, 0, 1)$ is $t = 0$.
$\mathbf{r}'(t) = e^t\,\mathbf{i} + e^t(\cos t + \sin t)\,\mathbf{j} + e^t(\cos t - \sin t)\,\mathbf{k}$
$\Rightarrow \quad |\mathbf{r}'(t)| = e^t\sqrt{1 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{3}e^t$
and $s(t) = \int_0^t e^u \sqrt{3}\, du = \sqrt{3}\left(e^t - 1\right) \Rightarrow t = \ln\left(1 + \frac{1}{\sqrt{3}}s\right)$. Therefore,
$\mathbf{r}(t(s)) = \left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{i} + \left(1 + \frac{1}{\sqrt{3}}s\right)\sin\ln\left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{j} + \left(1 + \frac{1}{\sqrt{3}}s\right)\cos\ln\left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{k}$.

11. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle t^2, t, 1\rangle}{|\langle t^2, t, 1\rangle|} = \dfrac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}$

(b) $\mathbf{T}'(t) = -\frac{1}{2}\left(t^4 + t^2 + 1\right)^{-3/2}\left(4t^3 + 2t\right)\langle t^2, t, 1\rangle + \left(t^4 + t^2 + 1\right)^{-1/2}\langle 2t, 1, 0\rangle$

$$= \frac{-2t^3 - t}{\left(t^4 + t^2 + 1\right)^{3/2}}\langle t^2, t, 1\rangle + \frac{1}{\left(t^4 + t^2 + 1\right)^{1/2}}\langle 2t, 1, 0\rangle$$

$$= \frac{\left\langle -2t^5 - t^3, -2t^4 - t^2, -2t^3 - t\right\rangle + \left\langle 2t^5 + 2t^3 + 2t, t^4 + t^2 + 1, 0\right\rangle}{\left(t^4 + t^2 + 1\right)^{3/2}}$$

$$= \frac{\left\langle 2t, -t^4 + 1, -2t^3 - t\right\rangle}{\left(t^4 + t^2 + 1\right)^{3/2}}$$

$|\mathbf{T}'(t)| = \dfrac{\sqrt{4t^2 + t^8 - 2t^4 + 1 + 4t^6 + 4t^4 + t^2}}{\left(t^4 + t^2 + 1\right)^{3/2}} = \dfrac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{\left(t^4 + t^2 + 1\right)^{3/2}}$, and

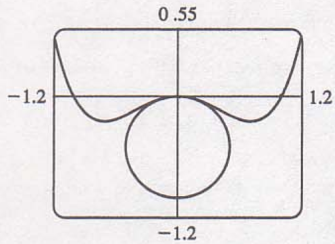
$\mathbf{N}(t) = \dfrac{\left\langle 2t, 1 - t^4, -2t^3 - t\right\rangle}{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}$.

(c) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{\left(t^4 + t^2 + 1\right)^2}$

12. Using Exercise 14.3.30 [ET 13.3.30], we have $\mathbf{r}'(t) = \langle -3\sin t, 4\cos t\rangle$, $\mathbf{r}''(t) = \langle -3\cos t, -4\sin t\rangle$,
$|\mathbf{r}'(t)|^3 = \left(\sqrt{9\sin^2 t + 4\cos^2 t}\right)^3$ and then
$$\kappa(t) = \frac{|(-3\sin t)(-4\sin t) - (-3\cos t)(4\cos t)|}{(9\sin^2 t + 16\cos^2 t)^{3/2}} = \frac{12}{(9\sin^2 t + 16\cos^2 t)^{3/2}}.$$
At $(3,0)$, $t = 0$ and $\kappa(0) = 12/(16)^{3/2} = \frac{12}{64} = \frac{3}{16}$. At $(0,4)$, $t = \frac{\pi}{2}$ and $\kappa\left(\frac{\pi}{2}\right) = 12/9^{3/2} = \frac{12}{27} = \frac{4}{9}$.

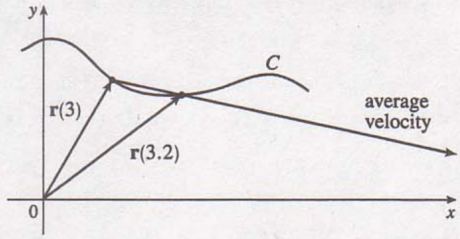
13. $y' = 4x^3$, $y'' = 12x^2$ and $\kappa(x) = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \dfrac{|12x^2|}{(1 + 16x^6)^{3/2}}$, so $\kappa(1) = \dfrac{12}{(17)^{3/2}}$.

14. $\kappa(x) = \dfrac{|12x^2 - 2|}{\left[1 + (4x^3 - 2x)^2\right]^{3/2}} \quad \Rightarrow \quad \kappa(0) = 2$. So the osculating circle has radius $\frac{1}{2}$ and center $\left(0, -\frac{1}{2}\right)$. Thus its equation is $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$.



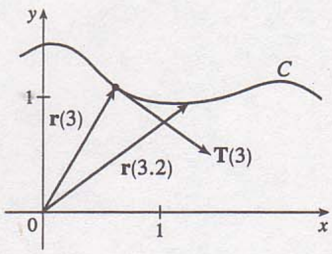
15. $\mathbf{r}(t) = \langle \sin 2t, t, \cos 2t\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 2\cos 2t, 1, -2\sin 2t\rangle \quad \Rightarrow \quad \mathbf{T}(t) = \frac{1}{\sqrt{5}}\langle 2\cos 2t, 1, -2\sin 2t\rangle \quad \Rightarrow$
$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -4\sin 2t, 0, -4\cos 2t\rangle \quad \Rightarrow \quad \mathbf{N}(t) = \langle -\sin 2t, 0, -\cos 2t\rangle$. So $\mathbf{N} = \mathbf{N}(\pi) = \langle 0, 0, -1\rangle$ and
$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}}\langle -1, 2, 0\rangle$. So a normal to the osculating plane is $\langle -1, 2, 0\rangle$ and an equation is
$-1(x - 0) + 2(y - \pi) + 0(z - 1) = 0$ or $x - 2y + 2\pi = 0$.

16. (a) The average velocity over $[3, 3.2]$ is given by
$\dfrac{\mathbf{r}(3.2) - \mathbf{r}(3)}{3.2 - 3} = 5\,[\mathbf{r}(3.2) - \mathbf{r}(3)]$, so we draw a vector with the same direction but 5 times the length of the vector $[\mathbf{r}(3.2) - \mathbf{r}(3)]$.



(b) $\mathbf{v}(3) = \mathbf{r}'(3) = \displaystyle\lim_{h \to 0} \frac{\mathbf{r}(3 + h) - \mathbf{r}(3)}{h}$.

(c) $\mathbf{T}(3) = \dfrac{\mathbf{r}'(3)}{|\mathbf{r}'(3)|}$, a unit vector in the same direction as $\mathbf{r}'(3)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(3)$, pointing in the direction corresponding to increasing t, and with length 1.



17. $\mathbf{v}(t) = 2\sqrt{2}\,\mathbf{i} + 2e^{2t}\,\mathbf{j} - 2e^{-2t}\,\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{8 + 4e^{4t} + 4e^{-4t}} = 2\left(e^{2t} + e^{-2t}\right)$, $\mathbf{a}(t) = 4e^{2t}\,\mathbf{j} + 4e^{-2t}\,\mathbf{k}$

18. $\mathbf{v}(t) = \int \left(t\,\mathbf{i} + \mathbf{j} + t^2\,\mathbf{k}\right) dt = \frac{1}{2}t^2\,\mathbf{i} + t\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{c}_1$, but $\mathbf{i} + 2\mathbf{j} + \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so
$\mathbf{c}_1 = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v}(t) = \left(1 + \frac{1}{2}t^2\right)\mathbf{i} + (2 + t)\,\mathbf{j} + \left(1 + \frac{1}{3}t^3\right)\mathbf{k}$.
$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \left(t + \frac{1}{6}t^3\right)\mathbf{i} + \left(2t + \frac{1}{2}t^2\right)\mathbf{j} + \left(t + \frac{1}{12}t^4\right)\mathbf{k} + \mathbf{c}_2$. But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{c}_2 = \mathbf{0}$ and
$\mathbf{r}(t) = \left(t + \frac{1}{6}t^3\right)\mathbf{i} + \left(2t + \frac{1}{2}t^2\right)\mathbf{j} + \left(t + \frac{1}{12}t^4\right)\mathbf{k}$.

19. We set up the axes so that the shot leaves the athlete's hand 7 ft above the origin. Then we are given $\mathbf{r}(0) = 7\mathbf{j}$, $|\mathbf{v}(0)| = 43$ ft/s, and $\mathbf{v}(0)$ has direction given by a $45°$ angle of elevation. Then a unit vector in the direction of $\mathbf{v}(0)$ is $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \Rightarrow \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$. Assuming air resistance is negligible, the only external force is due to gravity, so as in Example 14.4.5 [ET 13.4.5] we have $\mathbf{a} = -g\,\mathbf{j}$ where here $g \approx 32$ ft/s^2. Since $\mathbf{v}'(t) = \mathbf{a}(t)$, we integrate, giving $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$ where $\mathbf{C} = \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \Rightarrow \mathbf{v}(t) = \frac{43}{\sqrt{2}}\,\mathbf{i} + \left(\frac{43}{\sqrt{2}} - gt\right)\mathbf{j}$. Since $\mathbf{r}'(t) = \mathbf{v}(t)$ we integrate again, so $\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{D} = \mathbf{r}(0) = 7\mathbf{j} \Rightarrow \mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7\right)\mathbf{j}$.

(a) At 2 seconds, the shot is at $\mathbf{r}(2) = \frac{43}{\sqrt{2}}(2)\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 7\right)\mathbf{j} \approx 60.8\,\mathbf{i} + 3.8\,\mathbf{j}$, so the shot is about 3.8 ft above the ground, at a horizontal distance of 60.8 ft from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{43}{\sqrt{2}} - gt = 0 \Rightarrow$ $t = \dfrac{43}{\sqrt{2}g} \approx 0.95$ s. Then $\mathbf{r}(0.95) \approx 28.9\,\mathbf{i} + 21.4\,\mathbf{j}$, so the maximum height is approximately 21.4 ft.

(c) The shot hits the ground when the vertical component of $\mathbf{r}(t)$ is 0, so $\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7 = 0 \Rightarrow$ $-16t^2 + \frac{43}{\sqrt{2}}t + 7 = 0 \Rightarrow t \approx 2.11$ s. $\mathbf{r}(2.11) \approx 64.2\,\mathbf{i} - 0.08\,\mathbf{j}$, thus the shot lands approximately 64.2 ft from the athlete.

20. $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{r}''(t) = 2\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1+4+4t^2} = \sqrt{4t^2+5}$.

Then $a_T = \dfrac{4t}{\sqrt{4t^2+5}}$ and $a_N = \dfrac{|4\mathbf{i} - 2\mathbf{j}|}{\sqrt{4t^2+5}} = \dfrac{2\sqrt{5}}{\sqrt{4t^2+5}}$.

21. (a) Instead of proceeding directly, we use Formula 3 of Theorem 14.2.3 [ET 13.2.3]: $\mathbf{r}(t) = t\,\mathbf{R}(t) \Rightarrow$ $\mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\,\mathbf{R}'(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j} + t\,\mathbf{v}_d$.

(b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\,\mathbf{R}'(t)$, we have $\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{v}_d + t\,\mathbf{a}_d$.

(c) Here we have $\mathbf{r}(t) = e^{-t}\cos\omega t\,\mathbf{i} + e^{-t}\sin\omega t\,\mathbf{j} = e^{-t}\,\mathbf{R}(t)$. So, as in parts (a) and (b),

$$\mathbf{v} = \mathbf{r}'(t) = e^{-t}\,\mathbf{R}'(t) - e^{-t}\,\mathbf{R}(t) = e^{-t}\left[\mathbf{R}'(t) - \mathbf{R}(t)\right] \Rightarrow$$

$$\begin{aligned}\mathbf{a} = \mathbf{v}' &= e^{-t}\left[\mathbf{R}''(t) - \mathbf{R}'(t)\right] - e^{-t}\left[\mathbf{R}'(t) - \mathbf{R}(t)\right] = e^{-t}\left[\mathbf{R}''(t) - 2\,\mathbf{R}'(t) + \mathbf{R}(t)\right]\\ &= e^{-t}\,\mathbf{a}_d - 2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}\end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving $\mathbf{a}_d$) is $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$.

22. (a) $$F(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -x + \sqrt{2} & \text{if } x \ge \frac{1}{\sqrt{2}} \end{cases} \quad \Rightarrow \quad F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x/\sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -1 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

$$\Rightarrow \quad F''(x) = \begin{cases} 0 & \text{if } x < 0 \\ -1/(1-x^2)^{3/2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

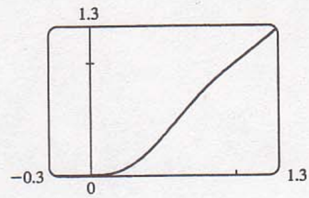
since $\dfrac{d}{dx}\left[-x\left(1-x^2\right)^{-1/2}\right] = -\left(1-x^2\right)^{-1/2} - x^2\left(1-x^2\right)^{-3/2} = -\left(1-x^2\right)^{-3/2}$.

Now $\lim\limits_{x\to 0^+}\sqrt{1-x^2} = 1 = F(0)$ and $\lim\limits_{x\to(1/\sqrt{2})^-}\sqrt{1-x^2} = \frac{1}{\sqrt{2}} = F\left(\frac{1}{\sqrt{2}}\right)$, so F is continuous. Also, since

$\lim\limits_{x\to 0+} F'(x) = 0 = \lim\limits_{x\to 0-} F'(x)$ and $\lim\limits_{x\to(1/\sqrt{2})^-} F'(x) = -1 = \lim\limits_{x\to(1/\sqrt{2})^+} F'(x)$, F' is continuous. But $\lim\limits_{x\to 0+} F''(x) = -1 \neq 0 = \lim\limits_{x\to 0-} F''(x)$, so F'' is not continuous at $x = 0$. (The same is true at $x = \frac{1}{\sqrt{2}}$.) So F does not have continuous curvature.

(b) Set $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. The continuity conditions on P are $P(0) = 0$, $P(1) = 1$, $P'(0) = 0$ and $P'(1) = 1$. Also the curvature must be continuous. For $x \leq 0$ and $x \geq 1$, $\kappa(x) = 0$; elsewhere $\kappa(x) = \dfrac{|P''(x)|}{\left(1 + [P'(x)]^2\right)^{3/2}}$, so we need $P''(0) = 0$ and $P''(1) = 0$.

The conditions $P(0) = P'(0) = P''(0) = 0$ imply that $d = e = f = 0$. The other conditions imply that $a + b + c = 1$, $5a + 4b + 3c = 1$, and $10a + 6b + 3c = 0$. From these, we find that $a = 3$, $b = -8$, and $c = 6$. Therefore $P(x) = 3x^5 - 8x^4 + 6x^3$. Since there was no solution with $a = 0$, this could not have been done with a polynomial of degree 4.



Problems Plus

1. (a) $\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j} \quad\Rightarrow\quad \mathbf{v} = \mathbf{r}'(t) = -\omega R\sin\omega t\,\mathbf{i} + \omega R\cos\omega t\,\mathbf{j}$, so $\mathbf{r} = R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$ and $\mathbf{v} = \omega R(-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j})$. $\mathbf{v}\cdot\mathbf{r} = \omega R^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$, so $\mathbf{v}\perp\mathbf{r}$. Since $\mathbf{r}$ points along a radius of the circle, and $\mathbf{v}\perp\mathbf{r}$, $\mathbf{v}$ is tangent to the circle. Because it is a velocity vector, $\mathbf{v}$ points in the direction of motion.

(b) In (a), we wrote $\mathbf{v}$ in the form $\omega R\,\mathbf{u}$, where $\mathbf{u}$ is the unit vector $-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j}$. Clearly $|\mathbf{v}| = \omega R\,|\mathbf{u}| = \omega R$. At speed ωR, the particle completes one revolution, a distance $2\pi R$, in time $T = \dfrac{2\pi R}{\omega R} = \dfrac{2\pi}{\omega}$.

(c) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\omega^2 R\cos\omega t\,\mathbf{i} - \omega^2 R\sin\omega t\,\mathbf{j} = -\omega^2 R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$, so $\mathbf{a} = -\omega^2\mathbf{r}$. This shows that $\mathbf{a}$ is proportional to $\mathbf{r}$ and points in the opposite direction (toward the origin). Also, $|\mathbf{a}| = \omega^2\,|\mathbf{r}| = \omega^2 R$.

(d) By Newton's Second Law (see Section 14.4 [ET 13.4]), $\mathbf{F} = m\mathbf{a}$, so
$|\mathbf{F}| = m\,|\mathbf{a}| = mR\omega^2 = \dfrac{m(\omega R)^2}{R} = \dfrac{m\,|\mathbf{v}|^2}{R}$.

2. (a) Dividing the equation $|\mathbf{F}|\sin\theta = \dfrac{mv_R^2}{R}$ by the equation $|\mathbf{F}|\cos\theta = mg$, we obtain $\tan\theta = \dfrac{v_R^2}{Rg}$, so $v_R^2 = Rg\tan\theta$.

(b) $R = 400$ ft and $\theta = 12^\circ$, so $v_R = \sqrt{Rg\tan\theta} \approx \sqrt{400\cdot 32\cdot\tan 12^\circ} \approx 52.16$ ft/s ≈ 36 mi/h.

(c) We want to choose a new radius R_1 for which the new rated speed is $\frac{3}{2}$ of the old one: $\sqrt{R_1 g\tan 12^\circ} = \frac{3}{2}\sqrt{Rg\tan 12^\circ}$. Squaring, we get $R_1 g\tan 12^\circ = \frac{9}{4}Rg\tan 12^\circ$, so $R_1 = \frac{9}{4}R = \frac{9}{4}(400) = 900$ ft.

3. (a) The projectile reaches maximum height when $0 = \dfrac{dy}{dt} = \dfrac{d}{dt}\left[(v_0\sin\alpha)\,t - \frac{1}{2}gt^2\right] = v_0\sin\alpha - gt$; that is, when $t = \dfrac{v_0\sin\alpha}{g}$ and $y = (v_0\sin\alpha)\left(\dfrac{v_0\sin\alpha}{g}\right) - \dfrac{1}{2}g\left(\dfrac{v_0\sin\alpha}{g}\right)^2 = \dfrac{v_0^2\sin^2\alpha}{2g}$. This is the maximum height attained when the projectile is fired with an angle of elevation α. This maximum height is largest when $\alpha = \frac{\pi}{2}$. In that case, $\sin\alpha = 1$ and the maximum height is $\dfrac{v_0^2}{2g}$.

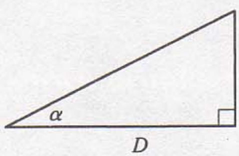
(b) Let $R = v_0^2/g$. We are asked to consider the parabola $x^2 + 2Ry - R^2 = 0$ which can be rewritten as $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. The points on or inside this parabola are those for which $-R \le x \le R$ and $0 \le y \le \dfrac{-1}{2R}x^2 + \dfrac{R}{2}$. When the projectile is fired at angle of elevation α, the points (x, y) along its path satisfy the relations $x = (v_0\cos\alpha)\,t$ and $y = (v_0\sin\alpha)\,t - \frac{1}{2}gt^2$, where $0 \le t \le (2v_0\sin\alpha)/g$ (as in Example 5 in Section 14.4 [ET 13.4]). Thus $|x| \le \left|v_0\cos\alpha\left(\dfrac{2v_0\sin\alpha}{g}\right)\right| = \left|\dfrac{v_0^2}{g}\sin 2\alpha\right| \le \left|\dfrac{v_0^2}{g}\right| = |R|$. This shows that $-R \le x \le R$.

For t in the specified range, we also have $y = t\left(v_0 \sin\alpha - \frac{1}{2}gt\right) = \frac{1}{2}gt\left(\dfrac{2v_0 \sin\alpha}{g} - t\right) \geq 0$ and

$y = (v_0 \sin\alpha)\dfrac{x}{v_0 \cos\alpha} - \dfrac{g}{2}\left(\dfrac{x}{v_0 \cos\alpha}\right)^2 = (\tan\alpha)\,x - \dfrac{g}{2v_0^2 \cos^2\alpha}x^2 = -\dfrac{1}{2R\cos^2\alpha}x^2 + (\tan\alpha)\,x$. Thus

$$\begin{aligned}
y - \left(\frac{-1}{2R}x^2 + \frac{R}{2}\right) &= \frac{-1}{2R\cos^2\alpha}x^2 + \frac{1}{2R}x^2 + (\tan\alpha)\,x - \frac{R}{2} \\
&= \frac{x^2}{2R}\left(1 - \frac{1}{\cos^2\alpha}\right) + (\tan\alpha)\,x - \frac{R}{2} = \frac{x^2\left(1 - \sec^2\alpha\right) + 2R(\tan\alpha)\,x - R^2}{2R} \\
&= \frac{-\left(\tan^2\alpha\right)x^2 + 2R(\tan\alpha)\,x - R^2}{2R} = \frac{-\left[(\tan\alpha)\,x - R\right]^2}{2R} \leq 0
\end{aligned}$$

We have shown that every target that can be hit by the projectile lies on or inside the parabola $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. Now let (a, b) be any point on or inside the parabola $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. Then $-R \leq a \leq R$ and $0 \leq b \leq -\dfrac{1}{2R}a^2 + \dfrac{R}{2}$. We seek an angle α such that (a, b) lies in the path of the projectile; that is, we wish to find an angle α such that $b = -\dfrac{1}{2R\cos^2\alpha}a^2 + (\tan\alpha)\,a$ or equivalently $b = \dfrac{-1}{2R}\left(\tan^2\alpha + 1\right)a^2 + (\tan\alpha)\,a$. Rearranging this equation we get $\dfrac{a^2}{2R}\tan^2\alpha - a\tan\alpha + \left(\dfrac{a^2}{2R} + b\right) = 0$ or $a^2(\tan\alpha)^2 - 2aR(\tan\alpha) + \left(a^2 + 2bR\right) = 0$ $(\bigstar)$. This quadratic equation for $\tan\alpha$ has real solutions exactly when the discriminant is nonnegative. Now $B^2 - 4AC \geq 0 \;\Leftrightarrow\; (-2aR)^2 - 4a^2\left(a^2 + 2bR\right) \geq 0 \;\Leftrightarrow\; 4a^2\left(R^2 - a^2 - 2bR\right) \geq 0 \;\Leftrightarrow\; -a^2 - 2bR + R^2 \geq 0 \;\Leftrightarrow\; b \leq \dfrac{1}{2R}\left(R^2 - a^2\right) \;\Leftrightarrow\; b \leq \dfrac{-1}{2R}a^2 + \dfrac{R}{2}$. This condition is satisfied since (a, b) is on or inside the parabola $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. It follows that (a, b) lies in the path of the projectile when $\tan\alpha$ satisfies $(\bigstar)$, that is, when $\tan\alpha = \dfrac{2aR \pm \sqrt{4a^2\left(R^2 - a^2 - 2bR\right)}}{2a^2} = \dfrac{R \pm \sqrt{R^2 - 2bR - a^2}}{a}$.

(c) 

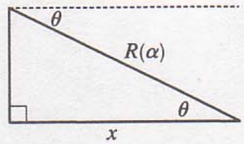
If the gun is pointed at a target with height h at a distance D downrange, then $\tan\alpha = h/D$. When the projectile reaches a distance D downrange (remember we are assuming that it doesn't hit the ground first), we have $D = x = (v_0 \cos\alpha)\,t$, so $t = \dfrac{D}{v_0 \cos\alpha}$ and $y = (v_0 \sin\alpha)\,t - \frac{1}{2}gt^2 = D\tan\alpha - \dfrac{gD^2}{2v_0^2 \cos^2\alpha}$. Meanwhile, the target, whose x-coordinate is also D, has fallen from height h to height $h - \frac{1}{2}gt^2 = D\tan\alpha - \dfrac{gD^2}{2v_0^2 \cos^2\alpha}$. Thus the projectile hits the target.

4. (a) As in Problem 3, $\mathbf{r}(t) = (v_0 \cos\alpha)\, t\, \mathbf{i} + \left[(v_0 \sin\alpha)\, t - \frac{1}{2}gt^2\right] \mathbf{j}$, so $x = (v_0 \cos\alpha)\, t$ and $y = (v_0 \sin\alpha)\, t - \frac{1}{2}gt^2$. The difference here is that the projectile travels until it reaches a point where $x > 0$ and $y = -(\tan\theta)\, x$ (Here $0 \le \theta \le \frac{\pi}{2}$.) From the parametric equations, we obtain $t = \dfrac{x}{v_0 \cos\alpha}$ and

$$y = \frac{(v_0 \sin\alpha)\, x}{v_0 \cos\alpha} - \frac{gx^2}{2v_0^2 \cos^2\alpha} = (\tan\alpha)\, x - \frac{gx^2}{2v_0^2 \cos^2\alpha}.$$

Thus the projectile hits the inclined plane at the point where $(\tan\alpha)\, x - \dfrac{gx^2}{2v_0^2 \cos^2\alpha} = -(\tan\theta)\, x$. Since $\dfrac{gx^2}{2v_0^2 \cos^2\alpha} = (\tan\alpha + \tan\theta)\, x$ and $x > 0$, we must have $\dfrac{gx}{2v_0^2 \cos^2\alpha} = \tan\alpha + \tan\theta$. It follows that $x = \dfrac{2v_0^2 \cos^2\alpha}{g}(\tan\alpha + \tan\theta)$ and $t = \dfrac{x}{v_0 \cos\alpha} = \dfrac{2v_0 \cos\alpha}{g}(\tan\alpha + \tan\theta)$. This means that the parametric equations are defined for t in the interval $\left[0, \dfrac{2v_0 \cos\alpha}{g}(\tan\alpha + \tan\theta)\right]$.

(b) The downhill range (that is, the distance to the projectile's landing point as measured along the inclined plane) is $R(\alpha) = x \sec\theta$, where x is the coordinate of the landing point calculated in part (a). Thus



$$\begin{aligned} R(\alpha) &= \frac{2v_0^2 \cos^2\alpha}{g}(\tan\alpha + \tan\theta) \sec\theta = \frac{2v_0^2}{g}\left(\frac{\sin\alpha\cos\alpha}{\cos\theta} + \frac{\cos^2\alpha \sin\theta}{\cos^2\theta}\right) \\ &= \frac{2v_0^2 \cos\alpha}{g\cos^2\theta}(\sin\alpha\cos\theta + \cos\alpha\sin\theta) = \frac{2v_0^2 \cos\alpha \sin(\alpha+\theta)}{g\cos^2\theta} \end{aligned}$$

$R(\alpha)$ is maximized when

$$\begin{aligned} 0 = R'(\alpha) &= \frac{2v_0^2}{g\cos^2\theta}\left[-\sin\alpha\sin(\alpha+\theta) + \cos\alpha\cos(\alpha+\theta)\right] \\ &= \frac{2v_0^2}{g\cos^2\theta}\cos\left[(\alpha+\theta)+\alpha\right] = \frac{2v_0^2\cos(2\alpha+\theta)}{g\cos^2\theta} \end{aligned}$$

This condition implies that $\cos(2\alpha+\theta) = 0 \quad\Rightarrow\quad 2\alpha+\theta = \frac{\pi}{2} \quad\Rightarrow\quad \alpha = \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)$.

(c) The solution is similar to the solutions to parts (a) and (b). This time the projectile travels until it reaches a point where $x > 0$ and $y = (\tan\theta)\, x$. Since $\tan\theta = -\tan(-\theta)$, we obtain the solution from the previous one by replacing θ with $-\theta$. The desired angle is $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)$.

5. (a) $m\dfrac{d^2\mathbf{R}}{dt^2} = -mg\,\mathbf{j} - k\dfrac{d\mathbf{R}}{dt} \quad\Rightarrow\quad \dfrac{d}{dt}\left(m\dfrac{d\mathbf{R}}{dt} + k\,\mathbf{R} + mgt\,\mathbf{j}\right) = \mathbf{0} \quad\Rightarrow\quad m\dfrac{d\mathbf{R}}{dt} + k\mathbf{R} + mgt\,\mathbf{j} = \mathbf{c}$ ($\mathbf{c}$ is a constant vector in the xy-plane). At $t = 0$, this says that $m\mathbf{v}(0) + k\mathbf{R}(0) = \mathbf{c}$. Since $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{R}(0) = \mathbf{0}$, we have $\mathbf{c} = m\mathbf{v}_0$. Therefore $\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}\mathbf{R} + gt\,\mathbf{j} = \mathbf{v}_0$, or $\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}\mathbf{R} = \mathbf{v}_0 - gt\,\mathbf{j}$.

(b) Multiplying by $e^{(k/m)t}$ gives $e^{(k/m)t}\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}e^{(k/m)t}\mathbf{R} = e^{(k/m)t}\mathbf{v}_0 - gte^{(k/m)t}\,\mathbf{j}$ or $\dfrac{d}{dt}\left(e^{(k/m)t}\mathbf{R}\right) = e^{(k/m)t}\mathbf{v}_0 - gte^{(k/m)t}\,\mathbf{j}$. Integrating gives $e^{(k/m)t}\mathbf{R} = \dfrac{m}{k}e^{(k/m)t}\mathbf{v}_0 - \left[\dfrac{mg}{k}te^{(k/m)t} - \dfrac{m^2g}{k^2}e^{(k/m)t}\right]\mathbf{j} + \mathbf{b}$ for some constant vector $\mathbf{b}$. Setting $t = 0$

yields the relation $\mathbf{R}(0) = \frac{m}{k}\mathbf{v}_0 + \frac{m^2 g}{k^2}\mathbf{j} + \mathbf{b}$, so $\mathbf{b} = -\frac{m}{k}\mathbf{v}_0 - \frac{m^2 g}{k^2}\mathbf{j}$. Thus

$e^{(k/m)t}\mathbf{R} = \frac{m}{k}\left[e^{(k/m)t} - 1\right]\mathbf{v}_0 - \left[\frac{mg}{k}te^{(k/m)t} - \frac{m^2 g}{k^2}\left(e^{(k/m)t} - 1\right)\right]\mathbf{j}$ and

$\mathbf{R}(t) = \frac{m}{k}\left[1 - e^{-kt/m}\right]\mathbf{v}_0 + \frac{mg}{k}\left[\frac{m}{k}\left(1 - e^{-kt/m}\right) - t\right]\mathbf{j}$.

6. By the Fundamental Theorem of Calculus, $\mathbf{r}'(t) = \langle \sin(\pi t^2/2), \cos(\pi t^2/2)\rangle$, $|\mathbf{r}'(t)| = 1$ and so $\mathbf{T}(t) = \mathbf{r}'(t)$. Thus $\mathbf{T}'(t) = \pi t\langle \sin(\pi t^2/2), \cos(\pi t^2/2)\rangle$ and the curvature is $\kappa = |\mathbf{T}'(t)| = \sqrt{(\pi t)^2(1)} = \pi|t|$.

7. (a) $\mathbf{a} = -g\mathbf{j} \;\Rightarrow\; \mathbf{v} = \mathbf{v}_0 - gt\mathbf{j} = 2\mathbf{i} - gt\mathbf{j} \;\Rightarrow\; \mathbf{s} = \mathbf{s}_0 + 2t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} = 3.5\mathbf{j} + 2t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} \;\Rightarrow$ $\mathbf{s} = 2t\mathbf{i} + \left(3.5 - \frac{1}{2}gt^2\right)\mathbf{j}$. Therefore $y = 0$ when $t = \sqrt{7/g}$ seconds. At that instant, the ball is $2\sqrt{7/g} \approx 0.94$ ft to the right of the table top. Its coordinates (relative to an origin on the floor directly under the table's edge) are $(0.94, 0)$. At impact, the velocity is $\mathbf{v} = 2\mathbf{i} - \sqrt{7g}\,\mathbf{j}$, so the speed is $|\mathbf{v}| = \sqrt{4 + 7g} \approx 15$ ft/s.

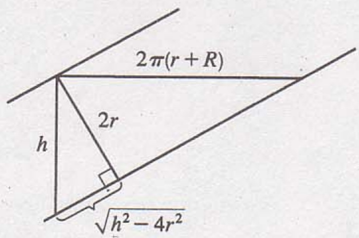
(b) The slope of the curve when $t = \sqrt{\frac{7}{g}}$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-gt}{2} = \frac{-g\sqrt{7/g}}{2} = \frac{-\sqrt{7g}}{2}$. Thus $\cot\theta = \sqrt{7g}/2$ and $\theta \approx 7.6°$.

(c) From (a), $|\mathbf{v}| = \sqrt{4 + 7g}$. So the ball rebounds with speed $0.8\sqrt{4 + 7g} \approx 12.08$ ft/s at angle of inclination $90° - \theta \approx 82.3886°$. By Example 14.4.5 [ET 13.4.5], the horizontal distance traveled between bounces is $d = \frac{v_0^2 \sin 2\alpha}{g}$, where $v_0 \approx 12.08$ ft/s and $\alpha \approx 82.3886°$. Therefore, $d \approx 1.197$ ft. So the ball strikes the floor at about $2\sqrt{7/g} + 1.197 \approx 2.13$ ft to the right of the table's edge.

8. As the cable is wrapped around the spool, think of the top or bottom of the cable forming a helix of radius $R + r$. Let h be the vertical distance between coils. Then, from similar triangles,

$$\frac{2r}{\sqrt{h^2 - 4r^2}} = \frac{2\pi(r + R)}{h} \;\Rightarrow\; h^2r^2 = \pi^2(r + R)^2(h^2 - 4r^2)$$

$$\Rightarrow\; h = \frac{2\pi r(r + R)}{\sqrt{\pi^2(r + R)^2 - r^2}}.$$



If we parametrize the helix by $x(t) = (R + r)\cos t$, $y(t) = (R + r)\sin t$, then we must have $z(t) = [h/(2\pi)]\,t$. The length of one complete cycle is

$$\ell = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \int_0^{2\pi} \sqrt{(R + r)^2 + \left(\frac{h}{2\pi}\right)^2}\,dt = 2\pi\sqrt{(R + r)^2 + \left(\frac{h}{2\pi}\right)^2}$$

$$= 2\pi\sqrt{(R + r)^2 + \frac{r^2(R + r)^2}{\pi^2(R + r)^2 - r^2}} = 2\pi(R + r)\sqrt{1 + \frac{r^2}{\pi^2(R + r)^2 - r^2}} = \frac{2\pi^2(R + r)^2}{\sqrt{\pi^2(R + r)^2 - r^2}}$$

The number of complete cycles is $[\![L/\ell]\!]$, and so the shortest length along the spool is

$$h\left[\!\!\left[\frac{L}{\ell}\right]\!\!\right] = \frac{2\pi r(R + r)}{\sqrt{\pi^2(R + r)^2 - r^2}}\left[\!\!\left[\frac{L\sqrt{\pi^2(R + r)^2 - r^2}}{2\pi^2(R + r)^2}\right]\!\!\right]$$

15 Partial Derivatives ET 14

15.1 Functions of Several Variables ET 14.1

1. (a) From Table 1, $f(8,60) = -7$, which means that if the temperature is $8\,^\circ\mathrm{C}$ and the wind speed is 60 km/h, then the air would feel equivalent to approximately $-7\,^\circ\mathrm{C}$ without wind.

(b) The question is asking: when the temperature is $-12\,^\circ\mathrm{C}$, what wind speed gives a wind-chill index of $-26\,^\circ\mathrm{C}$? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 80 km/h, what temperature gives a wind-chill index of $-14\,^\circ\mathrm{C}$? From Table 1, the temperature is $4\,^\circ\mathrm{C}$.

(d) The function $I = f(-4, v)$ means that we fix T at -4 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-4\,^\circ\mathrm{C}$. From Table 1 (look at the row corresponding to $T = -4$), the function decreases and appears to approach a constant value as v increases.

(e) The function $I = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h. From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

2. (a) From the table, $f(95, 70) = 124$, which means that when the actual temperature is $95\,^\circ\mathrm{F}$ and the relative humidity is 70%, the perceived air temperature is approximately $124\,^\circ\mathrm{F}$.

(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.

(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.

(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is $80\,^\circ\mathrm{F}$. Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is $100\,^\circ\mathrm{F}$. Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately $1\,^\circ\mathrm{F}$ per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of $5\,^\circ\mathrm{F}$ per 10% relative humidity) and at an increasing rate (approximately $12\,^\circ\mathrm{F}$ per 10% relative humidity for larger values of h).

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$P(2L, 2K) = 1.01\,(2L)^{0.75}\,(2K)^{0.25} = 1.01\,(2^{0.75})\,(2^{0.25})\,L^{0.75}K^{0.25} = (2^1)\,1.01L^{0.75}K^{0.25}$$
$$= 2P(L, K)$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^\alpha K^{1-\alpha}$:
$P(2L, 2K) = b\,(2L)^\alpha\,(2K)^{1-\alpha} = b\,(2^\alpha)\,(2^{1-\alpha})\,L^\alpha K^{1-\alpha} = (2^{\alpha+1-\alpha})\,bL^\alpha K^{1-\alpha} = 2P(L, K)$.

4. We compare the values for the temperature-humidity index given by Table 3 with those given by the model function:

Modeled Humidex Values $I(T, h)$

Relative humidity (%)

Actual temperature (°F) T \ h	20	30	40	50	60	70
80	78.6	79.2	79.9	80.8	81.8	83.0
85	82.0	82.9	84.3	86.5	89.3	92.7
90	86.3	87.9	90.7	94.6	99.7	105.9
95	91.5	94.4	99.0	105.2	113.1	122.6
100	97.5	102.3	109.3	118.3	129.5	142.8

Although the model is not a perfect fit, the values it gives appear to be fairly close to the values in Table 3, usually within 2 °F. For most purposes, the numerical representation of the function is much more convenient.

5. (a) According to the table, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

6. (a) $f(1, 1) = \ln(1 + 1 - 1) = \ln 1 = 0$

(b) $f(e, 1) = \ln(e + 1 - 1) = \ln e = 1$

(c) $\ln(x + y - 1)$ is defined only when $x + y - 1 > 0$, that is, $y > 1 - x$. So the domain of f is $\{(x, y) \mid y > 1 - x\}$.

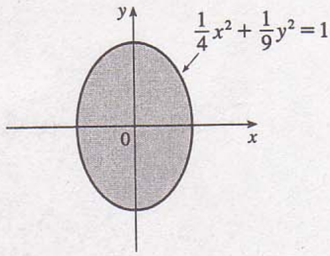
(d) Since $\ln(x + y - 1)$ can be any real number, the range is $\mathbb{R}$.

7. (a) $f(2, 4) = e^{2^2 - 4} = e^0 = 1$.

(b) The exponential function is defined everywhere, so no matter what values of x and y we use, $e^{x^2 - y}$ is defined. So the domain of f is $\mathbb{R}^2$.

(c) Because the range of $g(x, y) = x^2 - y$ is $\mathbb{R}$, and the range of e^x is $(0, \infty)$, the range of $e^{g(x,y)} = e^{x^2 - y}$ is $\{z \mid z > 0\}$.

8. (a) $g(1,2) = \sqrt{36 - 9(1)^2 - 4(2)^2} = \sqrt{11}$



(b) For the square root to be defined, we need $36 - 9x^2 - 4y^2 \geq 0$ or $\frac{1}{4}x^2 + \frac{1}{9}y^2 \leq 1$. Thus the domain is $\{(x,y) \mid \frac{1}{4}x^2 + \frac{1}{9}y^2 \leq 1\}$, the points on or inside the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

(c) Since $0 \leq \sqrt{36 - 9x^2 - 4y^2} \leq 6$, the range is $\{z \mid 0 \leq z \leq 6\}$.

9. (a) $f(3,6,4) = 3^2 \ln(3 - 6 + 4) = 9 \ln 1 = 0$.

(b) For the logarithmic function to be defined, we need $x - y + z > 0$. Thus the domain of f is $\{(x,y,z) \mid x + z > y\}$.

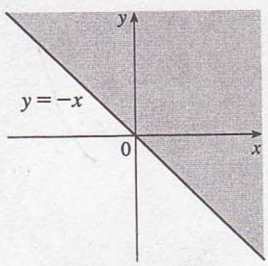
(c) Since $x^2 \ln(x - y + z)$ can be any real number, the range of f is $\mathbb{R}$.

10. (a) $f(1,3,-4) = 1 \Big/ \sqrt{1^2 + 3^2 + (-4)^2 - 1} = \frac{1}{\sqrt{25}} = \frac{1}{5}$.

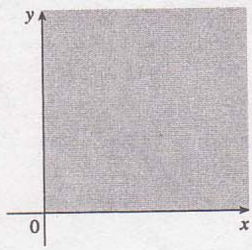
(b) The domain of f is $\{(x,y,z) \mid x^2 + y^2 + z^2 > 1\}$, the exterior of the sphere $x^2 + y^2 + z^2 = 1$.

(c) Since $\sqrt{x^2 + y^2 + z^2 - 1} > 0$, the range of f is $(0, \infty)$.

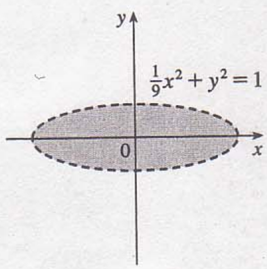
11. $\sqrt{x+y}$ is defined only when $x + y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x,y) \mid y \geq -x\}$.



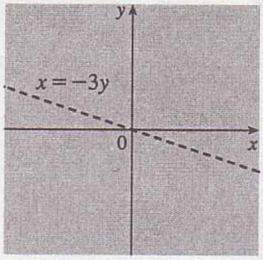
12. We need $x \geq 0$ and $y \geq 0$, so $D = \{(x,y) \mid x \geq 0 \text{ and } y \geq 0\}$, the first quadrant.



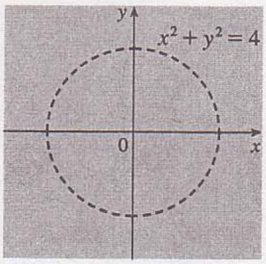
13. $\ln(9 - x^2 - 9y^2)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9}x^2 + y^2 < 1$. So the domain of f is $\{(x,y) \mid \frac{1}{9}x^2 + y^2 < 1\}$, the interior of an ellipse.



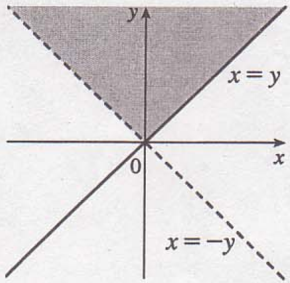
14. $\dfrac{x-3y}{x+3y}$ is defined only when $x+3y \neq 0$, or $x \neq -3y$. So the domain of f is $\{(x,y) \mid x \neq -3y\}$.



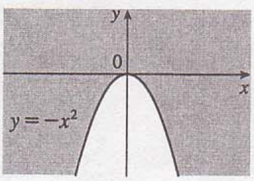
15. $\dfrac{3x+5y}{x^2+y^2-4}$ is defined only when $x^2+y^2-4 \neq 0$, or $x^2+y^2 \neq 4$. So the domain of f is $\{(x,y) \mid x^2+y^2 \neq 4\}$.



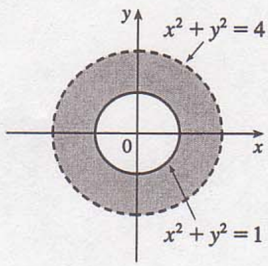
16. We need $y-x \geq 0$ or $y \geq x$ and $y+x > 0$ or $x > -y$. Thus $D = \{(x,y) \mid -y < x \leq y, y > 0\}$.



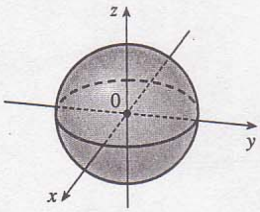
17. $D = \{(x,y) \mid x^2+y \geq 0\} = \{(x,y) \mid y \geq -x^2\}$



18. f is defined only when $x^2+y^2-1 \geq 0 \;\Rightarrow\; x^2+y^2 \geq 1$ and $4-x^2-y^2 > 0 \;\Rightarrow\; x^2+y^2 < 4$. Thus $D = \{(x,y) \mid 1 \leq x^2+y^2 < 4\}$.

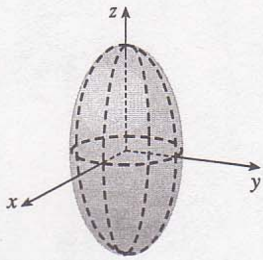


19. We need $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, so $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ (the points inside or on the sphere of radius 1, center the origin).

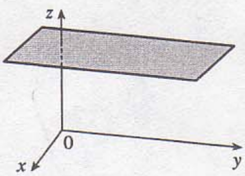


20. f is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0$ $\Rightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1$. Thus,

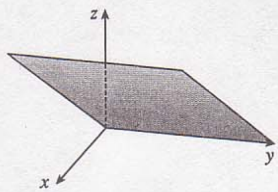
$$D = \left\{ (x, y, z) \,\middle|\, \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} < 1 \right\},$$ that is, the points inside the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$.



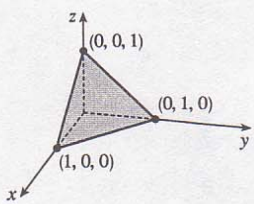
21. $z = 3$, a horizontal plane through the point $(0, 0, 3)$.



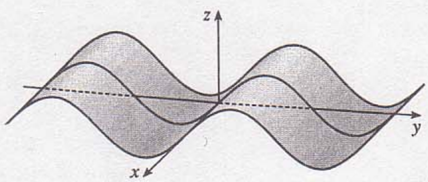
22. $z = x$, a plane which intersects the xz-plane in the line $z = x$, $y = 0$. The portion of this plane that lies in the first octant is shown.



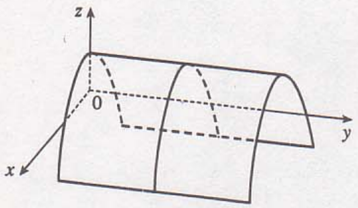
23. $z = 1 - x - y$ or $x + y + z = 1$, a plane with intercepts 1, 1, and 1.



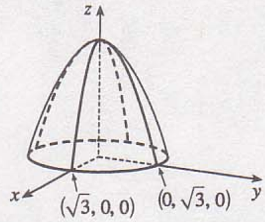
24. $z = \sin y$, a "wave."



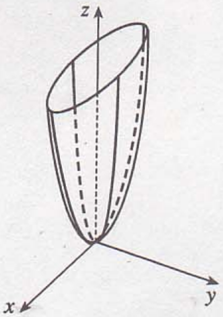
25. $z = 1 - x^2$, a parabolic cylinder.



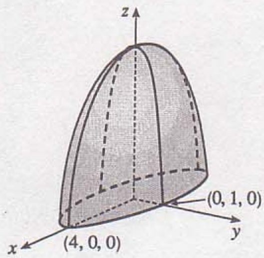
26. $z = 3 - x^2 - y^2$, a circular paraboloid with vertex at $(0, 0, 3)$.



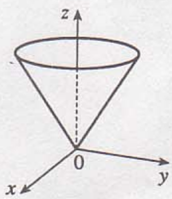
27. $z = x^2 + 9y^2$, an elliptic paraboloid with vertex the origin.



28. $z = \sqrt{16 - x^2 - 16y^2}$ so $z \geq 0$ and $z^2 + x^2 + 16y^2 = 16$, the top half of an ellipsoid.



29. $z = \sqrt{x^2 + y^2}$ so $x^2 + y^2 = z^2$ and $z \geq 0$, the top half of a right circular cone.



30. All six graphs have different traces in the planes $x = 0$ and $y = 0$, so we investigate these for each function.

(a) $f(x, y) = |x| + |y|$. The trace in $x = 0$ is $z = |y|$, and in $y = 0$ is $z = |x|$, so it must be graph VI.

(b) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and in $y = 0$ is $z = 0$, so it must be graph V.

(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. In addition, we can see that f is close to 0 for large values of x and y, so this is graph I.

(d) $f(x, y) = (x^2 - y^2)^2$. The trace in $x = 0$ is $z = y^4$, and in $y = 0$ is $z = x^4$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x^2 - y^2)^2 \Rightarrow y = \pm x$, so it must be graph IV.

(e) $f(x, y) = (x - y)^2$. The trace in $x = 0$ is $z = y^2$, and in $y = 0$ is $z = x^2$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x - y)^2 \Rightarrow y = x$, so it must be graph II.

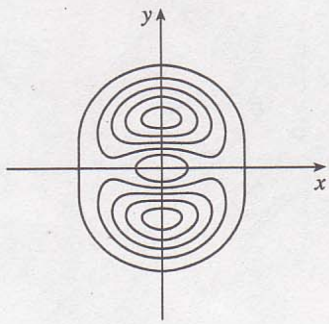
(f) $f(x, y) = \sin(|x| + |y|)$. The trace in $x = 0$ is $z = \sin|y|$, and in $y = 0$ is $z = \sin|x|$. In addition, notice that the oscillating nature of the graph is characteristic of trigonometric functions. So this is graph III.

31. The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

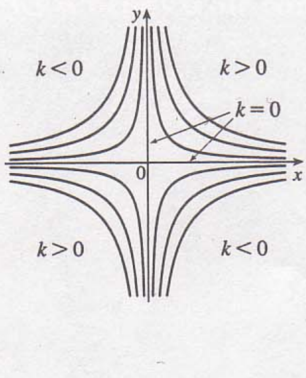
32. If we start at the origin and move along the x-axis, for example, the z-values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z-values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

33. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

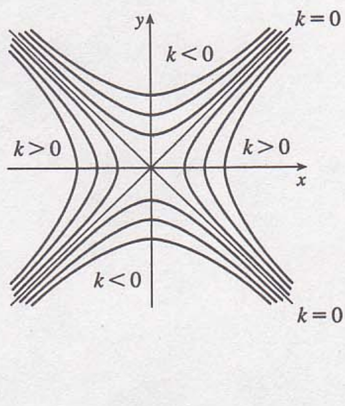
34.



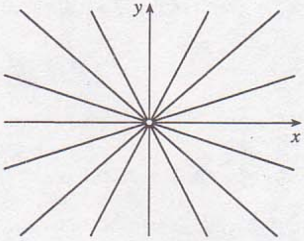
35. The level curves are $xy = k$. For $k = 0$ the curves are the coordinate axis; if $k > 0$, they are hyperbolas in the first and third quadrants; if $k < 0$, they are hyperbolas in the second and fourth quadrants.



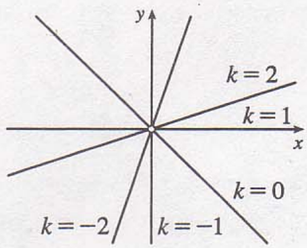
36. The level curves are $k = x^2 - y^2$. When $k = 0$, these are the lines $y = \pm x$. When $k > 0$, the curves are hyperbolas with axis the x-axis and when $k < 0$, they are hyperbolas with axis the y-axis.



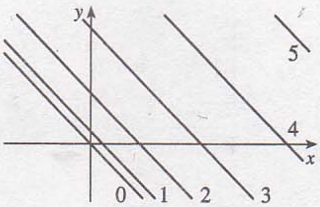
37. $k = x/y$ or $x = ky$ is a family of lines without the point $(0,0)$.



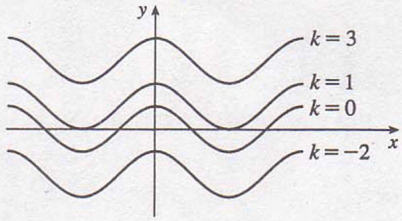
38. $k = \dfrac{x+y}{x-y}$ is a family of lines with slope $\dfrac{k-1}{k+1}$ (for $k \neq -1$) without the origin. For $k = -1$, the curve is the y-axis without the origin.



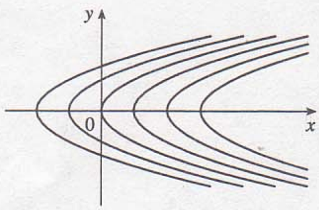
39. $k = \sqrt{x+y}$ or for $x + y \geq 0$, $k^2 = x + y$, or $y = -x + k^2$.
Note: $k \geq 0$ since $k = \sqrt{x+y}$.



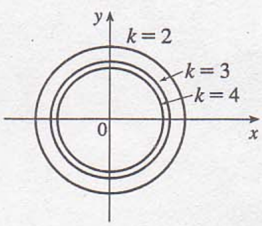
40. $k = y - \cos x$ or $y = k + \cos x$



41. $k = x - y^2$, or $x - k = y^2$, a family of parabolas with vertex $(k, 0)$.

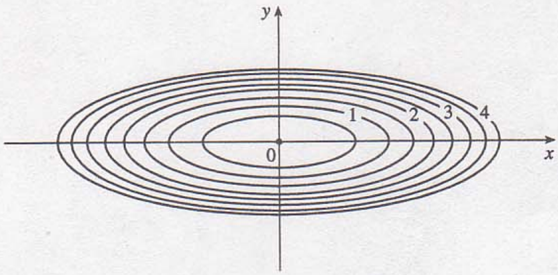


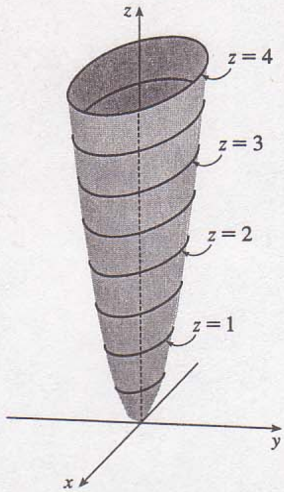
42. $k = e^{1/(x^2+y^2)}$, thus $k > 1$ and $1/(x^2+y^2) = \ln k$ or $x^2 + y^2 = 1/\ln k$, a family of circles.



43. The contour map consists of the level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.)

The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.

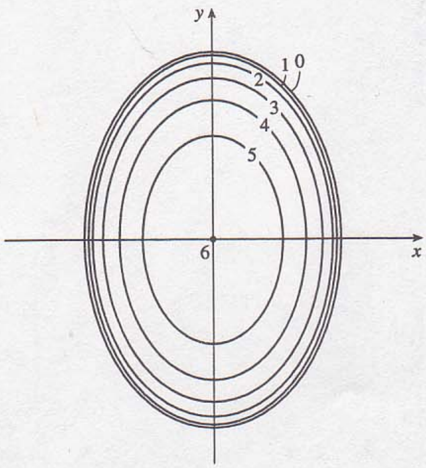


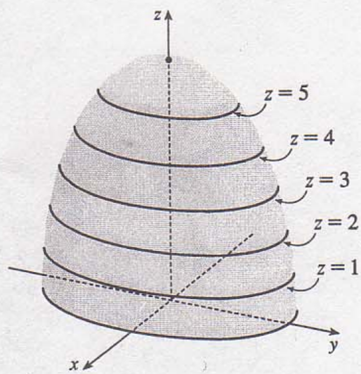


If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

44. The contour map consists of the level curves $k = \sqrt{36 - 9x^2 - 4y^2} \quad \Rightarrow \quad 9x^2 + 4y^2 = 36 - k^2$, $k \geq 0$, a family of ellipses with major axis the y-axis. (Or, if $k = 6$, the origin.)

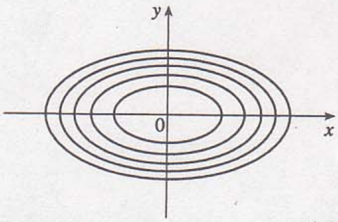
The graph of $f(x, y)$ is the surface $z = \sqrt{36 - 9x^2 - 4y^2}$, or equivalently the upper half of the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.



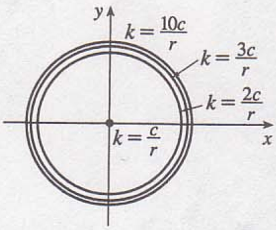


If we visualize lifting each ellipse $k = \sqrt{36 - 9x^2 - 4y^2}$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

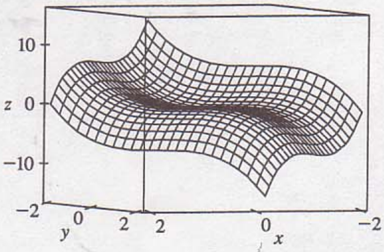
45. The isothermals are given by $k = 100/\left(1 + x^2 + 2y^2\right)$ or $x^2 + 2y^2 = (100 - k)/k$ $(0 < k \leq 100)$, a family of ellipses.

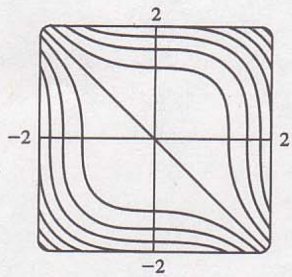


46. The equipotential curves are $k = c \big/ \sqrt{r^2 - x^2 - y^2}$ or $x^2 + y^2 = r^2 - (c/k)^2$, a family of circles $(k \geq c/r)$. *Note:* As $k \to \infty$, the radius of the circle approaches r.



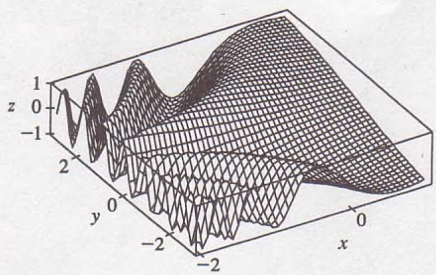
47. $f(x, y) = x^3 + y^3$

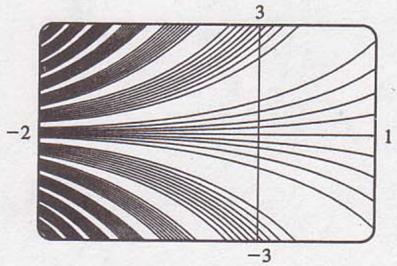




Note that the function is 0 along the line $y = -x$.

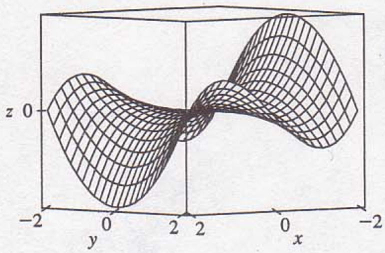
48. $f(x, y) = \sin\left(ye^{-x}\right)$

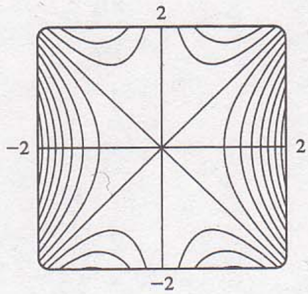




Cross-sections parallel to the yz-plane (such as the left-front trace in the first graph above) are sine-like curves. The periods of these curves decrease as x decreases.

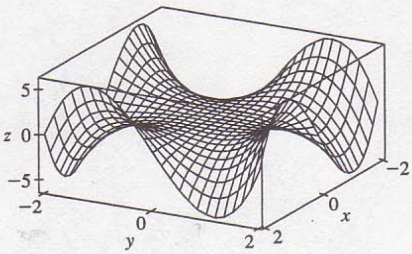
49. $f(x, y) = xy^2 - x^3$

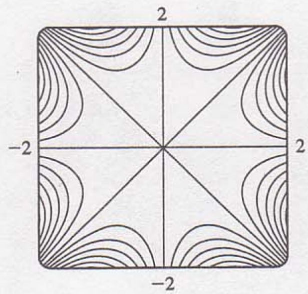




The cross-sections parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

50. $f(x, y) = xy^3 - yx^3$





The cross-sections parallel to either the yz-plane or the xz-plane are cubic curves.

51. (a) B (b) III *Reasons:* This function is constant on any circle centered at the origin, a description which matches only B and III.

52. (a) C (b) II *Reasons:* This function is the same if x is interchanged with y, so its graph is symmetric about the plane $x = y$. Also, $z(0, 0) = 0$ and the values of z approach 0 as we use points farther from the origin. These conditions are satisfied only by C and II.

53. (a) F (b) V *Reasons:* z increases without bound as we use points closer to the origin, a condition satisfied only by F and V.

54. (a) A (b) VI *Reasons:* Along the lines $y = \pm\frac{1}{\sqrt{3}}x$ and $x = 0$, this function is 0.

55. (a) D (b) IV *Reasons:* This function is periodic in both x and y, with period 2π in each variable.

56. (a) E (b) I *Reasons:* This function is periodic along the x-axis, and increases as $|y|$ increases.

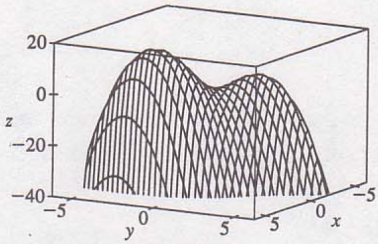
57. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

58. $k = x^2 + 3y^2 + 5z^2$ is a family of ellipsoids for $k > 0$ and the origin for $k = 0$.

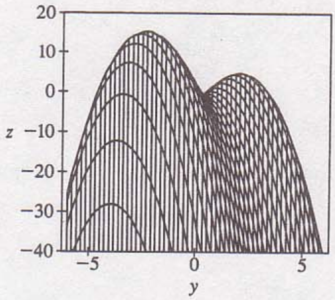
59. $k = x^2 - y^2 + z^2$ are the equations of the level surfaces. For $k = 0$, the surface is a right circular cone with vertex the origin and axis the y-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the y-axis. For $k < 0$, we have a family of hyperboloids of two sheets with axis the y-axis.

60. $k = x^2 - y^2$ is a family of hyperbolic cylinders. The cross section of this family in the xy-plane has the same graph as the level curves in Exercise 36.

61. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



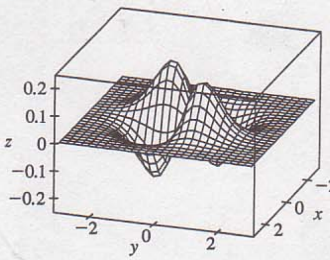
Three-dimensional view



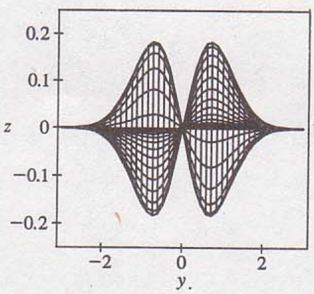
Front view

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

62. $f(x, y) = xye^{-x^2-y^2}$



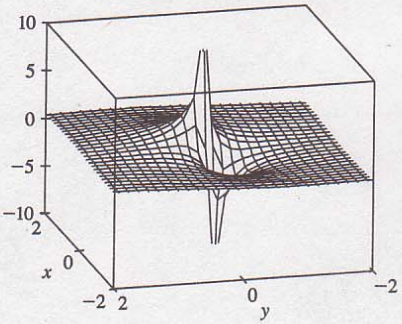
Three-dimensional view



Front view

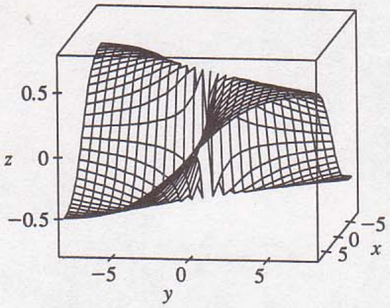
The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops.") From the front view graph, we can estimate the maximum value to be approximately 0.18. These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f.

63.



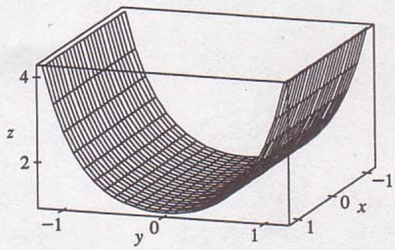
$f(x, y) = \dfrac{x + y}{x^2 + y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x, y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x, y) \to \infty$, while in others $f(x, y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x, y)$ approaches 0 along the line $y = -x$.

64.



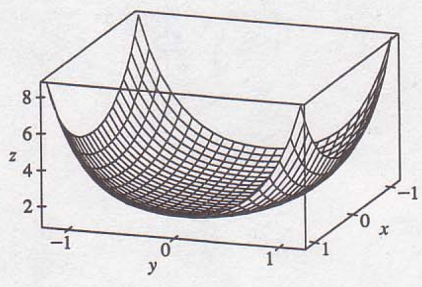
$f(x,y)=\dfrac{xy}{x^2+y^2}$. The graph exhibits different limiting values as x and y become large or as (x,y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y=x$, regardless of the distance from the origin. Along the line $y=-x$, the value is always $-\frac{1}{2}$. Along the axes, $f(x,y)=0$ for all values of (x,y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

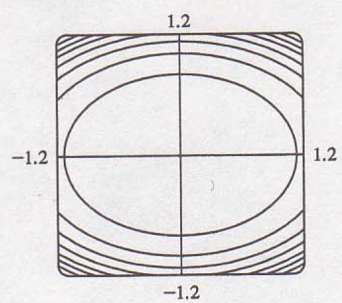
65. $f(x,y)=e^{cx^2+y^2}$. First, if $c=0$, the graph is the cylindrical surface $z=e^{y^2}$ (whose level curves are parallel lines). When $c>0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



$c=0$

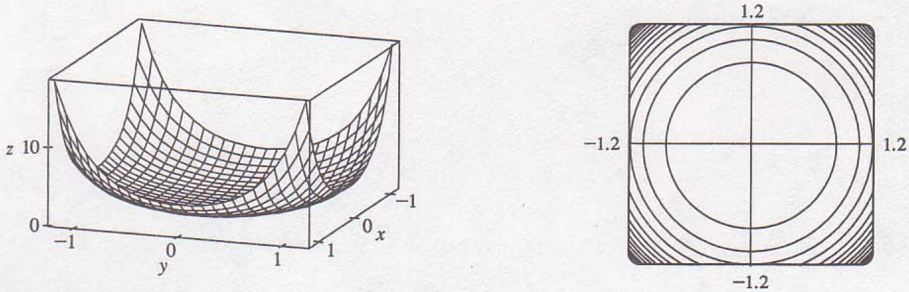
For $0<c<1$, the ellipses have major axis the x-axis and the eccentricity increases as $c\to 0$.





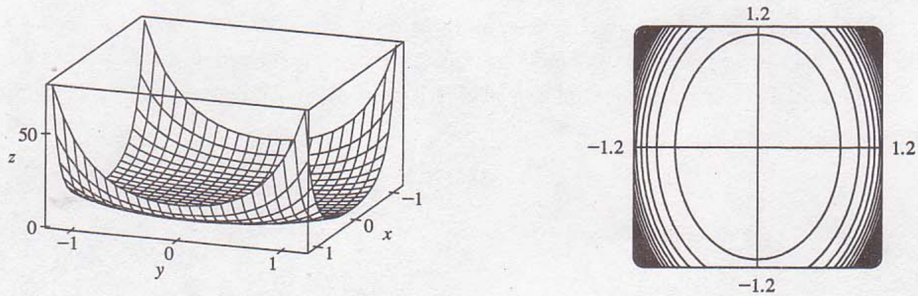
$c=0.5$ (level curves in increments of 1)

For $c = 1$ the level curves are circles centered at the origin.



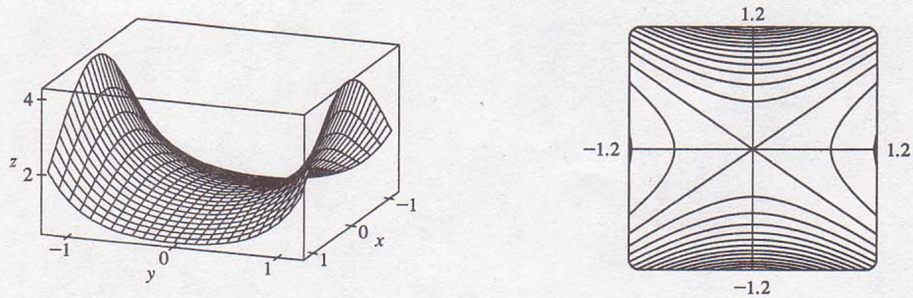
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.

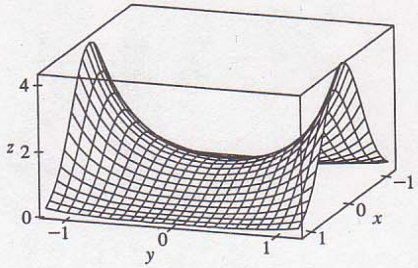


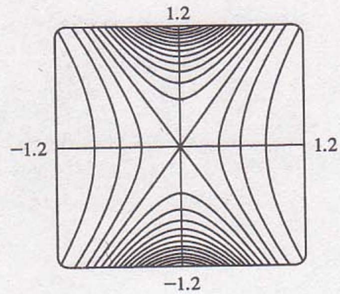
$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.



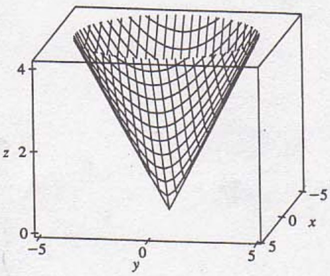
$c = -0.5$ (level curves in increments of 0.25)



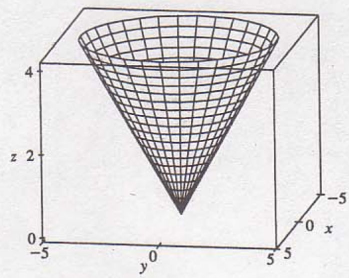


$c = -2$ (level curves in increments of 0.25)

66. First, we graph $f(x,y) = \sqrt{x^2 + y^2}$. As an alternative, the $x^2 + y^2$ expression suggests that cylindrical coordinates may be appropriate, giving the equivalent equation $z = \sqrt{r^2} = r$, $r \geq 0$ which we graph as well. Notice that the graph in cylindrical coordinates better demonstrates the symmetry of the surface.

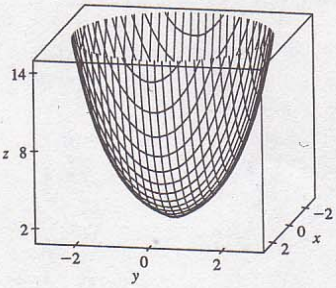


$f(x,y) = \sqrt{x^2 + y^2}$

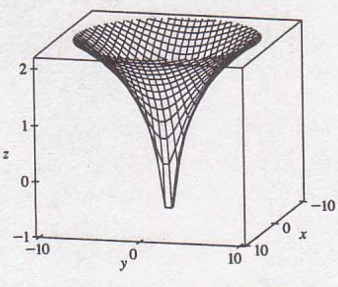


$z = r, r \geq 0$

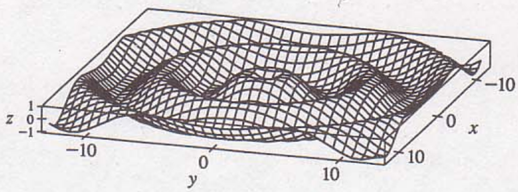
Graphs of the other four functions follow.



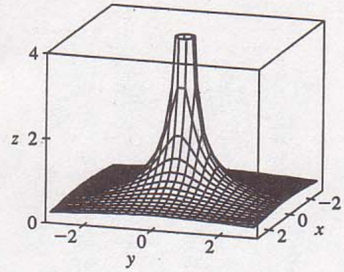
$f(x,y) = e^{\sqrt{x^2+y^2}}$



$f(x,y) = \ln\sqrt{x^2 + y^2}$



$f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$



$f(x, y) = 1/\sqrt{x^2 + y^2}$

Notice that each graph $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$ exhibits radial symmetry about the z-axis and the trace in the xz-plane for $x \geq 0$ is the graph of $z = g(x)$, $x \geq 0$. This suggests that the graph of $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$ is obtained from the graph of g by graphing $z = g(x)$ in the xz-plane and rotating the curve about the z-axis.

67. (a) $P = bL^\alpha K^{1-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = bL^\alpha K^{-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^\alpha \quad \Rightarrow \quad \ln \dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^\alpha\right) \quad \Rightarrow$

$\ln \dfrac{P}{K} = \ln b + \alpha \ln\left(\dfrac{L}{K}\right)$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899-1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1899	0	0
1900	−0.02	−0.06
1901	−0.04	−0.02
1902	−0.04	0
1903	−0.07	−0.05
1904	−0.13	−0.12
1905	−0.18	−0.04
1906	−0.20	−0.07
1907	−0.23	−0.15
1908	−0.41	−0.38
1909	−0.33	−0.24
1910	−0.35	−0.27

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1911	−0.38	−0.34
1912	−0.38	−0.24
1913	−0.41	−0.25
1914	−0.47	−0.37
1915	−0.53	−0.34
1916	−0.49	−0.28
1917	−0.53	−0.39
1918	−0.60	−0.50
1919	−0.68	−0.57
1920	−0.74	−0.57
1921	−1.05	−0.85
1922	−0.98	−0.59

After entering the (x, y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \quad \Rightarrow \quad b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^\alpha K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

15.2 Limits and Continuity

ET 14.2

1. In general, we can't say anything about $f(3,1)$! $\lim\limits_{(x,y)\to(3,1)} f(x,y) = 6$ means that the values of $f(x,y)$ approach 6 as (x,y) approaches, but is not equal to, $(3,1)$. If f is continuous, we know that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, so $\lim\limits_{(x,y)\to(3,1)} f(x,y) = f(3,1) = 6$.

2. (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x,y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of (x,y) points near the origin.

x \ y	−0.2	−0.1	−0.05	0	0.05	0.1	0.2
−0.2	−2.551	−2.525	−2.513	−2.500	−2.488	−2.475	−2.451
−0.1	−2.525	−2.513	−2.506	−2.500	−2.494	−2.488	−2.475
−0.05	−2.513	−2.506	−2.503	−2.500	−2.497	−2.494	−2.488
0	−2.500	−2.500	−2.500		−2.500	−2.500	−2.500
0.05	−2.488	−2.494	−2.497	−2.500	−2.503	−2.506	−2.513
0.1	−2.475	−2.488	−2.494	−2.500	−2.506	−2.513	−2.525
0.2	−2.451	−2.475	−2.488	−2.500	−2.513	−2.525	−2.551

As the table shows, the values of $f(x,y)$ seem to approach -2.5 as (x,y) approaches the origin from a variety of different directions. This suggests that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = -2.5$.

Since f is a rational function, it is continuous on its domain. f is defined at $(0,0)$, so we can use direct substitution to establish that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{0^2 0^3 + 0^3 0^2 - 5}{2 - 0 \cdot 0} = -\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y) = \dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

x \ y	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
-0.3	0.667	0.706	0.545	0.000	-0.545	-0.706	-0.667
-0.2	0.545	0.667	0.667	0.000	-0.667	-0.667	-0.545
-0.1	0.316	0.444	0.667	0.000	-0.667	-0.444	-0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	-0.316	-0.444	-0.667	0.000	0.667	0.444	0.316
0.2	-0.545	-0.667	-0.667	0.000	0.667	0.667	0.545
0.3	-0.667	-0.706	-0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach $(0,0)$ along the x-axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach $(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{2x^2}{x^2+2x^2} = \dfrac{2}{3}$ $(x \neq 0)$, so $f(x,y)\to \frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y) = x^5+4x^3y-5xy^2$ is a polynomial, and hence continuous, so $\displaystyle\lim_{(x,y)\to(5,-2)} f(x,y) = f(5,-2) = 5^5+4(5)^3(-2)-5(5)(-2)^2 = 2025.$

6. $x-2y$ is a polynomial and therefore continuous. Since $\cos t$ is a continuous function, the composition $\cos(x-2y)$ is also continuous. xy is also a polynomial, and hence continuous, so the product $f(x,y)=xy\cos(x-2y)$ is a continuous function. Then $\displaystyle\lim_{(x,y)\to(6,3)} f(x,y) = f(6,3) = (6)(3)\cos(6-2\cdot 3) = 18.$

7. $f(x,y) = x^2/(x^2+y^2)$. First approach $(0,0)$ along the x-axis. Then $f(x,0)=x^2/x^2=1$ for $x\neq 0$, so $f(x,y)\to 1$. Now approach $(0,0)$ along the y-axis. Then for $y\neq 0$, $f(0,y)=0$, so $f(x,y)\to 0$. Since f has two different limits along two different lines, the limit does not exist.

8. $f(x,y) = (x+y)^2/(x^2+y^2)$. As $(x,y)\to(0,0)$ along the x-axis, $f(x,y)\to 1$. But as $(x,y)\to(0,0)$ along the line $y=x$, $f(x,x)=4x^2/(2x^2)=2$ for $x\neq 0$, so $f(x,y)\to 2$. Thus, the limit does not exist.

9. $f(x,y) = 8x^2y^2/(x^4+y^4)$. Approaching $(0,0)$ along the x-axis gives $f(x,y)\to 0$. Approaching $(0,0)$ along the line $y=x$, $f(x,x)=8x^4/2x^4=4$ for $x\neq 0$, so along this line $f(x,y)\to 4$ as $(x,y)\to(0,0)$. Thus the limit doesn't exist.

10. $\displaystyle\lim_{(x,y)\to(0,0)} (x^3+xy^2)/(x^2+y^2) = \lim_{(x,y)\to(0,0)} x = 0$

11. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x=y^2$ and $y=x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. $0 \le \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \le |x|$ since $|y|\le\sqrt{x^2+y^2}$, and $|x|\to 0$ as $(x,y)\to(0,0)$. So $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0.$

12. Since $\dfrac{xy+1}{x^2+y^2+1}$ is a rational function defined at $(0,0)$ the limit is $\dfrac{0+1}{0+0+1}=1.$

13. Let $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x^2) = \dfrac{2x^4}{2x^4} = 1$ for $x \neq 0$, so $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$. Thus the limit doesn't exist.

14. $f(x,y) = \dfrac{x^3y^2}{x^2+y^2}$. We use the Squeeze Theorem: $0 \leq \dfrac{|x^3y^2|}{x^2+y^2} \leq |x^3|$ since $y^2 \leq x^2+y^2$, and $|x^3| \to 0$ as $(x,y) \to (0,0)$. So $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

15.
$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}\\ &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2}\\ &= \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = 2\end{aligned}$$

16. $f(x,y) = \dfrac{xy-2y}{x^2+y^2-4x+4} = \dfrac{y(x-2)}{y^2+(x-2)^2}$. Then $f(x,0) = 0$ for $x \neq 2$, so $f(x,y) \to 0$ as $(x,y) \to (2,0)$ along the x-axis. But $f(x,x-2) = \dfrac{(x-2)(x-2)}{(x-2)^2+(x-2)^2} = \dfrac{(x-2)^2}{2(x-2)^2} = \dfrac{1}{2}$ for $x \neq 2$, so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (2,0)$ along the line $y = x-2$ $(x \neq 2)$. Thus, the limit doesn't exist.

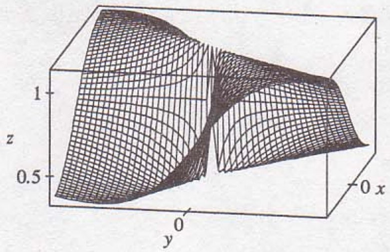
17. e^{-xy} and $\sin(\pi z/2)$ are each compositions of continuous functions, and hence continuous, so their product $f(x,y,z) = e^{-xy}\sin(\pi z/2)$ is a continuous function. Then $\lim\limits_{(x,y,z)\to(3,0,1)} f(x,y,z) = f(3,0,1) = e^{-(3)(0)}\sin(\pi\cdot 1/2) = 1$.

18. $f(x,y,z) = \dfrac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$. Then $f(x,0,0) = \dfrac{x^2+0+0}{x^2+0+0} = 1$ for $x \neq 0$, so $f(x,y,z) \to 1$ as $(x,y,z) \to (0,0,0)$ along the x-axis. But $f(0,y,0) = \dfrac{0+2y^2+0}{0+y^2+0} = 2$ for $y \neq 0$, so $f(x,y,z) \to 2$ as $(x,y,z) \to (0,0,0)$ along the y-axis. Thus, the limit doesn't exist.

19. $f(x,y,z) = \dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0) = 0/x^2 = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

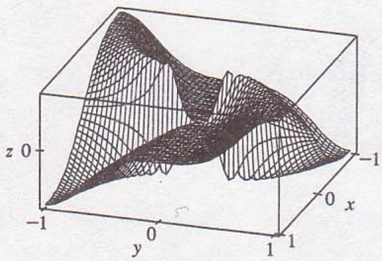
20. $f(x,y,z) = \dfrac{xy+yz+zx}{x^2+y^2+z^2}$. Then $f(x,0,0) = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

21.



From the ridges on the graph, we see that as $(x, y) \to (0, 0)$ along the lines under the two ridges, $f(x, y)$ approaches different values. So the limit does not exist.

22.

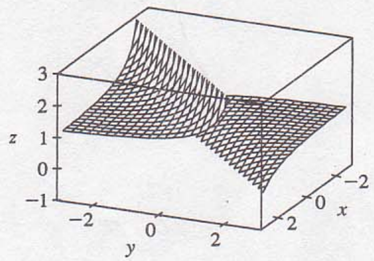


From the graph, it appears that as we approach the origin along the lines $x = 0$ or $y = 0$, the function is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3, y) = y^6/(2y^6) = \frac{1}{2}$ for $y \neq 0$, so $f(x, y) \to \frac{1}{2}$ as $(x, y) \to (0, 0)$ along the curve $x = y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

23. $h(x, y) = g(f(x, y)) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain $D = \{(x, y) \mid 2x + 3y - 6 \geq 0\} = \{(x, y) \mid y \geq -\frac{2}{3}x + 2\}$, which consists of all points on and above the line $y = -\frac{2}{3}x + 2$.

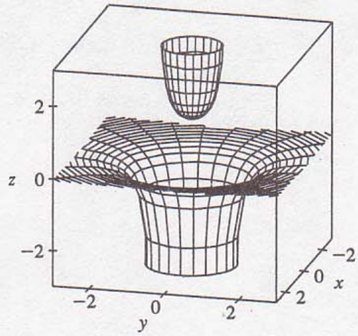
24. $h(x, y) = g(f(x, y)) = \left(\sqrt{x^2 - y} - 1\right) / \left(\sqrt{x^2 - y} + 1\right)$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain $D = \{(x, y) \mid x^2 - y \geq 0\} = \{(x, y) \mid y \leq x^2\}$ which consists of all points below or on the parabola $y = x^2$.

25.



From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x, y) = e^{1/(x-y)}$ as a composition of functions, $g(x, y) = 1/(x - y)$ is a rational function and therefore continuous except where $x - y = 0 \Rightarrow y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x, y)) = e^{1/(x-y)} = f(x, y)$ is continuous except along the line $y = x$, as we suspected.

26.



We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous. [*Note:* For a more accurate graph, try converting to cylindrical coordinates first.] Since $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$ is a rational function, it is continuous except where $1 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 1$, confirming our observation that f is discontinuous on the circle $x^2 + y^2 = 1$.

27. $F(x,y) = \dfrac{1}{x^2 - y}$ is a rational function and thus is continuous on its domain $\{(x,y) \mid x^2 - y \neq 0\} = \{(x,y) \mid y \neq x^2\}$, so F is continuous on $\mathbb{R}^2$ except the parabola $y = x^2$.

28. $F(x,y) = \dfrac{x - y}{1 + x^2 + y^2}$ is a rational function and thus is continuous on its domain $\mathbb{R}^2$ (since the denominator is never zero).

29. $F(x,y) = \arctan(x + \sqrt{y}) = g(f(x,y))$ where $f(x,y) = x + \sqrt{y}$, continuous on its domain $\{(x,y) \mid y \geq 0\}$, and $g(t) = \arctan t$ is continuous everywhere. Thus F is continuous on its domain $\{(x,y) \mid y \geq 0\}$.

30. $F(x,y) = \ln(2x + 3y) = g(f(x,y))$ where $f(x,y) = 2x + 3y$, continuous on $\mathbb{R}^2$ and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Thus F is continuous on its domain $D = \{(x,y) \mid 2x + 3y > 0\}$.

31. $G(x,y) = g_1(f_1(x,y)) - g_2(f_2(x,y))$ where $f_1(x,y) = x + y$ and $f_2(x,y) = x - y$, both of which are polynomials so continuous on $\mathbb{R}^2$, and $g_1(t) = \sqrt{t}$, $g_2(s) = \sqrt{s}$, both of which are continuous on their respective domains $\{t \mid t \geq 0\}$ and $\{s \mid s \geq 0\}$. Thus $g_1 \circ f_1$ is continuous on its domain $D_1 = \{(x,y) \mid x + y \geq 0\} = \{(x,y) \mid y \geq -x\}$ and $g_2 \circ f_2$ is continuous on its domain $D_2 = \{(x,y) \mid x - y \geq 0\} = \{(x,y) \mid y \leq x\}$. Then G, being the difference of these two composite functions, is continuous on its domain $D = D_1 \cap D_2 = \{(x,y) \mid -x \leq y \leq x\} = \{(x,y) \mid |y| \leq x\}$.

32. $G(x,y) = g(f(x,y))$ where $f(x,y) = x^2 + y^2$, continuous on $\mathbb{R}^2$, and $g(t) = \sin^{-1} t$, continuous on its domain $\{t \mid -1 \leq t \leq 1\}$. Thus G is continuous on its domain $D = \{(x,y) \mid -1 \leq x^2 + y^2 \leq 1\} = \{(x,y) \mid x^2 + y^2 \leq 1\}$, inside and on the circle $x^2 + y^2 = 1$.

33. $f(x,y,z) = \dfrac{xyz}{x^2 + y^2 - z}$ is a rational function and thus is continuous on its domain $\{(x,y,z) \mid x^2 + y^2 - z \neq 0\} = \{(x,y,z) \mid z \neq x^2 + y^2\}$, so f is continuous on $\mathbb{R}^3$ except on the circular paraboloid $z = x^2 + y^2$.

34. $f(x,y,z) = \sqrt{x + y + z} = h(g(x,y,z))$ where $g(x,y,z) = x + y + z$, continuous everywhere, and $h(t) = \sqrt{t}$ is continuous on its domain $\{t \mid t \geq 0\}$. Thus f is continuous on its domain $\{(x,y,z) \mid x + y + z \geq 0\}$, so f is continuous on and above the plane $z = -x - y$.

35. $f(x,y) = \begin{cases} \dfrac{x^2 y^3}{2x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2 \leq 2x^2 + y^2$, we have $\left|x^2 y^3/(2x^2 + y^2)\right| \leq |y^3|$. We know that $|y^3| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem,

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y^3}{2x^2 + y^2} = 0$. But $f(0,0) = 1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y) \mid (x,y) \neq (0,0)\}$.

36. $f(x,y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

37. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0+} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2} = \lim_{r\to 0+} \left(r\cos^3\theta + r\sin^3\theta\right) = 0$

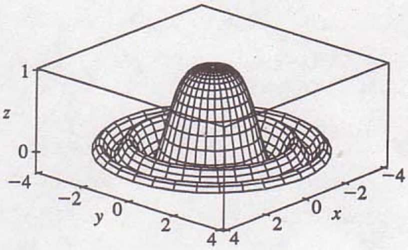
38. $$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\ln\left(x^2+y^2\right) = \lim_{r\to 0+} r^2\ln r^2 = \lim_{r\to 0+} \frac{\ln r^2}{1/r^2}$$
$$= \lim_{r\to 0+} \frac{\left(1/r^2\right)(2r)}{-2/r^3} \text{ (using l'Hospital's Rule)} = \lim_{r\to 0+} \left(-r^2\right) = 0$$

39. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0+} \frac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$$
$$= \lim_{\rho\to 0+} \left(\rho\sin^2\phi\cos\phi\sin\theta\cos\theta\right) = 0$$

40. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim_{r\to 0+} \frac{\sin\left(r^2\right)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

$$\lim_{r\to 0+} \frac{\sin\left(r^2\right)}{r^2} \overset{\text{H}}{=} \lim_{r\to 0+} \frac{2r\cos\left(r^2\right)}{2r} = \lim_{r\to 0+} \cos\left(r^2\right) = 1.$$

Or: Use the fact that $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.



41. Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}|-|\mathbf{a}|)^2$, we have $||\mathbf{x}|-|\mathbf{a}|| \leq |\mathbf{x}-\mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then whenever $0 < |\mathbf{x}-\mathbf{a}| < \delta$, $||\mathbf{x}|-|\mathbf{a}|| \leq |\mathbf{x}-\mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

42. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x}-\mathbf{a}| < \delta$ or $|\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| < \epsilon$ whenever $|\mathbf{x}-\mathbf{a}| < \delta$. But $|\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| = |\mathbf{c}\cdot(\mathbf{x}-\mathbf{a})|$ and $|\mathbf{c}\cdot(\mathbf{x}-\mathbf{a})| \leq |\mathbf{c}|\,|\mathbf{x}-\mathbf{a}|$ by Exercise 13.3.61 [ET 12.3.61] (the Cauchy-Schwartz Inequality). Let $\epsilon > 0$ be given and set $\delta = \epsilon/|\mathbf{c}|$. Then whenever $0 < |\mathbf{x}-\mathbf{a}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| \leq |\mathbf{c}|\,|\mathbf{x}-\mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\epsilon/|\mathbf{c}|) = \epsilon$. So f is continuous on $\mathbb{R}^n$.

15.3 Partial Derivatives ET 14.3

1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158 ° W, latitude 21 ° N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and

location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t\,(158, 21, 9)$ to be positive.

2. By Definition 4, $f_T\,(92, 60) = \lim\limits_{h\to 0} \dfrac{f\,(92+h, 60) - f\,(92, 60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T\,(92, 60) \approx \dfrac{f\,(94, 60) - f\,(92, 60)}{2} = \dfrac{111 - 105}{2} = 3$, $f_T\,(92, 60) \approx \dfrac{f\,(90, 60) - f\,(92, 60)}{-2} = \dfrac{100 - 105}{-2} = 2.5$. Averaging these values, we estimate $f_T\,(92, 60)$ to be approximately 2.75. Thus, when the actual temperature is $92\,^\circ\mathrm{F}$ and the relative humidity is 60%, the apparent temperature rises by about $2.75\,^\circ\mathrm{F}$ for every degree that the actual temperature rises.

Similarly, $f_H\,(92, 60) = \lim\limits_{h\to 0} \dfrac{f\,(92, 60+h) - f\,(92, 60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H\,(92, 60) \approx \dfrac{f\,(92, 65) - f\,(92, 60)}{5} = \dfrac{108 - 105}{5} = 0.6$,
$f_H\,(92, 60) \approx \dfrac{f\,(92, 55) - f\,(92, 60)}{-5} = \dfrac{103 - 105}{-5} = 0.4$. Averaging these values, we estimate $f_H\,(92, 60)$ to be approximately 0.5. Thus, when the actual temperature is $92\,^\circ\mathrm{F}$ and the relative humidity is 60%, the apparent temperature rises by about $0.5\,^\circ\mathrm{F}$ for every percent that the relative humidity increases.

3. (a) By Definition 4, $f_T\,(12, 20) = \lim\limits_{h\to 0} \dfrac{f\,(12+h, 20) - f\,(12, 20)}{h}$, which we can approximate by considering $h = 4$ and $h = -4$ and using the values given in the table:
$f_T\,(12, 20) \approx \dfrac{f\,(16, 20) - f\,(12, 20)}{4} = \dfrac{11 - 5}{4} = 1.5$,
$f_T\,(12, 20) \approx \dfrac{f\,(8, 20) - f\,(12, 20)}{-4} = \dfrac{0 - 5}{-4} = 1.25$. Averaging these values, we estimate $f_T\,(12, 20)$ to be approximately 1.375. Thus, when the actual temperature is $12\,^\circ\mathrm{C}$ and the wind speed is 20 km/h, the apparent temperature rises by about $1.375\,^\circ\mathrm{C}$ for every degree that the actual temperature rises.
Similarly, $f_v\,(12, 20) = \lim\limits_{h\to 0} \dfrac{f\,(12, 20+h) - f\,(12, 20)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v\,(12, 20) \approx \dfrac{f\,(12, 30) - f\,(12, 20)}{10} = \dfrac{3 - 5}{10} = -0.2$,
$f_v\,(12, 20) \approx \dfrac{f\,(12, 10) - f\,(12, 20)}{-10} = \dfrac{9 - 5}{-10} = -0.4$. Averaging these values, we estimate $f_v\,(12, 20)$ to be approximately -0.3. Thus, when the actual temperature is $12\,^\circ\mathrm{C}$ and the wind speed is 20 km/h, the apparent temperature decreases by about $0.3\,^\circ\mathrm{C}$ for every km/h that the wind speed increases.

(b) For a fixed wind speed v, the values of the wind-chill index I increase as temperature T increases (look at a column of the table), so $\dfrac{\partial I}{\partial T}$ is positive. For a fixed temperature T, the values of I decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial I}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T, the function values $f\,(T, v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim\limits_{v\to\infty} \left(\dfrac{\partial I}{\partial v}\right) = 0$.

4. (a) $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v, which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of h when we fix v and consider h as a function of t, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40, 15) = \lim\limits_{h\to 0} \dfrac{f(40+h, 15) - f(40, 15)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$ and using the values given in the table:

$f_v(40, 15) \approx \dfrac{f(50, 15) - f(40, 15)}{10} = \dfrac{36 - 25}{10} = 1.1,$

$f_v(40, 15) \approx \dfrac{f(30, 15) - f(40, 15)}{-10} = \dfrac{16 - 25}{-10} = 0.9$. Averaging these values, we have $f_v(40, 15) \approx 1.0$.

Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly,

$f_t(40, 15) = \lim\limits_{h\to 0} \dfrac{f(40, 15+h) - f(40, 15)}{h}$ which we can approximate by considering

$h = 5$ and $h = -5$: $f_t(40, 15) \approx \dfrac{f(40, 20) - f(40, 15)}{5} = \dfrac{28 - 25}{5} = 0.6,$

$f_t(40, 15) \approx \dfrac{f(40, 10) - f(40, 15)}{-5} = \dfrac{21 - 25}{-5} = 0.8$. Averaging these values, we have $f_t(40, 15) \approx 0.7$.

Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.

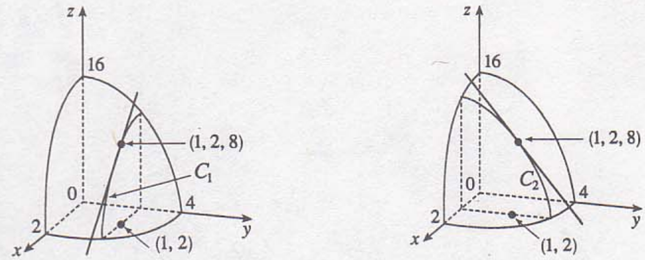
(c) For fixed values of v, the function values $f(v, t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim\limits_{t\to\infty} (\partial h/\partial t) = 0$.

5. First of all, if we start at the point $(3, -3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3, -1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3, -1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

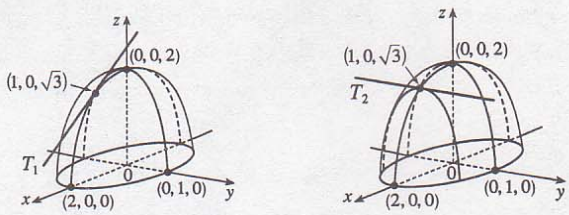
6. $f_x(2, 1)$ is the rate of change of f at $(2, 1)$ in the x-direction. Fixing y at 1 and moving along the contour map in the positive x-direction around $x = 2$, it appears that f increases about 4 units for each unit increase in x, so $f_x(2, 1) \approx 4$. Similarly fixing x at 2 and moving along the contour map in the positive y-direction around $y = 1$, it appears that f decreases about 2 units for each unit increase in y, so $f_y(2, 1) \approx -2$.

7. $f(x, y) = 16 - 4x^2 - y^2 \quad\Rightarrow\quad f_x(x, y) = -8x$ and $f_y(x, y) = -2y \quad\Rightarrow\quad f_x(1, 2) = -8$ and $f_y(1, 2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve C_1 in the first figure).

The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.

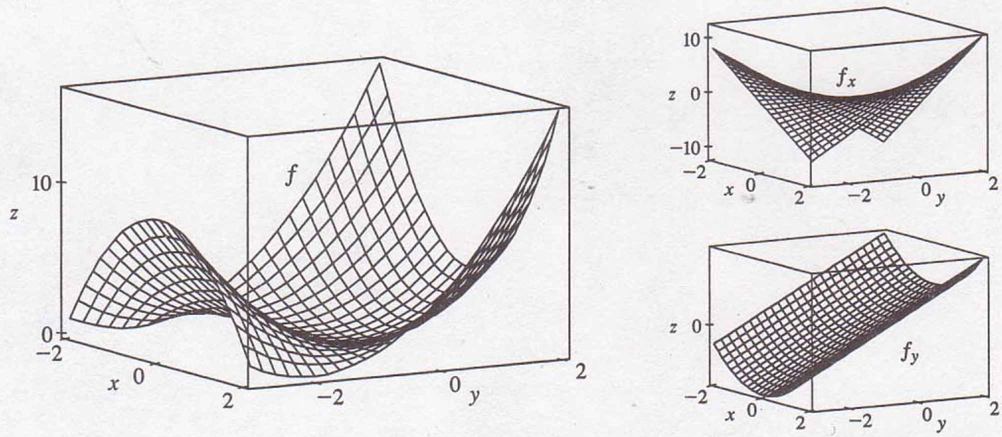


8. $f(x, y) = \left(4 - x^2 - 4y^2\right)^{1/2} \quad \Rightarrow \quad f_x(x, y) = -x\left(4 - x^2 - 4y^2\right)^{-1/2}$ and $f_y(x, y) = -4y\left(4 - x^2 - 4y^2\right)^{-1/2} \quad \Rightarrow \quad f_x(1, 0) = -\frac{1}{\sqrt{3}}$, $f_y(1, 0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$



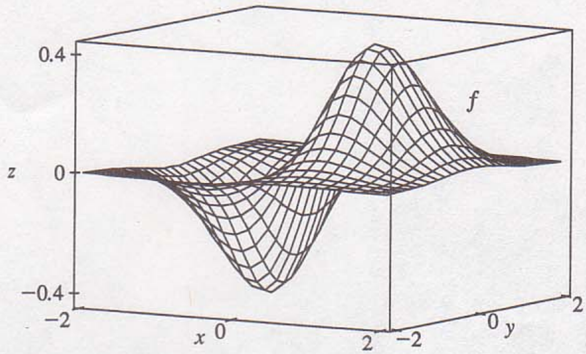
intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1, 0, \sqrt{3}\right)$ is $f_x(1, 0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $\left(1, 0, \sqrt{3}\right)$ is $f_y(1, 0) = 0$.

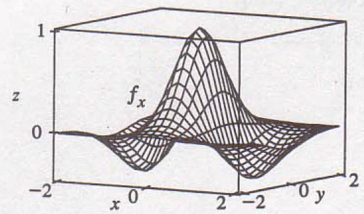
9. $f(x, y) = x^2 + y^2 + x^2 y \quad \Rightarrow \quad f_x = 2x + 2xy$, $f_y = 2y + x^2$

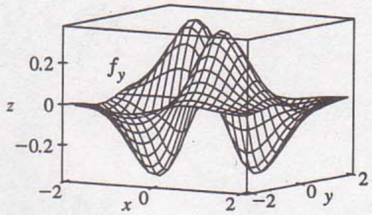


Note that the traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < -1$ and upward for $y > -1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of f in planes parallel to the yz-plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

10. $f(x,y) = xe^{-x^2-y^2} \quad\Rightarrow\quad f_x = x\left(-2xe^{-x^2-y^2}\right) + e^{-x^2-y^2} = e^{-x^2-y^2}\left(1-2x^2\right)$, $f_y = -2xye^{-x^2-y^2}$







Note that traces of f in planes parallel to the xz-plane have two extreme values, while traces of f_x in these planes have two zeros. Traces of f in planes parallel to the yz-plane have only one extreme value (a minimum if $x < 0$, a maximum if $x > 0$), and traces of f_y in these planes have only one zero (going from negative to positive if $x < 0$ and from positive to negative if $x > 0$).

11. $f(x,y) = 3x - 2y^4 \quad\Rightarrow\quad f_x(x,y) = 3 - 0 = 3$, $f_y(x,y) = 0 - 8y^3 = -8y^3$

12. $f(x,y) = x^5 + 3x^3y^2 + 3xy^4 \quad\Rightarrow\quad f_x(x,y) = 5x^4 + 3\cdot 3x^2\cdot y^2 + 3\cdot 1\cdot y^4 = 5x^4 + 9x^2y^2 + 3y^4$,
$f_y(x,y) = 0 + 3x^3\cdot 2y + 3x\cdot 4y^3 = 6x^3y + 12xy^3$.

13. $z = xe^{3y} \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = e^{3y}, \dfrac{\partial z}{\partial y} = 3xe^{3y}$

14. $z = y\ln x \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{y}{x}, \dfrac{\partial z}{\partial y} = \ln x$

15. $f(x,y) = \dfrac{x-y}{x+y} \quad\Rightarrow\quad f_x(x,y) = \dfrac{(1)(x+y)-(x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$,
$f_y(x,y) = \dfrac{(-1)(x+y)-(x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

16. $f(x,y) = x^y \quad\Rightarrow\quad f_x(x,y) = yx^{y-1}$, $f_y(x,y) = x^y\ln x$

17. $w = \sin\alpha\cos\beta \quad\Rightarrow\quad \dfrac{\partial w}{\partial\alpha} = \cos\alpha\cos\beta, \dfrac{\partial w}{\partial\beta} = -\sin\alpha\sin\beta$

18. $f(s,t) = \dfrac{st^2}{s^2+t^2} \quad\Rightarrow\quad f_s(s,t) = \dfrac{t^2\left(s^2+t^2\right)-st^2(2s)}{(s^2+t^2)^2} = \dfrac{t^4-s^2t^2}{(s^2+t^2)^2}$,
$f_t(s,t) = \dfrac{2st\left(s^2+t^2\right)-st^2(2t)}{(s^2+t^2)^2} = \dfrac{2s^3t}{(s^2+t^2)^2}$

19. $f(u,v) = \tan^{-1}\left(\dfrac{u}{v}\right) \quad\Rightarrow\quad f_u(u,v) = \dfrac{1}{1+(u/v)^2}\left(\dfrac{1}{v}\right) = \dfrac{1}{v}\left(\dfrac{v^2}{u^2+v^2}\right) = \dfrac{v}{u^2+v^2}$,
$f_v(u,v) = \dfrac{1}{1+(u/v)^2}\left(-\dfrac{u}{v^2}\right) = -\dfrac{u}{v^2}\left(\dfrac{v^2}{u^2+v^2}\right) = -\dfrac{u}{u^2+v^2}$

20. $f(x,t) = e^{\sin(t/x)} \Rightarrow f_x(x,t) = e^{\sin(t/x)} \cos\left(\frac{t}{x}\right)\left(-\frac{t}{x^2}\right) = -t\cos\left(\frac{t}{x}\right)\frac{e^{\sin(t/x)}}{x^2}$,

$f_t(x,t) = e^{\sin(t/x)} \cos\left(\frac{t}{x}\right)\left(\frac{1}{x}\right) = \frac{e^{\sin(t/x)}}{x}\cos\left(\frac{t}{x}\right)$

21. $z = \ln\left(x + \sqrt{x^2+y^2}\right) \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{1}{x+\sqrt{x^2+y^2}}\left[1+\tfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x)\right] = \frac{\left(\sqrt{x^2+y^2}+x\right)\Big/\sqrt{x^2+y^2}}{\left(x+\sqrt{x^2+y^2}\right)} = \frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+\sqrt{x^2+y^2}}\left(\frac{1}{2}\right)\left(x^2+y^2\right)^{-1/2}(2y) = \frac{y}{x\sqrt{x^2+y^2}+x^2+y^2}$$

22. $f(x,y) = \int_y^x \cos\left(t^2\right)\,dt \Rightarrow f_x(x,y) = \frac{\partial}{\partial x}\int_y^x \cos\left(t^2\right)\,dt = \cos\left(x^2\right)$ by the Fundamental Theorem of Calculus, Part 1; $f_y(x,y) = \frac{\partial}{\partial y}\int_y^x \cos\left(t^2\right)\,dt = -\frac{\partial}{\partial y}\int_x^y \cos\left(t^2\right)\,dt = -\cos\left(y^2\right)$.

23. $f(x,y,z) = xy^2z^3 + 3yz \Rightarrow f_x(x,y,z) = y^2z^3$, $f_y(x,y,z) = 2xyz^3 + 3z$, $f_z(x,y,z) = 3xy^2z^2 + 3y$

24. $f(x,y,z) = x^2e^{yz} \Rightarrow f_x(x,y,z) = 2xe^{yz}$, $f_y(x,y,z) = x^2e^{yz}(z) = x^2ze^{yz}$,
$f_z(x,y,z) = x^2e^{yz}(y) = x^2ye^{yz}$.

25. $w = \ln(x+2y+3z) \Rightarrow \frac{\partial w}{\partial x} = \frac{1}{x+2y+3z}$, $\frac{\partial w}{\partial y} = \frac{2}{x+2y+3z}$, $\frac{\partial w}{\partial z} = \frac{3}{x+2y+3z}$

26. $w = \sqrt{r^2+s^2+t^2} \Rightarrow \frac{\partial w}{\partial r} = \frac{1}{2}\left(r^2+s^2+t^2\right)^{-1/2}(2r) = \frac{r}{\sqrt{r^2+s^2+t^2}}$, $\frac{\partial w}{\partial s} = \frac{s}{\sqrt{r^2+s^2+t^2}}$,
$\frac{\partial w}{\partial t} = \frac{t}{\sqrt{r^2+s^2+t^2}}$.

27. $u = xe^{-t}\sin\theta \Rightarrow \frac{\partial u}{\partial x} = e^{-t}\sin\theta$, $\frac{\partial u}{\partial t} = -xe^{-t}\sin\theta$, $\frac{\partial u}{\partial \theta} = xe^{-t}\cos\theta$

28. $u = x^{y/z} \Rightarrow u_x = \frac{y}{z}x^{(y/z)-1}$, $u_y = x^{y/z}\ln x \cdot \frac{1}{z} = \frac{x^{y/z}}{z}\ln x$, $u_z = x^{y/z}\ln x \cdot \frac{-y}{z^2} = -\frac{yx^{y/z}}{z^2}\ln x$

29. $f(x,y,z,t) = \frac{x-y}{z-t} \Rightarrow f_x(x,y,z,t) = \frac{1}{z-t}$, $f_y(x,y,z,t) = -\frac{1}{z-t}$,
$f_z(x,y,z,t) = (x-y)(-1)(z-t)^{-2} = \frac{y-x}{(z-t)^2}$, and
$f_t(x,y,z,t) = (x-y)(-1)(z-t)^{-2}(-1) = \frac{x-y}{(z-t)^2}$.

30. $f(x,y,z,t) = xy^2z^3t^4 \Rightarrow f_x(x,y,z,t) = y^2z^3t^4$, $f_y(x,y,z,t) = 2xyz^3t^4$, $f_z(x,y,z,t) = 3xy^2z^2t^4$, and $f_t(x,y,z,t) = 4xy^2z^3t^3$.

31. $u = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$. For each $i = 1, \ldots, n$,
$u_{x_i} = \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{-1/2}(2x_i) = \frac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$.

32. $u = \sin(x_1+2x_2+\cdots+nx_n)$. For each $i = 1, \ldots, n$, $u_{x_i} = i\cos(x_1+2x_2+\cdots+nx_n)$.

33. $f(x,y) = \sqrt{x^2+y^2} \Rightarrow f_x(x,y) = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}$, so $f_x(3,4) = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$.

34. $f(x,y) = \sin(2x+3y) \quad \Rightarrow \quad f_y(x,y) = \cos(2x+3y)\cdot 3 = 3\cos(2x+3y)$, so

$f_y(-6,4) = 3\cos[2(-6)+3(4)] = 3\cos 0 = 3.$

35. $f(x,y,z) = \dfrac{x}{y+z} = x(y+z)^{-1} \quad \Rightarrow \quad f_z(x,y,z) = x(-1)(y+z)^{-2} = -\dfrac{x}{(y+z)^2}$, so

$f_z(3,2,1) = -\dfrac{3}{(2+1)^2} = -\dfrac{1}{3}.$

36. $f(u,v,w) = w\tan(uv) \quad \Rightarrow \quad f_v(u,v,w) = w\sec^2(uv)\cdot u = uw\sec^2(uv)$, so

$f_v(2,0,3) = (2)(3)\sec^2(2\cdot 0) = 6.$

37. $f(x,y) = x^2 - xy + 2y^2 \quad \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x+h)y+2y^2-(x^2-xy+2y^2)}{h}$$

$$= \lim_{h\to 0}\frac{h(2x-y+h)}{h} = \lim_{h\to 0}(2x-y+h) = 2x-y$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0}\frac{x^2-x(y+h)+2(y+h)^2-(x^2-xy+2y^2)}{h}$$

$$= \lim_{h\to 0}\frac{h(4y-x+2h)}{h} = \lim_{h\to 0}(4y-x+2h) = 4y-x$$

38. $f(x,y) = \sqrt{3x-y} \quad \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3(x+h)-y}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3(x+h)-y}+\sqrt{3x-y}}{\sqrt{3(x+h)-y}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)-y}+\sqrt{3x-y}} = \frac{3}{2\sqrt{3x-y}}$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3x-(y+h)}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3x-(y+h)}+\sqrt{3x-y}}{\sqrt{3x-(y+h)}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{3x-(y+h)}+\sqrt{3x-y}} = \frac{-1}{2\sqrt{3x-y}}$$

39. $xy+yz=xz \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(xy+yz) = \dfrac{\partial}{\partial x}(xz) \quad \Leftrightarrow \quad y+y\dfrac{\partial z}{\partial x} = z+x\dfrac{\partial z}{\partial x} \quad \Leftrightarrow \quad (y-x)\dfrac{\partial z}{\partial x} = z-y$, so $\dfrac{\partial z}{\partial x} = \dfrac{z-y}{y-x}$. $\dfrac{\partial}{\partial y}(xy+yz) = \dfrac{\partial}{\partial y}(xz) \quad \Leftrightarrow \quad x+z+y\dfrac{\partial z}{\partial y} = x\dfrac{\partial z}{\partial y} \quad \Leftrightarrow \quad (y-x)\dfrac{\partial z}{\partial y} = -(x+z)$, so $\dfrac{\partial z}{\partial y} = \dfrac{x+z}{x-y}$.

40. $xyz = \cos(x+y+z) \Rightarrow \dfrac{\partial}{\partial x}(xyz) = \dfrac{\partial}{\partial x}[\cos(x+y+z)] \Leftrightarrow$
$yz + xy\dfrac{\partial z}{\partial x} = [-\sin(x+y+z)]\left(1+\dfrac{\partial z}{\partial x}\right)$, $[xy+\sin(x+y+z)]\dfrac{\partial z}{\partial x} = -[yz+\sin(x+y+z)]$, so
$\dfrac{\partial z}{\partial x} = -\dfrac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}$, $\dfrac{\partial}{\partial y}(xyz) = \dfrac{\partial}{\partial y}(\cos(x+y+z))$, and so by symmetry,
$\dfrac{\partial z}{\partial y} = -\dfrac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$.

41. $x^2+y^2-z^2 = 2x(y+z) \Leftrightarrow \dfrac{\partial}{\partial x}(x^2+y^2-z^2) = \dfrac{\partial}{\partial x}[2x(y+z)] \Leftrightarrow$
$2x - 2z\dfrac{\partial z}{\partial x} = 2(y+z) + 2x\dfrac{\partial z}{\partial x} \Leftrightarrow 2(x+z)\dfrac{\partial z}{\partial x} = 2(x-y-z)$, so $\dfrac{\partial z}{\partial x} = \dfrac{x-y-z}{x+z}$.
$\dfrac{\partial}{\partial y}(x^2+y^2-z^2) = \dfrac{\partial}{\partial y}[2x(y+z)] \Leftrightarrow 2y - 2z\dfrac{\partial z}{\partial y} = 2x\left(1+\dfrac{\partial z}{\partial y}\right) \Leftrightarrow 2(x+z)\dfrac{\partial z}{\partial y} = 2(y-x)$, so
$\dfrac{\partial z}{\partial y} = \dfrac{y-x}{x+z}$.

42. $xy^2z^3 + x^3y^2z = x+y+z \Rightarrow \dfrac{\partial}{\partial x}(xy^2z^3+x^3y^2z) = \dfrac{\partial}{\partial x}(x+y+z) \Leftrightarrow$
$y^2z^3 + 3xy^2z^2\dfrac{\partial z}{\partial x} + 3x^2y^2z + x^3y^2\dfrac{\partial z}{\partial x} = 1 + \dfrac{\partial z}{\partial x}$, so $(3xy^2z^2+x^3y^2-1)\dfrac{\partial z}{\partial x} = 1 - y^2z^3 - 3x^2y^2z$ and
$\dfrac{\partial z}{\partial x} = \dfrac{1-y^2z^3-3x^2y^2z}{3xy^2z^2+x^3y^2-1}$.
$\dfrac{\partial}{\partial y}(xy^2z^3+x^3y^2z) = \dfrac{\partial}{\partial y}(x+y+z) \Leftrightarrow 2xyz^3 + 3xy^2z^2\dfrac{\partial z}{\partial y} + 2x^3yz + x^3y^2\dfrac{\partial z}{\partial y} = 1 + \dfrac{\partial z}{\partial y}$, so
$(3xy^2z^2+x^3y^2-1)\dfrac{\partial z}{\partial y} = 1 - 2xyz^3 - 2x^3yz$ and $\dfrac{\partial z}{\partial y} = \dfrac{1-2xyz^3-2x^3yz}{3xy^2z^2+x^3y^2-1}$.

43. (a) $z = f(x) + g(y) \Rightarrow \dfrac{\partial z}{\partial x} = f'(x)$, $\dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x+y)$. Let $u = x+y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x+y)$,
$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}(1) = f'(u) = f'(x+y)$.

44. (a) $z = f(x)\,g(y) \Rightarrow \dfrac{\partial z}{\partial x} = f'(x)\,g(y)$, $\dfrac{\partial z}{\partial y} = f(x)\,g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\dfrac{\partial u}{\partial x} = y$ and $\dfrac{\partial u}{\partial y} = x$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}\cdot y = yf'(u) = yf'(xy)$
and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}\cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\dfrac{x}{y}\right)$. Let $u = \dfrac{x}{y}$. Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = f'(u)\dfrac{1}{y} = \dfrac{f'(x/y)}{y}$ and
$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = f'(u)\left(-\dfrac{x}{y^2}\right) = -\dfrac{xf'(x/y)}{y^2}$.

45. $f(x,y) = x^4 - 3x^2y^3 \Rightarrow f_x(x,y) = 4x^3 - 6xy^3$, $f_y(x,y) = -9x^2y^2$. Then $f_{xx}(x,y) = 12x^2 - 6y^3$,
$f_{xy}(x,y) = -18xy^2$, $f_{yx}(x,y) = -18xy^2$, and $f_{yy}(x,y) = -18x^2y$.

46. $f(x,y) = \ln(3x+5y) \quad \Rightarrow \quad f_x(x,y) = \dfrac{3}{3x+5y}$, $f_y(x,y) = \dfrac{5}{3x+5y}$. Then

$f_{xx}(x,y) = 3(-1)(3x+5y)^{-2}(3) = -\dfrac{9}{(3x+5y)^2}$, $f_{xy}(x,y) = -\dfrac{15}{(3x+5y)^2}$, $f_{yx}(x,y) = -\dfrac{15}{(3x+5y)^2}$,

and $f_{yy}(x,y) = -\dfrac{25}{(3x+5y)^2}$.

47. $z = \dfrac{x}{x+y} = x(x+y)^{-1} \quad \Rightarrow \quad z_x = \dfrac{1(x+y)-1(x)}{(x+y)^2} = \dfrac{y}{(x+y)^2}$, $z_y = x(-1)(x+y)^{-2} = -\dfrac{x}{(x+y)^2}$.

Then $z_{xx} = y(-2)(x+y)^{-3} = -\dfrac{2y}{(x+y)^3}$, $z_{xy} = \dfrac{1(x+y)^2 - y(2)(x+y)}{\left[(x+y)^2\right]^2} = \dfrac{x+y-2y}{(x+y)^3} = \dfrac{x-y}{(x+y)^3}$,

$z_{yx} = -\dfrac{1(x+y)^2 - x(2)(x+y)}{\left[(x+y)^2\right]^2} = -\dfrac{-x^2+xy+y^2}{(x+y)^2} = \dfrac{(x+y)(x-y)}{(x+y)^2} = \dfrac{x-y}{(x+y)^3}$, and

$z_{yy} = -x(-2)(x+y)^{-3} = \dfrac{2x}{(x+y)^3}$.

48. $z = y\tan 2x \quad \Rightarrow \quad z_x = y\sec^2(2x)\cdot 2 = 2y\sec^2(2x)$, $z_y = \tan 2x$. Then

$z_{xx} = 2y(2)\sec(2x)\cdot\sec(2x)\tan(2x)\cdot 2 = 8y\sec^2(2x)\tan(2x)$, $z_{xy} = 2\sec^2(2x)$,

$z_{yx} = \sec^2(2x)\cdot 2 = 2\sec^2(2x)$, and $z_{yy} = 0$.

49. $u = e^{-s}\sin t \quad \Rightarrow \quad u_s = -e^{-s}\sin t$, $u_t = e^{-s}\cos t$. Then $u_{ss} = e^{-s}\sin t$, $u_{st} = -e^{-s}\cos t$,

$u_{ts} = -e^{-s}\cos t$, and $u_{tt} = -e^{-s}\sin t$.

50. $v = \sqrt{x+y^2} \quad \Rightarrow \quad v_x = \frac{1}{2}\left(x+y^2\right)^{-1/2} = \dfrac{1}{2\sqrt{x+y^2}}$,

$v_y = \frac{1}{2}\left(x+y^2\right)^{-1/2}(2y) = \dfrac{y}{\sqrt{x+y^2}}$. Then $v_{xx} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(x+y^2\right)^{-3/2} = -\dfrac{1}{4(x+y^2)^{3/2}}$,

$v_{xy} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(x+y^2\right)^{-3/2}(2y) = -\dfrac{y}{2(x+y^2)^{3/2}}$, $v_{yx} = y\left(-\frac{1}{2}\right)\left(x+y^2\right)^{-3/2} = -\dfrac{y}{2(x+y^2)^{3/2}}$, and

$v_{yy} = \dfrac{1\sqrt{x+y^2} - y\left(\frac{1}{2}\right)\left(x+y^2\right)^{-1/2}(2y)}{\left(\sqrt{x+y^2}\right)^2} = \dfrac{(x+y^2)-y^2}{(x+y^2)^{3/2}} = \dfrac{x}{(x+y^2)^{3/2}}$.

51. $u = x^5y^4 - 3x^2y^3 + 2x^2 \quad \Rightarrow \quad u_x = 5x^4y^4 - 6xy^3 + 4x$, $u_{xy} = 20x^4y^3 - 18xy^2$ and

$u_y = 4x^5y^3 - 9x^2y^2$, $u_{yx} = 20x^4y^3 - 18xy^2$. Thus $u_{xy} = u_{yx}$.

52. $u = \sin^2 x\cos y \quad \Rightarrow \quad u_x = 2\sin x\cos x\cos y$, $u_{xy} = -2\sin x\cos x\sin y$ and $u_y = -\sin^2 x\sin y$,

$u_{yx} = -2\sin x\cos x\sin y$. Thus $u_{xy} = u_{yx}$.

53. $u = \ln\sqrt{x^2+y^2} = \ln\left(x^2+y^2\right)^{1/2} = \frac{1}{2}\ln\left(x^2+y^2\right) \quad \Rightarrow \quad u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,

$u_{xy} = x(-1)\left(x^2+y^2\right)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,

$u_{yx} = y(-1)\left(x^2+y^2\right)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

54. $u = xye^y \quad \Rightarrow \quad u_x = ye^y$, $u_{xy} = ye^y + e^y = (y+1)e^y$ and $u_y = x(ye^y + e^y) = x(y+1)e^y$,

$u_{yx} = (y+1)e^y$. Thus $u_{xy} = u_{yx}$.

55. $f(x,y) = x^2y^3 - 2x^4y \quad \Rightarrow \quad f_x = 2xy^3 - 8x^3y$, $f_{xx} = 2y^3 - 24x^2y$, $f_{xxx} = -48xy$

56. $f(x,y) = e^{xy^2} \quad \Rightarrow \quad f_x = y^2e^{xy^2}$, $f_{xx} = y^4e^{xy^2}$, $f_{xxy} = 4y^3e^{xy^2} + 2xy^5e^{xy^2} = 2y^3e^{xy^2}\left(2+xy^2\right)$

57. $f(x,y,z) = x^5 + x^4y^4z^3 + yz^2 \;\Rightarrow\; f_x = 5x^4 + 4x^3y^4z^3$, $f_{xy} = 16x^3y^3z^3$, and $f_{xyz} = 48x^3y^3z^2$

58. $f(x,y,z) = e^{xyz} \;\Rightarrow\; f_y = xze^{xyz}$, $f_{yz} = xe^{xyz} + xz\,(xy)\,e^{xyz} = xe^{xyz}\,(1 + yxz)$, and
$f_{yzy} = x\,(xz)\,e^{xyz}\,(1 + xyz) + xe^{xyz}\,(xz) = x^2z\,(2 + xyz)\,e^{xyz}$.

59. $z = x\sin y \;\Rightarrow\; \dfrac{\partial z}{\partial x} = \sin y$, $\dfrac{\partial^2 z}{\partial y\partial x} = \cos y$, and $\dfrac{\partial^3 z}{\partial y^2\partial x} = -\sin y$.

60. $z = \ln\sin(x-y) \;\Rightarrow\; \dfrac{\partial z}{\partial x} = \dfrac{1}{\sin(x-y)}\cos(x-y) = \cot(x-y)$, $\dfrac{\partial^2 z}{\partial x^2} = -\csc^2(x-y)$ and
$\dfrac{\partial^3 z}{\partial y\partial x^2} = -2\csc(x-y)\,[-\csc(x-y)\cot(x-y)\,(-1)] = -2\csc^2(x-y)\cot(x-y)$.

61. $u = \ln\left(x + 2y^2 + 3z^3\right) \;\Rightarrow\; \dfrac{\partial u}{\partial z} = \dfrac{1}{x+2y^2+3z^3}\left(9z^2\right) = \dfrac{9z^2}{x+2y^2+3z^3}$,
$\dfrac{\partial^2 u}{\partial y\,\partial z} = -9z^2\left(x+2y^2+3z^3\right)^{-2}(4y) = -\dfrac{36yz^2}{\left(x+2y^2+3z^3\right)^2}$, and $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \dfrac{72yz^2}{\left(x+2y^2+3z^3\right)^3}$.

62. $u = x^ay^bz^c$. If $a = 0$, or if $b = 0$ or 1, or if $c = 0, 1,$ or 2, then $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = 0$. Otherwise
$\dfrac{\partial u}{\partial z} = cx^ay^bz^{c-1}$, $\dfrac{\partial^2 u}{\partial z^2} = c\,(c-1)\,x^ay^bz^{c-2}$, $\dfrac{\partial^3 u}{\partial z^3} = c\,(c-1)\,(c-2)\,x^ay^bz^{c-3}$,
$\dfrac{\partial^4 u}{\partial y\,\partial z^3} = bc\,(c-1)\,(c-2)\,x^ay^{b-1}z^{c-3}$, $\dfrac{\partial^5 u}{\partial y^2\,\partial z^3} = b\,(b-1)\,c\,(c-1)\,(c-2)\,x^ay^{b-2}z^{c-3}$, and
$\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = ab\,(b-1)\,c\,(c-1)\,(c-2)\,x^{a-1}y^{b-2}z^{c-3}$.

63. By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$,
$f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$,
$f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.
To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:
$f_x(3,1.8) \approx \dfrac{f(3.5,1.8) - f(3,1.8)}{0.5} = \dfrac{20.0 - 18.1}{0.5} = 3.8$,
$f_x(3,1.8) \approx \dfrac{f(2.5,1.8) - f(3,1.8)}{-0.5} = \dfrac{12.5 - 18.1}{-0.5} = 11.2$. Averaging these values, we get $f_x(3,1.8) \approx 7.5$.
Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)]$ and $f_x(x,y)$ is itself a function of 2 variables, so Definition 4 says that
$f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)] = \lim\limits_{h\to 0}\dfrac{f_x(x,y+h) - f_x(x,y)}{h} \;\Rightarrow\; f_{xy}(3,2) = \lim\limits_{h\to 0}\dfrac{f_x(3,2+h) - f_x(3,2)}{h}$.

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$f_{xy}(3,2) \approx \dfrac{f_x(3,2.2) - f_x(3,2)}{0.2} = \dfrac{16.8 - 12.2}{0.2} = 23$,

$f_{xy}(3,2) \approx \dfrac{f_x(3,1.8) - f_x(3,2)}{-0.2} = \dfrac{7.5 - 12.2}{-0.2} = 23.5$. Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

64. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.

(c) $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\dfrac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at P.

(d) $f_{xy} = \dfrac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \dfrac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $\dfrac{\partial}{\partial y}(f_y) = f_{yy}$ is positive at P.

65. $u = e^{-\alpha^2 k^2 t}\sin kx \quad\Rightarrow\quad u_x = ke^{-\alpha^2 k^2 t}\cos kx$, $u_{xx} = -k^2 e^{-\alpha^2 k^2 t}\sin kx$, and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t}\sin kx$. Thus $\alpha^2 u_{xx} = u_t$.

66. (a) $u = x^2 + y^2 \quad\Rightarrow\quad u_x = 2x$, $u_{xx} = 2$; $u_y = 2y$, $u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2$, $u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2$, $u_{xx} = 6x$; $u_y = 6xy$, $u_{yy} = 6x$.

(d) $u = \ln\sqrt{x^2+y^2}$ is a solution: $u_x = \dfrac{1}{\sqrt{x^2+y^2}}\left(\dfrac{1}{2}\right)(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{x^2+y^2}$,

$u_{xx} = \dfrac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$. By symmetry, $u_{yy} = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$, so $u_{xx} + u_{yy} = 0$.

(e) $u = \sin x\cosh y + \cos x\sinh y$ is a solution:
$u_x = \cos x\cosh y - \sin x\sinh y$, $u_{xx} = -\sin x\cosh y - \cos x\sinh y$, and $u_y = \sin x\sinh y + \cos x\cosh y$, $u_{yy} = \sin x\cosh y + \cos x\sinh y$.

(f) $u = e^{-x}\cos y - e^{-y}\cos x$ is a solution: $u_x = -e^{-x}\cos y + e^{-y}\sin x$, $u_{xx} = e^{-x}\cos y + e^{-y}\cos x$, and $u_y = -e^{-x}\sin y + e^{-y}\cos x$, $u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$.

67. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\frac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-3/2}(2x) = -x\left(x^2+y^2+z^2\right)^{-3/2}$ and

$$u_{xx} = -\left(x^2+y^2+z^2\right)^{-3/2} - x\left(-\tfrac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(2x) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$. Thus

$$u_{xx}+u_{yy}+u_{zz} = \frac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0.$$

68. (a) $u = \sin(kx)\sin(akt) \quad\Rightarrow\quad u_t = ak\sin(kx)\cos(akt)$, $u_{tt} = -a^2k^2\sin(kx)\sin(akt)$,
$u_x = k\cos(kx)\sin(akt)$, $u_{xx} = -k^2\sin(kx)\sin(akt)$. Thus $u_{tt} = a^2u_{xx}$.

(b) $u = \dfrac{t}{a^2t^2-x^2} \quad\Rightarrow\quad u_t = \dfrac{(a^2t^2-x^2)-t(2a^2t)}{(a^2t^2-x^2)^2} = -\dfrac{a^2t^2+x^2}{(a^2t^2-x^2)^2}$,

$$u_{tt} = \frac{-2a^2t\left(a^2t^2-x^2\right)^2+\left(a^2t^2+x^2\right)(2)\left(a^2t^2-x^2\right)\left(2a^2t\right)}{(a^2t^2-x^2)^4} = \frac{2a^4t^3+6a^2tx^2}{a^2t^2-x^2},$$

$$u_x = t(-1)\left(a^2t^2-x^2\right)^{-2}(2x) = \frac{2tx}{(a^2t^2-x^2)^2},$$

$$u_{xx} = \frac{2t\left(a^2t^2-x^2\right)^2-2tx(2)\left(a^2t^2-x^2\right)(-2x)}{(a^2t^2-x^2)^4} = \frac{2a^2t^3-2tx^2+8tx^2}{(a^2t^2-x^2)^3} = \frac{2a^2t^3+6tx^2}{(a^2t^2-x^2)^4}.$$

Thus $u_{tt} = a^2u_{xx}$.

(c) $u = (x-at)^6+(x+at)^6 \quad\Rightarrow\quad u_t = -6a(x-at)^5+6a(x+at)^5$,
$u_{tt} = 30a^2(x-at)^4+30a^2(x+at)^4$, $u_x = 6(x-at)^5+6(x+at)^5$,
$u_{xx} = 30(x-at)^4+30(x+at)^4$. Thus $u_{tt} = a^2u_{xx}$.

(d) $u = \sin(x-at)+\ln(x+at) \quad\Rightarrow\quad u_t = -a\cos(x-at)+\dfrac{a}{x+at}$, $u_{tt} = -a^2\sin(x-at)-\dfrac{a^2}{(x+at)^2}$,
$u_x = \cos(x-at)+\dfrac{1}{x+at}$, $u_{xx} = -\sin(x-at)-\dfrac{1}{(x+at)^2}$. Thus $u_{tt} = a^2u_{xx}$.

69. Let $v = x+at$, $w = x-at$. Then $u_t = \dfrac{\partial[f(v)+g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t}+\dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v)-ag'(w)$ and
$u_{tt} = \dfrac{\partial[af'(v)-ag'(w)]}{\partial t} = a[af''(v)+ag''(w)] = a^2[f''(v)+g''(w)]$. Similarly, by using the Chain Rule we have $u_x = f'(v)+g'(w)$ and $u_{xx} = f''(v)+g''(w)$. Thus $u_{tt} = a^2u_{xx}$.

70. For each i, $i = 1,\ldots,n$, $\partial u/\partial x_i = a_ie^{a_1x_1+a_2x_2+\cdots+a_nx_n}$ and $\partial^2u/\partial x_i^2 = a_i^2e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$. Then
$\dfrac{\partial^2u}{\partial x_1^2}+\dfrac{\partial^2u}{\partial x_2^2}+\cdots+\dfrac{\partial^2u}{\partial x_n^2} = \left(a_1^2+a_2^2+\cdots+a_n^2\right)e^{a_1x_1+a_2x_2+\cdots+a_nx_n} = e^{a_1x_1+a_2x_2+\cdots+a_nx_n} = u$ since $a_1^2+a_2^2+\cdots+a_n^2 = 1$.

71. $z_x = e^y+ye^x$, $z_{xx} = ye^x$, $\partial^3z/\partial x^3 = ye^x$. By symmetry $z_y = xe^y+e^x$, $z_{yy} = xe^y$ and $\partial^3z/\partial y^3 = xe^y$. Then $\partial^3z/\partial x\partial y^2 = e^y$ and $\partial^3z/\partial x^2\partial y = e^x$. Thus $z = xe^y+ye^x$ satisfies the given partial differential equation.

72. $P = bL^{\alpha}K^{\beta}$, so $\dfrac{\partial P}{\partial L} = \alpha bL^{\alpha-1}K^{\beta}$ and $\dfrac{\partial P}{\partial K} = \beta bL^{\alpha}K^{\beta-1}$. Then

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = L\left(\alpha bL^{\alpha-1}K^{\beta}\right) + K\left(\beta bL^{\alpha}K^{\beta-1}\right) = \alpha bL^{1+\alpha-1}K^{\beta} + \beta bL^{\alpha}K^{1+\beta-1}$$
$$= (\alpha+\beta)\,bL^{\alpha}K^{\beta} = (\alpha+\beta)\,P$$

73. If we fix $K = K_0$, $P(L, K_0)$ is a function of a single variable L, and $\dfrac{dP}{dL} = \alpha\dfrac{P}{L}$ is a separable differential equation. Then $\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \;\Rightarrow\; \displaystyle\int\frac{dP}{P} = \int\alpha\frac{dL}{L} \;\Rightarrow\; \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on K_0. Then $|P| = e^{\alpha\ln|L| + C(K_0)}$, and since $P > 0$ and $L > 0$, we have $P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^{\alpha}} = C_1(K_0)\,L^{\alpha}$ where $C_1(K_0) = e^{C(K_0)}$.

74. (a) $\partial T/\partial x = -60\,(2x)\,/\left(1+x^2+y^2\right)^2$, so at $(2,1)$, $T_x = -240/(1+4+1)^2 = -\frac{20}{3}$.

(b) $\partial T/\partial y = -60\,(2y)\,/\left(1+x^2+y^2\right)^2$, so at $(2,1)$, $T_y = -120/36 = -\frac{10}{3}$. Thus from the point $(2,1)$ the temperature is decreasing at a rate of $\frac{20}{3}$ °C/m in the x-direction and is decreasing at a rate of $\frac{10}{3}$ °C/m in the y-direction.

75. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives $\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

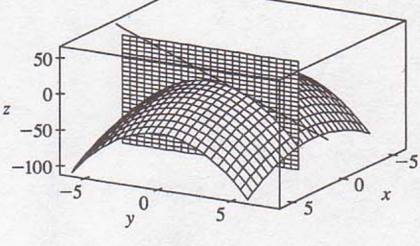
76. $P = \dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V} = \dfrac{-mRT}{V^2}$; $V = \dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$; $T = \dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P} = \dfrac{V}{mR}$. Thus $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR} = \dfrac{-mRT}{PV} = -1$, since $PV = mRT$.

77. $\dfrac{\partial K}{\partial m} = \frac{1}{2}V^2$, $\dfrac{\partial K}{\partial V} = mV$, $\dfrac{\partial^2 K}{\partial V^2} = m$. Thus $\dfrac{\partial K}{\partial m}\cdot\dfrac{\partial^2 K}{\partial V^2} = \frac{1}{2}V^2 m = K$.

78. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc\cos A$. Thus $\dfrac{\partial\left(a^2\right)}{\partial a} = \dfrac{\partial\left(b^2+c^2-2ab\cos A\right)}{\partial a}$ or $2a = -2bc\,(-\sin A)\,\dfrac{\partial A}{\partial a}$, implying that $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives $0 = 2b - 2c\,(\cos A) - 2bc\,(-\sin A)\,\dfrac{\partial A}{\partial b}$. Thus $\dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A}$. By symmetry $\dfrac{\partial A}{\partial c} = \dfrac{b\cos A - c}{bc\sin A}$.

79. $f_x(x,y) = x + 4y \;\Rightarrow\; f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \;\Rightarrow\; f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

80. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1,2,-4)$ the slope is -8. Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



81. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2+2y^2+z^2=16$, we get $8x+2z\,(\partial z/\partial x)=0 \quad\Rightarrow\quad \partial z/\partial x=-4x/z$, so when $x=1$ and $z=2$ we have $\partial z/\partial x=-2$. So the slope is $f_x(1,2)=-2$. Thus the tangent line is given by $z-2=-2\,(x-1)$, $y=2$. Taking the parameter to be $t=x-1$, we can write parametric equations for this line: $x=1+t$, $y=2$, $z=2-2t$.

82. $T(x,t)=T_0+T_1e^{-\lambda x}\sin(\omega t-\lambda x)$

(a) $$\begin{aligned}\partial T/\partial x &= T_1e^{-\lambda x}\left[\cos(\omega t-\lambda x)(-\lambda)\right]+T_1\left(-\lambda e^{-\lambda x}\right)\sin(\omega t-\lambda x)\\ &= -\lambda T_1e^{-\lambda x}\left[\sin(\omega t-\lambda x)+\cos(\omega t-\lambda x)\right]\end{aligned}$$

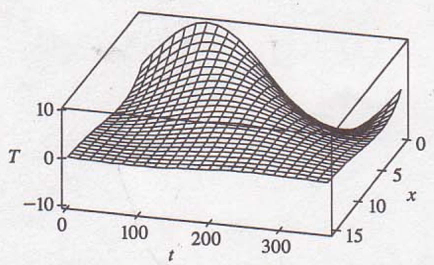
This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

(b) $\partial T/\partial t=T_1e^{-\lambda x}\left[\cos(\omega t-\lambda x)(\omega)\right]=\omega T_1e^{-\lambda x}\cos(\omega t-\lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c) $$\begin{aligned}T_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\\ &= -\lambda T_1\left(e^{-\lambda x}\left[\cos(\omega t-\lambda x)(-\lambda)-\sin(\omega t-\lambda x)(-\lambda)\right]\right.\\ &\qquad\qquad \left.+\,e^{-\lambda x}(-\lambda)\left[\sin(\omega t-\lambda x)+\cos(\omega t-\lambda x)\right]\right)\\ &= 2\lambda^2T_1e^{-\lambda x}\cos(\omega t-\lambda x)\end{aligned}$$

But from part (b), $T_t=\omega T_1e^{-\lambda x}\cos(\omega t-\lambda x)=\dfrac{\omega}{2\lambda^2}T_{xx}$. So with $k=\dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t\approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t\approx 150$, and at a depth of 10 feet, at $t\approx 220$.

83. By Clairaut's Theorem, $f_{xyy}=(f_{xy})_y=(f_{yx})_y=f_{yxy}=(f_y)_{xy}=(f_y)_{yx}=f_{yyx}$.

84. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n nth order partial derivatives.

(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all nth order partial derivatives with p partials with respect to x and $n-p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an nth order partial derivative can range from 0 to n, a function of two variables has $n+1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

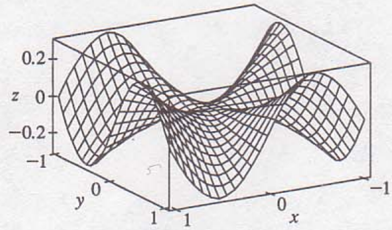
(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n nth order partial derivatives of a function of three variables.

85. Let $g(x) = f(x,0) = x(x^2)^{-3/2} e^0 = x|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1,0) = g'(1) = -2$.

86. $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(h^3+0)^{1/3} - 0}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = 1.$

Or: Let $g(x) = f(x,0) = \sqrt[3]{x^3+0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0,0) = g'(0) = 1$.

87. (a)



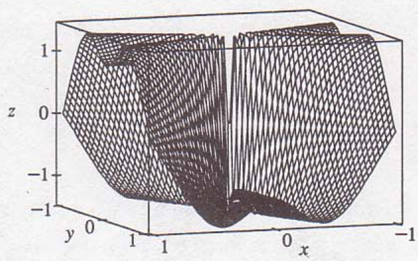
(b) For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$, and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0$.

(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h\to 0} \dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$f_{yx}(0,0) = \dfrac{\partial f_y}{\partial x} = \lim\limits_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{h^5/h^4}{h} = 1.$

(e) For $(x,y) \neq (0,0)$, we use a CAS to compute $f_{xy}(x,y) = \dfrac{x^6 + 9x^4y^2 - 4x^2y^4 + 4y^6}{(x^2+y^2)^3}$. Now as $(x,y) \to (0,0)$ along the x-axis, $f_{xy}(x,y) \to 1$ while as $(x,y) \to (0,0)$ along the y-axis, $f_{xy}(x,y) \to 4$. Thus f_{xy} isn't continuous at $(0,0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



15.4 Tangent Planes and Linear Approximations — ET 14.4

1. $z = f(x,y) = y^2 - x^2 \quad \Rightarrow \quad f_x(x,y) = -2x$, $f_y(x,y) = 2y$, so $f_x(-4,5) = 8$, $f_y(-4,5) = 10$.
By Equation 2, an equation of the tangent plane is $z - 9 = f_x(-4,5)[x - (-4)] + f_y(-4,5)(y-5) \quad \Rightarrow$
$z - 9 = 8(x+4) + 10(y-5)$ or $z = 8x + 10y - 9$.

2. $z = f(x,y) = 9x^2 + y^2 + 6x - 3y + 5 \quad \Rightarrow \quad f_x(x,y) = 18x + 6$, $f_y(x,y) = 2y - 3$, so
$f_x(1,2) = 24$ and $f_y(1,2) = 1$. By Equation 2, an equation of the tangent plane is
$z - 18 = f_x(1,2)(x-1) + f_y(1,2)(y-2) \quad \Rightarrow \quad z - 18 = 24(x-1) + 1(y-2)$ or $z = 24x + y - 8$.

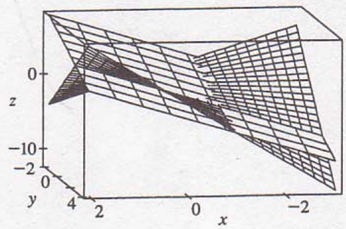
3. $z = f(x,y) = \sqrt{4 - x^2 - 2y^2} \quad \Rightarrow \quad f_x(x,y) = \frac{1}{2}\left(4 - x^2 - 2y^2\right)^{-1/2}(-2x) = -\dfrac{x}{\sqrt{4 - x^2 - 2y^2}}$,
$f_y(x,y) = \frac{1}{2}\left(4 - x^2 - 2y^2\right)^{-1/2}(-4y) = -\dfrac{2y}{\sqrt{4 - x^2 - 2y^2}}$, so $f_x(1,-1) = -1$ and $f_y(1,-1) = 2$. Thus,
an equation of the tangent plane is
$z - 1 = f_x(1,-1)(x-1) + f_y(1,-1)[y - (-1)] \quad \Rightarrow \quad z - 1 = -1(x-1) + 2(y+1)$ or $x - 2y + z = 4$.

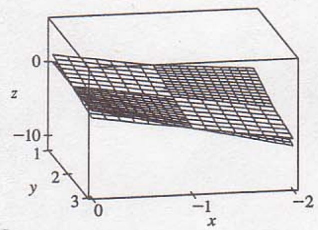
4. $z = f(x,y) = \sin(x+y) \quad \Rightarrow \quad f_x(x,y) = \cos(x+y)$, $f_y(x,y) = \cos(x+y)$, $f_x(1,-1) = 1 = f_y(1,-1)$
and an equation of the tangent plane is $z = (x-1) + (y+1)$ or $z = x + y$.

5. $z = f(x,y) = \ln(2x+y) \quad \Rightarrow \quad f_x(x,y) = \dfrac{2}{2x+y}$, $f_y(x,y) = \dfrac{1}{2x+y}$, $f_x(-1,3) = 2$, $f_y(-1,3) = 1$.
Thus an equation of the tangent plane is $z = 2(x+1) + (y-3)$ or $z = 2x + y - 1$.

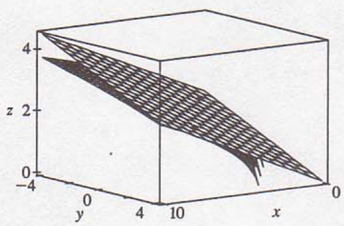
6. $z = f(x,y) = e^x \ln y \quad \Rightarrow \quad f_x(x,y) = e^x \ln y$, $f_y(x,y) = e^x/y$, $f_x(3,1) = 0$, $f_y(3,1) = e^3$, and an equation
of the tangent plane is $z = e^3(y-1)$ or $z = e^3 y - e^3$.

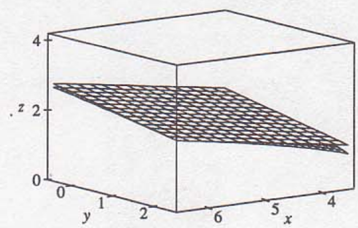
7. $z = f(x,y) = xy$, so $f_x(x,y) = y \quad \Rightarrow \quad f_x(-1,2) = 2$, $f_y(x,y) = x \quad \Rightarrow \quad f_y(-1,2) = -1$ and an equation
of the tangent plane is $z + 2 = 2(x+1) + (-1)(y-2)$ or $z = 2x - y + 2$. After zooming in, the surface and the
tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the
surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



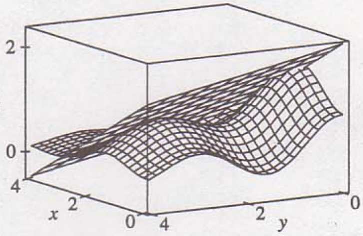


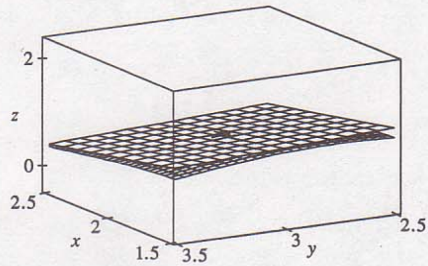
8. $z = f(x,y) = \sqrt{x-y}$, so $f_x(x,y) = \frac{1}{2}(x-y)^{-1/2}$, $f_x(5,1) = \frac{1}{4}$, $f_y(x,y) = -\frac{1}{2}(x-y)^{-1/2}$,
$f_y(5,1) = -\frac{1}{4}$, and an equation of the tangent plane is $z - 2 = \frac{1}{4}(x-5) - \frac{1}{4}(y-1)$ or $z = \frac{1}{4}x - \frac{1}{4}y + 1$. After
zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above
the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



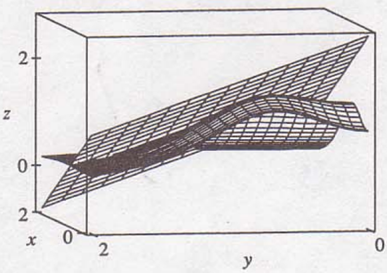


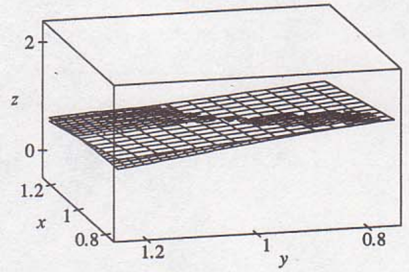
9. $f(x,y) = e^{-(x^2+y^2)/15}\left(\sin^2 x + \cos^2 y\right)$. A CAS gives
$f_x = -\frac{2}{15}e^{-(x^2+y^2)/15}\left(x\sin^2 x + x\cos^2 y - 15\sin x\cos x\right)$ and
$f_y = -\frac{2}{15}e^{-(x^2+y^2)/15}\left(y\sin^2 x + y\cos^2 y + 15\sin y\cos y\right)$. We use the CAS to evaluate these at $(2,3)$, and then substitute the results into Equation 2 in order to plot the tangent plane. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





10. $f(x,y) = \dfrac{\sqrt{1+4x^2+4y^2}}{1+x^4+y^4}$. A CAS gives $f_x = \dfrac{4x\left(1-3x^4+y^4-x^2-4x^2y^2\right)}{\sqrt{1+4x^2+4y^2}\,(1+x^4+y^4)^2}$ and
$f_y = \dfrac{4y\left(1-3y^4+x^4-y^2-4x^2y^2\right)}{\sqrt{1+4x^2+4y^2}\,(1+x^4+y^4)^2}$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane: $z = \dfrac{25-8x-8y}{9}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





11. $f(x,y) = x\sqrt{y}$. The partial derivatives are $f_x(x,y) = \sqrt{y}$ and $f_y(x,y) = \dfrac{x}{2\sqrt{y}}$, so $f_x(1,4) = 2$ and $f_y(1,4) = \frac{1}{4}$. Both f_x and f_y are continuous functions for $y > 0$, so by Theorem 8, f is differentiable at $(1,4)$. By Equation 3, the linearization of f at $(1,4)$ is given by
$L(x,y) = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4) = 2 + 2(x-1) + \frac{1}{4}(y-4) = 2x + \frac{1}{4}y - 1$.

12. $f(x,y) = y\ln x$. The partial derivatives are $f_x(x,y) = y/x$ and $f_y(x,y) = \ln x$, so $f_x(2,1) = \frac{1}{2}$ and $f_y(2,1) = \ln 2$. Both f_x and f_y are continuous functions for $x > 0$, so f is differentiable at $(2,1)$ by Theorem 8. The linearization of f at $(2,1)$ is given by

$$\begin{aligned} L(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = \ln 2 + \tfrac{1}{2}(x-2) + \ln 2\,(y-1) \\ &= \tfrac{1}{2}x + (\ln 2)\,y - 1 \end{aligned}$$

13. $f(x,y) = e^x \cos xy$. The partial derivatives are $f_x(x,y) = e^x(\cos xy - y\sin xy)$ and $f_y(x,y) = -xe^x \sin xy$, so $f_x(0,0) = 1$ and $f_y(0,0) = 0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0,0)$ by Theorem 8. The linearization of f at $(0,0)$ is given by $L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + 1(x-0) + 0(y-0) = x+1$.

14. $f(x,y) = \dfrac{x}{y}$. The partial derivatives are $f_x(x,y) = \dfrac{1}{y}$ and $f_y(x,y) = -\dfrac{x}{y^2}$, so $f_x(6,3) = \frac{1}{3}$ and $f_y(6,3) = -\frac{2}{3}$. Both f_x and f_y are continuous functions for $y \neq 0$, so f is differentiable at $(6,3)$ by Theorem 8. The linearization of f at $(6,3)$ is given by $L(x,y) = f(6,3) + f_x(6,3)(x-6) + f_y(6,3)(y-3) = 2 + \frac{1}{3}(x-6) - \frac{2}{3}(y-3) = \frac{1}{3}x - \frac{2}{3}y + 2$

15. $f(x,y) = \tan^{-1}(x+2y)$. The partial derivatives are $f_x(x,y) = \dfrac{1}{1+(x+2y)^2}$ and $f_y(x,y) = \dfrac{2}{1+(x+2y)^2}$, so $f_x(1,0) = \frac{1}{2}$ and $f_y(1,0) = 1$. Both f_x and f_y are continuous functions, so f is differentiable at $(1,0)$, and the linearization of f at $(1,0)$ is $L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = \frac{\pi}{4} + \frac{1}{2}(x-1) + 1(y) = \frac{1}{2}x + y + \frac{\pi}{4} - \frac{1}{2}$.

16. $f(x,y) = \sqrt{1+x^2y^2}$. The partial derivatives are $f_x(x,y) = \frac{1}{2}\left(1+x^2y^2\right)^{-1/2}\left(2xy^2\right) = \dfrac{xy^2}{\sqrt{1+x^2y^2}}$ and $f_y(x,y) = \frac{1}{2}\left(1+x^2y^2\right)^{-1/2}\left(2x^2y\right) = \dfrac{x^2y}{\sqrt{1+x^2y^2}}$, so $f_x(0,2) = 0$ and $f_y(0,2) = 0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0,2)$, and the linearization of f at $(0,2)$ is $L(x,y) = f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2) = 1 + 0(x) + 0(y-2) = 1$.

17. $f(x,y) = \sqrt{20-x^2-7y^2} \quad\Rightarrow\quad f_x(x,y) = -\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y) = -\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1) = -\frac{2}{3}$ and $f_y(2,1) = -\frac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by

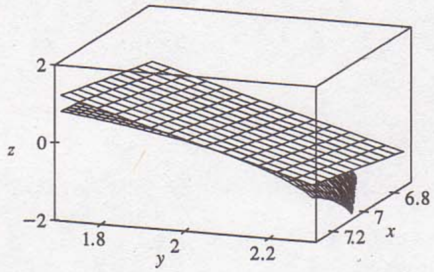
$$\begin{aligned} f(x,y) &\approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 3 - \tfrac{2}{3}(x-2) - \tfrac{7}{3}(y-1) \\ &= -\tfrac{2}{3}x - \tfrac{7}{3}y + \tfrac{20}{3} \end{aligned}$$

Thus $f(1.95, 1.08) \approx -\frac{2}{3}(1.95) - \frac{7}{3}(1.08) + \frac{20}{3} = 2.84\overline{6}$.

18. $f(x,y) = \ln(x-3y) \quad\Rightarrow\quad f_x(x,y) = \dfrac{1}{x-3y}$ and $f_y(x,y) = -\dfrac{3}{x-3y}$, so $f_x(7,2) = 1$ and $f_y(7,2) = -3$. Then the linear approximation of f at $(7,2)$ is given by

$$\begin{aligned} f(x,y) &\approx f(7,2) + f_x(7,2)(x-7) + f_y(7,2)(y-2) \\ &= 0 + 1(x-7) - 3(y-2) = x - 3y - 1 \end{aligned}$$

Thus $f(6.9, 2.06) \approx 6.9 - 3(2.06) - 1 = -0.28$. The graph shows that our approximated value is slightly greater than the actual value.



19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad f_x(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$, $f_y(x, y, z) = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$, and

$f_z(x, y, z) = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$, so $f_x(3, 2, 6) = \frac{3}{7}$, $f_y(3, 2, 6) = \frac{2}{7}$, and $f_z(3, 2, 6) = \frac{6}{7}$. Then the linear approximation of f at $(3, 2, 6)$ is given by

$$\begin{aligned} f(x, y, z) &\approx f(3, 2, 6) + f_x(3, 2, 6)(x - 3) + f_y(3, 2, 6)(y - 2) + f_z(3, 2, 6)(z - 6) \\ &= 7 + \tfrac{3}{7}(x - 3) + \tfrac{2}{7}(y - 2) + \tfrac{6}{7}(z - 6) = \tfrac{3}{7}x + \tfrac{2}{7}y + \tfrac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

20. From the table, $f(40, 20) = 28$. To estimate $f_v(40, 20)$ and $f_t(40, 20)$ we follow the procedure used in Exercise 15.3.4 [ET 14.3.4]. Since $f_v(40, 20) = \lim\limits_{h \to 0} \dfrac{f(40 + h, 20) - f(40, 20)}{h}$, we approximate this quantity with $h = \pm 10$ and use the values given in the table: $f_v(40, 20) \approx \dfrac{f(50, 20) - f(40, 20)}{10} = \dfrac{40 - 28}{10} = 1.2$,

$f_v(40, 20) \approx \dfrac{f(30, 20) - f(40, 20)}{-10} = \dfrac{17 - 28}{-10} = 1.1$. Averaging these values gives

$f_v(40, 20) \approx 1.15$. Similarly, $f_t(40, 20) = \lim\limits_{h \to 0} \dfrac{f(40, 20 + h) - f(40, 20)}{h}$, so we use

$h = 10$ and $h = -5$: $f_t(40, 20) \approx \dfrac{f(40, 30) - f(40, 20)}{10} = \dfrac{31 - 28}{10} = 0.3$,

$f_t(40, 20) \approx \dfrac{f(40, 15) - f(40, 20)}{-5} = \dfrac{25 - 28}{-5} = 0.6$. Averaging these values gives $f_t(40, 15) \approx 0.45$. The linear approximation, then, is

$$\begin{aligned} f(v, t) &\approx f(40, 20) + f_v(40, 20)(v - 40) + f_t(40, 20)(t - 20) \\ &\approx 28 + 1.15(v - 40) + 0.45(t - 20) \end{aligned}$$

When $v = 43$ and $t = 24$, we estimate $f(43, 24) \approx 28 + 1.15(43 - 40) + 0.45(24 - 20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.

21. From the table, $f(94,80) = 127$. To estimate $f_T(94,80)$ and $f_H(94,80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94,80) = \lim\limits_{h\to 0} \dfrac{f(94+h,80) - f(94,80)}{h}$, we approximate this quantity with $h = \pm 2$ and use the values given in the table: $f_T(94,80) \approx \dfrac{f(96,80) - f(94,80)}{2} = \dfrac{135 - 127}{2} = 4$,

$f_T(94,80) \approx \dfrac{f(92,80) - f(94,80)}{-2} = \dfrac{119 - 127}{-2} = 4.$

Averaging these values gives $f_T(94,80) \approx 4$. Similarly, $f_H(94,80) = \lim\limits_{h\to 0} \dfrac{f(94,80+h) - f(94,80)}{h}$,

so we use $h = \pm 5$: $f_H(94,80) \approx \dfrac{f(94,85) - f(94,80)}{5} = \dfrac{132 - 127}{5} = 1$,

$f_H(94,80) \approx \dfrac{f(94,75) - f(94,80)}{-5} = \dfrac{122 - 127}{-5} = 1$. Averaging these values gives $f_H(94,80) \approx 1$. The linear approximation, then, is

$$\begin{aligned} f(T,H) &\approx f(94,80) + f_T(94,80)(T - 94) + f_H(94,80)(H - 80) \\ &\approx 127 + 4(T - 94) + 1(H - 80) \end{aligned}$$

Thus when $T = 95$ and $H = 78$, $f(95,78) \approx 127 + 4(95 - 94) + 1(78 - 80) = 129$, so we estimate the heat index to be approximately $129\,^\circ\mathrm{F}$.

22. From the table, $f(16,30) = 9$. To estimate $f_T(16,30)$ and $f_v(16,30)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(16,30) = \lim\limits_{h\to 0} \dfrac{f(16+h,30) - f(16,30)}{h}$, we approximate this quantity with $h = \pm 4$ and use the values given in the table: $f_T(16,30) \approx \dfrac{f(20,30) - f(16,30)}{4} = \dfrac{14 - 9}{4} = 1.25$,

$f_T(16,30) \approx \dfrac{f(12,30) - f(16,30)}{-4} = \dfrac{3 - 9}{-4} = 1.5$. Averaging these values gives

$f_T(16,30) \approx 1.375$. Similarly, $f_v(16,30) = \lim\limits_{h\to 0} \dfrac{f(16,30+h) - f(16,30)}{h}$,

so we use $h = \pm 10$: $f_v(16,30) \approx \dfrac{f(16,40) - f(16,30)}{10} = \dfrac{7 - 9}{10} = -0.2$,

$f_v(16,30) \approx \dfrac{f(16,20) - f(16,30)}{-10} = \dfrac{11 - 9}{-10} = -0.2$. Averaging these values gives $f_v(16,30) \approx -0.2$. The linear approximation, then, is

$$\begin{aligned} f(T,v) &\approx f(16,30) + f_T(16,30)(T - 16) + f_v(16,30)(v - 30) \\ &\approx 9 + 1.375(T - 16) - 0.2(v - 30) \end{aligned}$$

Thus when $T = 14$ and $v = 27$, $f(14,27) \approx 9 + 1.375(14 - 16) - 0.2(27 - 30) = 6.85$, so we estimate the wind-chill index to be approximately $6.85\,^\circ\mathrm{C}$.

23. $z = x^2y^3 \;\Rightarrow\; dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = 2xy^3\,dx + 3x^2y^2\,dy$

24. $v = \ln(2x - 3y) \;\Rightarrow\; dv = \left(\dfrac{2}{2x - 3y}\right)dx + \left(\dfrac{-3}{2x - 3y}\right)dy = \dfrac{1}{2x - 3y}(2\,dx - 3\,dy)$

25. $u = e^t \sin\theta \;\Rightarrow\; du = \dfrac{\partial u}{\partial t}\,dt + \dfrac{\partial u}{\partial \theta}\,d\theta = e^t \sin\theta\,dt + e^t\cos\theta\,d\theta$

26. $u = \dfrac{r}{s+2t} \quad \Rightarrow$

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt = \frac{1}{s+2t}\,dr + r\,(-1)\,(s+2t)^{-2}\,ds + r\,(-1)\,(s+2t)^{-2}\,(2)\,dt$$
$$= \frac{1}{s+2t}\,dr - \frac{r}{(s+2t)^2}\,ds - \frac{2r}{(s+2t)^2}\,dt$$

27. $w = \ln\sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$
$$= \left(\frac{1}{2}\right)\frac{2x\left(x^2+y^2+z^2\right)^{-1/2}\,dx + 2y\left(x^2+y^2+z^2\right)^{-1/2}\,dy + 2z\left(x^2+y^2+z^2\right)^{-1/2}\,dz}{\left(x^2+y^2+z^2\right)^{1/2}}$$
$$= \frac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2}$$

28. $w = x\sin yz \quad \Rightarrow \quad dw = \dfrac{\partial w}{\partial x}\,dx + \dfrac{\partial w}{\partial y}\,dy + \dfrac{\partial w}{\partial z}\,dz = (\sin yz)\,dx + (xz\cos yz)\,dy + (xy\cos yz)\,dz$

29. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$, $dz = z_x\,(1,2)\,dx + z_y\,(1,2)\,dy = (10)\,(0.05) + (4)\,(0.1) = 0.9$ while $\Delta z = f\,(1.05, 2.1) - f\,(1,2) = 5\,(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

30. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$, $dz = (7)\,(-0.04) + (-9)\,(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)\,(-0.95) + 3\,(-0.95)^2 - (9+3+3) = -0.7189$.

31. $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24\,(0.1) + 30\,(0.1) = 5.4\text{ cm}^2$.

32. Let S be surface area. Then $S = 2\,(xy + xz + yz)$ and $dS = 2\,(y+z)\,dx + 2\,(x+z)\,dy + 2\,(x+y)\,dz$. The maximum error occurs with $\Delta x = \Delta y = \Delta z = 0.2$. Using $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$ we find the maximum error in calculated surface area to be about $dS = (220)\,(0.2) + (260)\,(0.2) + (280)\,(0.2) = 152\text{ cm}^2$.

33. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi rh\,dr + \pi r^2\,dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi\,(48)\,(0.04) + \pi\,(16)\,(0.08) \approx 16.08\text{ cm}^3$. Thus the amount of tin is about 16 cm^3.

34. Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi rh\,dr + \pi r^2\,dh$ is an estimate of the amount of metal. With $dr = 0.05$ and $dh = 0.20$ we get $dV = 2\pi\,(2)\,(10)\,(0.05) + \pi\,(2)^2\,(0.20) = 2.80\pi \approx 8.8\text{ cm}^3$.

35. The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y\,dx + x\,dy$, so with $dx = dy = \frac{3+3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150\text{ ft}^2$. Thus there are approximately 150 ft^2 of paint in the stripe.

36. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31\dfrac{T}{V}$, so

$$dP = \left(\frac{8.31}{V}\right)dT - \frac{8.31\cdot T}{V^2}\,dV = 8.31\left[-\frac{5}{12} - \frac{310}{144}\cdot\frac{3}{10}\right] \approx -8.83.$$

Thus the pressure will drop by about 8.83 kPa.

37. First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:

$\dfrac{\partial}{\partial R_1}\left[\dfrac{1}{R}\right] = \dfrac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$. Then by

symmetry, $\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \quad\Leftrightarrow\quad R = \frac{200}{17}$ ohms.

Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005 R_i$. So

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\,\Delta R_1 + \frac{\partial R}{\partial R_2}\,\Delta R_2 + \frac{\partial R}{\partial R_3}\,\Delta R_3 = (0.005)\,R^2\left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right]$$

$$= (0.005)\,R = \frac{1}{17} \approx 0.059 \text{ ohms}$$

38. Let x, y, z and w be the four numbers with $p(x,y,z,w) = xyzw$. Since the largest error due to rounding for each number is 0.05, the maximum error in the calculated product is approximated by

$dp = (yzw)(0.05) + (xzw)(0.05) + (xyw)(0.05) + (xyz)(0.05)$. Furthermore, each of the numbers is positive but less than 50, so the product of any three is between 0 and $(50)^3$. Thus $dp \le 4(50)^3(0.05) = 25{,}000$.

39. $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = (a+\Delta x)^2 + (b+\Delta y)^2 - (a^2+b^2)$

$$= a^2 + 2a\,\Delta x + (\Delta x)^2 + b^2 + 2b\,\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\,\Delta x + (\Delta x)^2 + 2b\,\Delta y + (\Delta y)^2$$

But $f_x(a,b) = 2a$ and $f_y(a,b) = 2b$ and so $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \Delta x\Delta x + \Delta y\Delta y$, which is Definition 7 with $\epsilon_1 = \Delta x$ and $\epsilon_2 = \Delta y$. Hence f is differentiable.

40. $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = (a+\Delta x)(b+\Delta y) - 5(b+\Delta y)^2 - (ab - 5b^2)$

$$= ab + a\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5b^2 - 10b\,\Delta y - 5(\Delta y)^2 - ab + 5b^2$$

$$= (a - 10b)\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5\,\Delta y\,\Delta y,$$

but $f_x(a,b) = b$ and $f_y(a,b) = a - 10b$ and so $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \Delta x\,\Delta y - 5\Delta y\,\Delta y$, which is Definition 7 with $\epsilon_1 = \Delta y$ and $\epsilon_2 = -5\,\Delta y$. Hence f is differentiable.

41. To show that f is continuous at (a,b) we need to show that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ or equivalently

$\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Since f is differentiable at (a,b),

$f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Thus $f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x,\Delta y)\to(0,0)$ gives $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Thus f is continuous at (a,b).

42. $\lim\limits_{h\to 0}\dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{0-0}{h} = 0$ and $\lim\limits_{h\to 0}\dfrac{f(0,h)-f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{0-0}{h} = 0$. Thus $f_x(0,0) = f_y(0,0) = 0$. To show that f isn't differentiable at $(0,0)$ we need only show that f is not continuous at $(0,0)$ and apply Exercise 41. As $(x,y)\to(0,0)$ along the x-axis $f(x,y) = 0/x^2 = 0$ for $x \neq 0$ so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x-axis. But as $(x,y)\to(0,0)$ along the line $y = x$, $f(x,x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x,y) \to \frac{1}{2}$ as $(x,y)\to(0,0)$ along this line. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is discontinuous at $(0,0)$ and thus not differentiable there.

15.5 The Chain Rule

1. $z = x^2y + xy^2$, $x = 2 + t^4$, $y = 1 - t^3$ $\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(2xy + y^2\right)\left(4t^3\right) + \left(x^2 + 2xy\right)\left(-3t^2\right) = 4\left(2xy + y^2\right)t^3 - 3\left(x^2 + 2xy\right)t^2$$

2. $z = \sqrt{x^2 + y^2}$, $x = e^{2t}$, $y = e^{-2t}$ $\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2x)\cdot e^{2t}(2) + \tfrac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2y)\cdot e^{-2t}(-2) = \frac{2xe^{2t} - 2ye^{2t}}{\sqrt{x^2 + y^2}}$$

3. $z = \sin x \cos y$, $x = \pi t$, $y = \sqrt{t}$ $\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \cos x \cos y \cdot \pi + \sin x\,(-\sin y)\cdot \tfrac{1}{2}t^{-1/2} = \pi \cos x \cos y - \frac{1}{2\sqrt{t}}\sin x \sin y$$

4. $z = x \ln(x + 2y)$, $x = \sin t$, $y = \cos t$ $\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left[x\cdot\frac{1}{x + 2y} + 1\cdot\ln(x + 2y)\right]\cos t + x\cdot\frac{1}{x + 2y}(2)\cdot(-\sin t)$$

$$= \left[\frac{x}{x + 2y} + \ln(x + 2y)\right]\cos t - \frac{2x}{x + 2y}(\sin t)$$

5. $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$ $\Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

6. $w = xy + yz^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$ $\Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = y\cdot e^t + \left(x + z^2\right)\cdot\left(e^t\cos t + e^t\sin t\right) + 2yz\cdot\left(-e^t\sin t + e^t\cos t\right)$$

$$= e^t\left[y + \left(x + z^2\right)(\cos t + \sin t) + 2yz(\cos t - \sin t)\right]$$

7. $z = x^2 + xy + y^2$, $x = s + t$, $y = st$ $\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(t) = 2x + y + xt + 2yt$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x + y)(1) + (x + 2y)(s) = 2x + y + xs + 2ys$$

8. $z = \dfrac{x}{y}$, $x = se^t$, $y = 1 + se^{-t}$ $\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{y}\left(e^t\right) + \left(-\frac{x}{y^2}\right)\left(e^{-t}\right) = \frac{1}{y}e^t - \frac{x}{y^2}e^{-t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{y}\left(se^t\right) + \left(-\frac{x}{y^2}\right)\left(-se^{-t}\right) = \frac{s}{y}e^t + \frac{xs}{y^2}e^{-t}$$

9. $z = \arctan(2x + y)$, $x = s^2t$, $y = s\ln t$ $\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2}{1 + (2x + y)^2}\cdot 2st + \frac{1}{1 + (2x + y)^2}\cdot \ln t = \frac{4st + \ln t}{1 + (2x + y)^2}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2}{1 + (2x + y)^2}\cdot s^2 + \frac{1}{1 + (2x + y)^2}\cdot\frac{s}{t} = \frac{2s^2 + s/t}{1 + (2x + y)^2}$$

10. $z = e^{xy}\tan y,\ x = s + 2t,\ y = \dfrac{s}{t} \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = ye^{xy}\tan y \cdot 1 + \left(e^{xy}\sec^2 y + xe^{xy}\tan y\right)\cdot\frac{1}{t} = ye^{xy}\tan y + \frac{e^{xy}}{t}\left(\sec^2 y + x\tan y\right)$$

$$\frac{\partial z}{\partial t} = ye^{xy}\tan y \cdot 2 + \left(e^{xy}\sec^2 y + xe^{xy}\tan y\right)\left(\frac{-s}{t^2}\right) = 2ye^{xy}\tan y - \frac{se^{xy}}{t^2}\left(\sec^2 y + x\tan y\right)$$

11. $z = e^r\cos\theta,\ r = st,\ \theta = \sqrt{s^2+t^2} \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + e^r\left(-\sin\theta\right)\cdot\tfrac{1}{2}\left(s^2+t^2\right)^{-1/2}(2s)\\ &= te^r\cos\theta - e^r\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}} = e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + e^r\left(-\sin\theta\right)\cdot\tfrac{1}{2}\left(s^2+t^2\right)^{-1/2}(2t)\\ &= se^r\cos\theta - e^r\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}} = e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)\end{aligned}$$

12. $z = \sin\alpha\tan\beta,\ \alpha = 3s + t,\ \beta = s - t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial s} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial s} = \cos\alpha\tan\beta\cdot 3 + \sin\alpha\sec^2\beta\cdot 1 = 3\cos\alpha\tan\beta + \sin\alpha\sec^2\beta$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial t} = \cos\alpha\tan\beta\cdot 1 + \sin\alpha\sec^2\beta\cdot(-1) = \cos\alpha\tan\beta - \sin\alpha\sec^2\beta$$

13. When $t = 3$, $x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)\,g'(3) + f_y(2,7)\,h'(3) = (6)(5) + (-8)(-4) = 62.$$

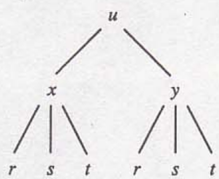
14. By the Chain Rule (3), $\dfrac{\partial W}{\partial s} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial s}$. Then

$$\begin{aligned}W_s(1,0) &= F_u\left(u(1,0), v(1,0)\right)u_s(1,0) + F_v\left(u(1,0), v(1,0)\right)v_s(1,0)\\ &= F_u(2,3)\,u_s(1,0) + F_v(2,3)\,v_s(1,0) = (-1)(-2) + (10)(5) = 52\end{aligned}$$

Similarly, $\dfrac{\partial W}{\partial t} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial t} \quad \Rightarrow$

$$\begin{aligned}W_t(1,0) &= F_u\left(u(1,0), v(1,0)\right)u_t(1,0) + F_v\left(u(1,0), v(1,0)\right)v_t(1,0)\\ &= F_u(2,3)\,u_t(1,0) + F_v(2,3)\,v_t(1,0) = (-1)(6) + (10)(4) = 34\end{aligned}$$

15.

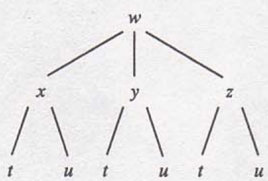


$u = f(x,y),\ x = x(r,s,t),\ y = y(r,s,t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},\quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

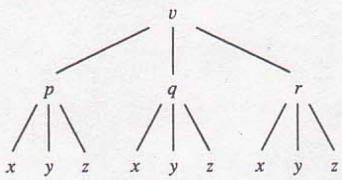
16.



$w = f(x,y,z),\ x = x(t,u),\ y = y(t,u),\ z = z(t,u) \quad \Rightarrow$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t},\quad \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

17.

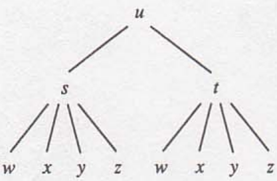


$v = f(p,q,r)$, $p = p(x,y,z)$, $q = q(x,y,z)$, $r = r(x,y,z) \quad \Rightarrow$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x},\ \frac{\partial v}{\partial y} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial y},$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial z}$$

18.



$u = f(s,t)$, $s = s(w,x,y,z)$, $t = t(w,x,y,z) \quad \Rightarrow$

$$\frac{\partial u}{\partial w} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial w} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial w},\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y},\ \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

19. $w = x^2 + y^2 + z^2$, $x = st$, $y = s\cos t$, $z = s\sin t \quad \Rightarrow$

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial s} = 2xt + 2y\cos t + 2z\sin t$. When $s = 1$, $t = 0$, we have $x = 0$, $y = 1$ and $z = 0$, so $\partial w/\partial s = 2\cos 0 = 2$. Similarly $\partial w/\partial t = 2xs + 2y\,(-s\sin t) + 2z\,(s\cos t) = 0 + (-2)\sin 0 + 0 = 0$ when $s = 1$ and $t = 0$.

20. $u = xy + yz + zx$, $x = st$, $y = e^{st}$, $z = t^2 \quad \Rightarrow$

$\partial u/\partial s = (y+z)\,t + (x+z)\,te^{st} + (x+y)\,(0)$ and $\partial u/\partial t = (y+z)\,s + (x+z)\,se^{st} + (x+y)\,(2t)$. When $s = 0$, $t = 1$, we have $x = 0$, $y = 1$, $z = 1$, so $\partial u/\partial s = 2 + 1 + 0 = 3$ and $\partial u/\partial t = 0 + 0 + (1)\,(2) = 2$.

21. $z = y^2\tan x$, $x = t^2uv$, $y = u + tv^2 \quad \Rightarrow$

$\partial z/\partial t = \left(y^2\sec^2 x\right)2tuv + (2y\tan x)\,v^2$, $\partial z/\partial u = \left(y^2\sec^2 x\right)t^2v + 2y\tan x$,

$\partial z/\partial v = \left(y^2\sec^2 x\right)t^2u + (2y\tan x)\,2tv$. When $t = 2$, $u = 1$ and $v = 0$, we have $x = 0$, $y = 1$, so $\partial z/\partial t = 0$, $\partial z/\partial u = 0$, $\partial z/\partial v = 4$.

22. $z = \dfrac{x}{y}$, $x = re^{st}$, $y = rse^t \quad \Rightarrow$

$\dfrac{\partial z}{\partial r} = \dfrac{1}{y}e^{st} + \dfrac{-x}{y^2}se^t$, $\dfrac{\partial z}{\partial s} = \dfrac{1}{y}rte^{st} - \dfrac{x}{y^2}re^t$, $\dfrac{\partial z}{\partial t} = \dfrac{1}{y}rse^{st} - \dfrac{x}{y^2}rse^t$. When $r = 1$, $s = 2$ and $t = 0$, we have $x = 1$, $y = 2$, so $\partial z/\partial r = \frac{1}{2} + \frac{-1}{4}\cdot 2 = 0$, $\partial z/\partial s = 0 - \frac{1}{4} = -\frac{1}{4}$ and $\partial z/\partial t = \frac{1}{2}\cdot 2 - \frac{1}{4}\cdot 2 = \frac{1}{2}$.

23. $u = \dfrac{x+y}{y+z}$, $x = p + r + t$, $y = p - r + t$, $z = p + r - t \quad \Rightarrow$

$$\frac{\partial u}{\partial p} = \frac{1}{y+z} + \frac{(y+z)-(x+y)}{(y+z)^2} - \frac{x+y}{(y+z)^2} = \frac{(y+z)+(z-x)-(x+y)}{(y+z)^2} = 2\frac{z-x}{(y+z)^2}$$

$$= 2\frac{-2t}{4p^2} = -\frac{t}{p^2},$$

$$\frac{\partial u}{\partial r} = \frac{1}{y+z} + \frac{z-x}{(y+z)^2}(-1) - \frac{x+y}{(y+z)^2} = 0, \text{ and}$$

$$\frac{\partial u}{\partial t} = \frac{1}{y+z} + \frac{z-x}{(y+z)^2} + \frac{x+y}{(y+z)^2} = 2\frac{y+z}{(y+z)^2} = \frac{2}{2p} = \frac{1}{p}.$$

24. $t = z\sec(xy)$, $x = uv$, $y = vw$, $z = wu$ $\Rightarrow$

$$\frac{\partial t}{\partial u} = [zy\sec(xy)\tan(xy)]\,v + [zx\sec(xy)\tan(xy)]\,(0) + [\sec(xy)]\,w = \sec(xy)\,[w + vzy\tan(xy)],$$

$$\frac{\partial t}{\partial v} = [zy\sec(xy)\tan(xy)]\,u + [zx\sec(xy)\tan(xy)]\,w + [\sec(xy)]\,(0) = z\sec(xy)\tan(xy)\,[yu + xw],$$

$$\frac{\partial t}{\partial w} = [zy\sec(xy)\tan(xy)]\,(0) + [zx\sec(xy)\tan(xy)]\,v + [\sec(xy)]\,u = \sec(xy)\,[u + vzx\tan(xy)].$$

25. $x^2 - xy + y^3 = 8$, so let $F(x,y) = x^2 - xy + y^3 - 8 = 0$. Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = \dfrac{-(2x-y)}{-x+3y^2} = \dfrac{y-2x}{3y^2-x}$.

26. $y^5 + 3x^2y^2 + 5x^4 = 12$, so let $F(x,y) = y^5 + 3x^2y^2 + 5x^4 - 12 = 0$. Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{6xy^2 + 20x^3}{5y^4 + 6x^2y}$

27. $\cos(x-y) = xe^y$, so let $F(x,y) = \cos(x-y) - xe^y = 0$.

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{-\sin(x-y) - e^y}{-\sin(x-y)(-1) - xe^y} = \dfrac{\sin(x-y) + e^y}{\sin(x-y) - xe^y}$.

28. $x\cos y + y\cos x = 1$, so let $F(x,y) = x\cos y + y\cos x - 1 = 0$. Then

$\dfrac{dy}{dx} = -\dfrac{\cos y - y\sin x}{-x\sin y + \cos x} = \dfrac{y\sin x - \cos y}{\cos x - x\sin y}$

29. $xy^2 + yz^2 + zx^2 = 3$, so let $F(x,y) = xy^2 + yz^2 + zx^2 - 3 = 0$.

Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{y^2 + 2zx}{2yz + x^2}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2xy + z^2}{2yz + x^2}$.

30. $xyz = \cos(x+y+z)$. Let $F(x,y,z) = xyz - \cos(x+y+z) = 0$, so

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$, $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$.

31. Let $F(x,y,z) = xe^y + yz + ze^x = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{e^y + ze^x}{y + e^x}$, $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xe^y + z}{y + e^x}$.

32. $\ln(x + yz) = 1 + xy^2z^3$, so let $F(x,y) = \ln(x+yz) - 1 - xy^2z^3 = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1/(x+yz) - y^2z^3}{y/(x+yz) - 3xy^2z^2} = \frac{y^2z^3(x+yz) - 1}{y - 3xy^2z^2(x+yz)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z/(x+yz) - 2xyz^3}{y/(x+yz) - 3xy^2z^2} = \frac{2xyz^3(x+yz) - z}{y - 3xy^2z^2(x+yz)}$$

33. Since x and y are each functions of t, $T(x,y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$.

After 3 seconds, $x = \sqrt{1+t} = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$,

and $\dfrac{dy}{dt} = \dfrac{1}{3}$. Then $\dfrac{dT}{dt} = T_x(2,3)\,\dfrac{dx}{dt} + T_y(2,3)\,\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2 degrees Celsius per second.

34. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of $0.15\,^\circ\mathrm{C}$ /year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule, $\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)\,(0.15) + (8)\,(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.

35. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$. According to the graph, the diver is experiencing a temperature of approximately $12.5\,^\circ\mathrm{C}$ at $t = 20$ minutes, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11\,(12.5) + 0.00087\,(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate $\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$. Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

36. $V = \pi r^2 h/3$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = \dfrac{2\pi rh}{3}1.8 + \dfrac{\pi r^2}{3}(-2.5) = 20{,}160\pi - 12{,}000\pi = 8160\pi\ \mathrm{in}^3/\mathrm{s}$.

37. (a) $V = \ell wh$, so by the Chain Rule,

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt}\\ &= 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6\ \mathrm{m}^3/\mathrm{s}\end{aligned}$$

(b) $S = 2\,(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2\,(w+h)\,\frac{d\ell}{dt} + 2\,(\ell+h)\,\frac{dw}{dt} + 2\,(\ell+w)\,\frac{dh}{dt}\\ &= 2\,(2+2)\,2 + 2\,(1+2)\,2 + 2\,(1+2)\,(-3) = 10\ \mathrm{m}^2/\mathrm{s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \quad\Rightarrow\quad 2L\dfrac{dL}{dt} = 2\ell\dfrac{d\ell}{dt} + 2w\dfrac{dw}{dt} + 2h\dfrac{dh}{dt} = 2\,(1)\,(2) + 2\,(2)\,(2) + 2\,(2)\,(-3) = 0 \quad\Rightarrow$ $dL/dt = 0$ m/s.

38. $I = \dfrac{V}{R} \quad\Rightarrow\quad \dfrac{dI}{dt} = \dfrac{\partial I}{\partial V}\dfrac{dV}{dt} + \dfrac{\partial I}{\partial R}\dfrac{dR}{dt} = \dfrac{1}{R}\dfrac{dV}{dt} - \dfrac{V}{R^2}\dfrac{dR}{dt} = \dfrac{1}{R}\dfrac{dV}{dt} - \dfrac{I}{R}\dfrac{dR}{dt}$
$= \frac{1}{400}\,(-0.01) - \frac{0.08}{400}\,(0.03) = -0.000031$ A/s

39. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$, $\dfrac{dV}{dt} = 8.31\left[\dfrac{0.15}{20} - \dfrac{(0.05)\,(320)}{400}\right] \approx -0.27$ L/s.

40. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt = -90$, $dy/dt = -80$ and $z^2 = x^2 + y^2$. When $x = 0.3$ and $y = 0.4$, $z = \sqrt{0.25} = 0.5$ and $2z\,(dz/dt) = 2x\,(dx/dt) + 2y\,(dy/dt)$ or $dz/dt = 0.6\,(-90) + 0.8\,(-80) = -118$ km/h.

41. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\,(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta$,

$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}r^2\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$.

42. By the Chain Rule, $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}e^s\cos t + \frac{\partial u}{\partial y}e^s\sin t$, $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}(-e^s\sin t) + \frac{\partial u}{\partial y}e^s\cos t$. Then

$\left(\frac{\partial u}{\partial s}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t + \left(\frac{\partial u}{\partial y}\right)^2 e^{2t}\sin^2 t$ and

$\left(\frac{\partial u}{\partial t}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t + \left(\frac{\partial u}{\partial y}\right)^2 e^{2t}\sin^2 t$. Thus

$\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]e^{-2t} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$.

43. Let $u = x - y$. Then $\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du}(-1)$. Thus $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

44. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$. Thus $\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$.

45. Let $u = x + at$, $v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial}{\partial t}[f'(u) - g'(v)] = a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t} - \frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v)$. Similarly

$\frac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\frac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$.

46. By the Chain Rule, $\frac{\partial u}{\partial s} = e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t} = -e^s\sin t\frac{\partial u}{\partial x} + e^s\cos t\frac{\partial u}{\partial y}$.

Then $\frac{\partial^2 u}{\partial s^2} = e^s\cos t\frac{\partial u}{\partial x} + e^s\cos t\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) + e^s\sin t\frac{\partial u}{\partial y} + e^s\sin t\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)$.

But $\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial s} = e^s\cos t\frac{\partial^2 u}{\partial x^2} + e^s\sin t\frac{\partial^2 u}{\partial y\,\partial x}$ and

$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial s} = e^s\sin t\frac{\partial^2 u}{\partial y^2} + e^s\cos t\frac{\partial^2 u}{\partial x\,\partial y}$. Also, by continuity of the partials,

$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$. Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= e^s\cos t\frac{\partial u}{\partial x} + e^s\cos t\left(e^s\cos t\frac{\partial^2 u}{\partial x^2} + e^s\sin t\frac{\partial^2 u}{\partial x\,\partial y}\right) + e^s\sin t\frac{\partial u}{\partial y}\\ &\quad + e^s\sin t\left(e^s\sin t\frac{\partial^2 u}{\partial y^2} + e^s\cos t\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\frac{\partial u}{\partial y} + e^{2s}\cos^2 t\frac{\partial^2 u}{\partial x^2} + 2e^{2s}\cos t\sin t\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\sin^2 t\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - e^s \sin t \frac{\partial u}{\partial y} + e^s \cos t \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) \\
&= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \left(-e^s \sin t \frac{\partial^2 u}{\partial x^2} + e^s \cos t \frac{\partial^2 u}{\partial x\, \partial y}\right) \\
&\qquad -e^s \sin t \frac{\partial u}{\partial y} + e^s \cos t \left(e^s \cos t \frac{\partial^2 u}{\partial y^2} - e^s \sin t \frac{\partial^2 u}{\partial x\, \partial y}\right) \\
&= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \frac{\partial u}{\partial y} + e^{2s} \sin^2 t \frac{\partial^2 u}{\partial x^2} - 2e^{2s} \cos t \sin t \frac{\partial^2 u}{\partial x\, \partial y} + e^{2s} \cos^2 t \frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Thus $e^{-2s}\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right) = \left(\cos^2 t + \sin^2 t\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, as desired.

47. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} 2s + \frac{\partial z}{\partial y} 2r$. Then

$$\begin{aligned}
\frac{\partial^2 z}{\partial r \partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x} 2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y} 2r\right) \\
&= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} 2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} 2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}(2s) + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r} 2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} 2r + \frac{\partial z}{\partial y} 2 \\
&= 4rs \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\, \partial x} 4s^2 + 0 + 4rs \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\, \partial y} 4r^2 + 2\frac{\partial z}{\partial y}
\end{aligned}$$

By the continuity of the partials, $\frac{\partial^2 z}{\partial r \partial s} = 4rs \frac{\partial^2 z}{\partial x^2} + 4rs \frac{\partial^2 z}{\partial y^2} + \left(4r^2 + 4s^2\right)\frac{\partial^2 z}{\partial x\, \partial y} + 2\frac{\partial z}{\partial y}$.

48. By the Chain Rule,

(a) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$

(b) $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$

(c)
$$\begin{aligned}
\frac{\partial^2 z}{\partial r \partial \theta} &= \frac{\partial^2 z}{\partial \theta \partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) \\
&= -\sin\theta \frac{\partial z}{\partial x} + \cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta \frac{\partial z}{\partial y} + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right) \\
&= -\sin\theta \frac{\partial z}{\partial x} + \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y\, \partial x}\frac{\partial y}{\partial \theta}\right) + \cos\theta \frac{\partial z}{\partial y} + \sin\theta \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial x\, \partial y}\frac{\partial x}{\partial \theta}
\end{aligned}$$

$$= -\sin\theta \frac{\partial z}{\partial x} + \cos\theta \left(-r\sin\theta \frac{\partial^2 z}{\partial x^2} + r\cos\theta \frac{\partial^2 z}{\partial y\,\partial x} \right) + \cos\theta \frac{\partial z}{\partial y}$$
$$+ \sin\theta \left(r\cos\theta \frac{\partial^2 z}{\partial y^2} - r\sin\theta \frac{\partial^2 z}{\partial x\,\partial y} \right)$$
$$= -\sin\theta \frac{\partial z}{\partial x} - r\cos\theta\sin\theta \frac{\partial^2 z}{\partial x^2} + r\cos^2\theta \frac{\partial^2 z}{\partial y\,\partial x} + \cos\theta \frac{\partial z}{\partial y}$$
$$+ r\cos\theta\sin\theta \frac{\partial^2 z}{\partial y^2} - r\sin^2\theta \frac{\partial^2 z}{\partial y\,\partial x}$$
$$= \cos\theta \frac{\partial z}{\partial y} - \sin\theta \frac{\partial z}{\partial x} + r\cos\theta\sin\theta \left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} \right) + r\left(\cos^2\theta - \sin^2\theta\right) \frac{\partial^2 z}{\partial y\,\partial x}$$

49. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x} r\sin\theta + \dfrac{\partial z}{\partial y} r\cos\theta$. Then

$$\frac{\partial^2 z}{\partial r^2} = \cos\theta \left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta \right) + \sin\theta \left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta \right)$$
$$= \cos^2\theta \frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta \frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta \frac{\partial^2 z}{\partial y^2}$$

and

$$\frac{\partial^2 z}{\partial \theta^2} = -r\cos\theta \frac{\partial z}{\partial x} + (-r\sin\theta) \left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x} r\cos\theta \right)$$
$$-r\sin\theta \frac{\partial z}{\partial y} + r\cos\theta \left(\frac{\partial^2 z}{\partial y^2} r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta) \right)$$
$$= -r\cos\theta \frac{\partial z}{\partial x} - r\sin\theta \frac{\partial z}{\partial y} + r^2\sin^2\theta \frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\sin\theta \frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta \frac{\partial^2 z}{\partial y^2}$$

Thus

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \left(\cos^2\theta + \sin^2\theta\right) \frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right) \frac{\partial^2 z}{\partial y^2} - \frac{1}{r}\cos\theta \frac{\partial z}{\partial x}$$
$$-\frac{1}{r}\sin\theta \frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y} \right)$$
$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}$$

50. (a) $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \right)$$
$$= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x} \right)\frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y} \right)\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$
$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$
$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t} \right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

(b)
$$\begin{aligned}
\frac{\partial^2 z}{\partial s\,\partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) \\
&= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t} \\
&= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x\,\partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}
\end{aligned}$$

51. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$$\begin{aligned}
f(tx, ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 &= t^3x^2y + 2t^3xy^2 + 5t^3y^3 \\
&= t^3\left(x^2y + 2xy^2 + 5y^3\right) = t^3 f(x, y)
\end{aligned}$$

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(tx, ty) = \frac{\partial}{\partial t}\left[t^n f(x, y)\right] \quad \Leftrightarrow$$

$$\frac{\partial}{\partial (tx)} f(tx, ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial}{\partial (ty)} f(tx, ty)\cdot\frac{\partial (ty)}{\partial t} = x\frac{\partial}{\partial (tx)} f(tx, ty) + y\frac{\partial}{\partial (ty)} f(tx, ty) = nt^{n-1} f(x, y).$$

Setting $t = 1$: $x\dfrac{\partial}{\partial x} f(x, y) + y\dfrac{\partial}{\partial y} f(x, y) = nf(x, y)$.

52. Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial (tx)} f(tx, ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial}{\partial (ty)} f(tx, ty)\cdot\frac{\partial (ty)}{\partial t} = x\frac{\partial}{\partial (tx)} f(tx, ty) + y\frac{\partial}{\partial (ty)} f(tx, ty) = nt^{n-1} f(x, y)$$

and differentiating again with respect to t gives

$$\begin{aligned}
&x\left[\frac{\partial^2}{\partial (tx)^2} f(tx, ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial^2}{\partial (ty)\,\partial (tx)} f(tx, ty)\cdot\frac{\partial (ty)}{\partial t}\right] \\
&\qquad + y\left[\frac{\partial^2}{\partial (tx)\,\partial (ty)} f(tx, ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial^2}{\partial (ty)^2} f(tx, ty)\cdot\frac{\partial (ty)}{\partial t}\right] = n(n-1)t^{n-1} f(x, y).
\end{aligned}$$

Setting $t = 1$ and using the fact that $f_{yx} = f_{xy}$, we have $x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = n(n-1) f(x, y)$.

53. Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x} f(tx, ty) = \frac{\partial}{\partial x}\left[t^n f(x, y)\right] \quad \Leftrightarrow$$

$$\frac{\partial}{\partial (tx)} f(tx, ty)\cdot\frac{\partial (tx)}{\partial x} + \frac{\partial}{\partial (ty)} f(tx, ty)\cdot\frac{\partial (ty)}{\partial x} = t^n\frac{\partial}{\partial x} f(x, y) \quad \Leftrightarrow \quad tf_x(tx, ty) = t^n f_x(x, y).$$

Thus $f_x(tx, ty) = t^{n-1} f_x(x, y)$.

54. $F(x, y, z) = 0$ is assumed to define z as a function of x and y, that is, $z = f(x, y)$. So by (7), $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ since $F_z \neq 0$. Similarly, it is assumed that $F(x, y, z) = 0$ defines x as a function of y and z, that is $x = h(x, z)$. Then $F(h(y, z), y, z) = 0$ and by the Chain Rule, $F_x\dfrac{\partial x}{\partial y} + F_y\dfrac{\partial y}{\partial y} + F_z\dfrac{\partial z}{\partial y} = 0$. But $\dfrac{\partial z}{\partial y} = 0$ and $\dfrac{\partial y}{\partial y} = 1$, so $F_x\dfrac{\partial x}{\partial y} + F_y = 0 \;\Rightarrow\; \dfrac{\partial x}{\partial y} = -\dfrac{F_y}{F_x}$. A similar calculation shows that $\dfrac{\partial y}{\partial z} = -\dfrac{F_z}{F_y}$. Thus

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right) = -1.$$

15.6 Directional Derivatives and the Gradient Vector

ET 14.6

1. First we draw a line passing through Raleigh and the eye of the hurricane. We can approximate the directional derivative at Raleigh in the direction of the eye of the hurricane by the average rate of change of pressure between the points where this line intersects the contour lines closest to Raleigh. In the direction of the eye of the hurricane, the pressure changes from 996 millibars to 992 millibars. We estimate the distance between these two points to be approximately 40 miles, so the rate of change of pressure in the direction given is approximately $\frac{992-996}{40} = -0.1$ millibars/mi.

2. First we draw a line passing through Muskegon and Ludington. We approximate the directional derivative at Muskegon in the direction of Ludington by the average rate of change of snowfall between the points where the line intersects the contour lines closest to Muskegon. In the direction of Ludington, the snowfall changes from 60 to 70 inches. We estimate the distance between these two points to be approximately 28 miles, so the rate of change of annual snowfall in the direction given is approximately $\frac{70-60}{28} \approx 0.36$ in/mi. [If we talk of snowfall (rather than annual snowfall), the units are (in/year) /mi.]

3. $f(x,y) = x^2y^3 + 2x^4y \Rightarrow f_x(x,y) = 2xy^3 + 8x^3y$ and $f_y(x,y) = 3x^2y^2 + 2x^4$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{3}$, then from Equation 6,
$D_{\mathbf{u}}f(1,-2) = f_x(1,-2)\cos\frac{\pi}{3} + f_y(1,-2)\sin\frac{\pi}{3} = (-32)\left(\frac{1}{2}\right) + (14)\left(\frac{\sqrt{3}}{2}\right) = 7\sqrt{3} - 16.$

4. $f(x,y) = \sin(x+2y) \Rightarrow f_x(x,y) = \cos(x+2y)$ and $f_y(x,y) = 2\cos(x+2y)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{3\pi}{4}$, then from Equation 6,
$D_{\mathbf{u}}f(4,-2) = f_x(4,-2)\cos\frac{3\pi}{4} + f_y(4,-2)\sin\frac{3\pi}{4} = (\cos 0)\left(-\frac{\sqrt{2}}{2}\right) + 2(\cos 0)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}.$

5. $f(x,y) = \sqrt{5x-4y} \Rightarrow f_x(x,y) = \frac{1}{2}(5x-4y)^{-1/2}(5) = \dfrac{5}{2\sqrt{5x-4y}}$ and
$f_y(x,y) = \frac{1}{2}(5x-4y)^{-1/2}(-4) = -\dfrac{2}{\sqrt{5x-4y}}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\frac{\pi}{6}$, then from Equation 6, $D_{\mathbf{u}}f(4,1) = f_x(4,1)\cos\left(-\frac{\pi}{6}\right) + f_y(4,1)\sin\left(-\frac{\pi}{6}\right) = \frac{5}{8}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{5\sqrt{3}}{16} + \frac{1}{4}.$

6. $f(x,y) = xe^{-2y} \Rightarrow f_x(x,y) = e^{-2y}$ and $f_y(x,y) = -2xe^{-2y}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{2}$, then $D_{\mathbf{u}}f(5,0) = f_x(5,0)\cos\frac{\pi}{2} + f_y(5,0)\sin\frac{\pi}{2} = 1\cdot 0 + (-10)\,1 = -10.$

7. $f(x,y) = 5xy^2 - 4x^3y$
(a) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 5y^2 - 12x^2y, 10xy - 4x^3\rangle$
(b) $\nabla f(1,2) = \langle 5(2)^2 - 12(1)^2(2), 10(1)(2) - 4(1)^3\rangle = \langle -4, 16\rangle$
(c) By Equation 9, $D_{\mathbf{u}}f(1,2) = \nabla f(1,2)\cdot\mathbf{u} = \langle -4,16\rangle\cdot\left\langle \frac{5}{13}, \frac{12}{13}\right\rangle = (-4)\left(\frac{5}{13}\right) + (16)\left(\frac{12}{13}\right) = \frac{172}{13}.$

8. $f(x,y) = y\ln x$
(a) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle y/x, \ln x\rangle$ (b) $\nabla f(1,-3) = \left\langle \frac{-3}{1}, \ln 1\right\rangle = \langle -3, 0\rangle$
(c) By Equation 9, $D_{\mathbf{u}}f(1,-3) = \nabla f(1,-3)\cdot\mathbf{u} = \langle -3,0\rangle\cdot\left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle = \frac{12}{5}.$

9. $f(x,y,z) = xy^2z^3$
(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$
(b) $\nabla f(1,-2,1) = \langle 4,-4,12\rangle$
(c) $\nabla f(1,-2,1)\cdot\mathbf{u} = \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{12}{\sqrt{3}} = \frac{20}{\sqrt{3}}$

10. $f(x,y,z) = xy + yz^2 + xz^3$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle y + z^3, x + z^2, 2yz + 3xz^2\rangle$

(b) $\nabla f(2,0,3) = \langle 27, 11, 54\rangle$

(c) $\nabla f(2,0,3) \cdot \mathbf{u} = \frac{1}{3}(-54 - 11 + 108) = \frac{43}{3}$

11. $f(x,y) = 1 + 2x\sqrt{y} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 2\sqrt{y}, 2x \cdot \frac{1}{2}y^{-1/2}\right\rangle = \langle 2\sqrt{y}, x/\sqrt{y}\rangle$, $\nabla f(3,4) = \langle 4, \frac{3}{2}\rangle$,
and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle = \langle \frac{4}{5}, -\frac{3}{5}\rangle$, so
$D_{\mathbf{u}}f(3,4) = \nabla f(3,4) \cdot \mathbf{u} = \langle 4, \frac{3}{2}\rangle \cdot \langle \frac{4}{5}, -\frac{3}{5}\rangle = \frac{23}{10}$.

12. $f(x,y) = x/y \quad \Rightarrow \quad \nabla f(x,y) = \langle 1/y, -x/y^2\rangle$, $\nabla f(6,-2) = \langle -\frac{1}{2}, -\frac{3}{2}\rangle$, $\mathbf{u} = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\rangle$ and
$D_{\mathbf{u}}f(6,-2) = \frac{1}{2\sqrt{10}} - \frac{9}{2\sqrt{10}} = -\frac{4}{\sqrt{10}} = -\frac{2\sqrt{10}}{5}$.

13. $g(s,t) = s^2e^t \quad \Rightarrow \quad \nabla g(s,t) = 2se^t\,\mathbf{i} + s^2e^t\,\mathbf{j}$, $\nabla g(2,0) = 4\,\mathbf{i} + 4\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is
$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$, so $D_{\mathbf{u}}g(2,0) = \nabla g(2,0)\cdot\mathbf{u} = (4\,\mathbf{i}+4\,\mathbf{j})\cdot\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \frac{8}{\sqrt{2}} = 4\sqrt{2}$.

14. $g(r,\theta) = e^{-r}\sin\theta \quad \Rightarrow \quad \nabla g(r,\theta) = \left(-e^{-r}\sin\theta\right)\mathbf{i} + \left(e^{-r}\cos\theta\right)\mathbf{j}$, $\nabla g\left(0,\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$,
and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$, so
$D_{\mathbf{u}}g\left(0,\frac{\pi}{3}\right) = \nabla g\left(0,\frac{\pi}{3}\right)\cdot\mathbf{u} = \left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)\cdot\frac{1}{\sqrt{13}}(3\mathbf{i}-2\mathbf{j}) = -\frac{3\sqrt{3}}{2\sqrt{13}} - \frac{1}{\sqrt{13}} = -\frac{3\sqrt{3}+2}{2\sqrt{13}}$.

15. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle \dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle$,
$\nabla f(1,2,-2) = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{9}\langle -6,6,-3\rangle = \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle$, so
$D_{\mathbf{u}}f(1,2,-2) = \nabla f(1,2,-2)\cdot\mathbf{u} = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle \cdot \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle = \frac{4}{9}$.

16. $f(x,y,z) = \dfrac{x}{y+z} \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle \dfrac{1}{y+z}, -\dfrac{x}{(y+z)^2}, -\dfrac{x}{(y+z)^2}\right\rangle$,
$\nabla f(4,1,1) = \langle \frac{1}{2}, -1, -1\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle$, so
$D_{\mathbf{u}}f(4,1,1) = \nabla f(4,1,1)\cdot\mathbf{u} = \langle \frac{1}{2}, -1, -1\rangle \cdot \frac{1}{\sqrt{14}}\langle 1,2,3\rangle = -\frac{9}{2\sqrt{14}}$.

17. $g(x,y,z) = x\tan^{-1}(y/z) \quad \Rightarrow \quad \nabla g(x,y,z) = \langle \tan^{-1}(y/z), xz/\left(y^2+z^2\right), -xy/\left(y^2+z^2\right)\rangle$,
$\nabla g(1,2,-2) = \langle -\frac{\pi}{4}, -\frac{1}{4}, -\frac{1}{4}\rangle$, $\mathbf{u} = \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle$ and
$D_{\mathbf{u}}g(1,2,-2) = \dfrac{(-\pi)(1)}{4\sqrt{3}} + \dfrac{(-1)(1)}{4\sqrt{3}} + \dfrac{(-1)(-1)}{4\sqrt{3}} = -\dfrac{\pi}{4\sqrt{3}}$.

18. $g(x,y,z) = z^3 - x^2y \quad \Rightarrow \quad \nabla g(x,y,z) = \langle -2xy, -x^2, 3z^2\rangle$, $\nabla g(1,6,2) = \langle -12,-1,12\rangle$,
$\mathbf{u} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$, and $D_{\mathbf{u}}g(1,6,2) = \dfrac{(-12)(3)}{13} + \dfrac{(-1)(4)}{13} + \dfrac{(12)(12)}{13} = 8$.

19. $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x)\right\rangle = \left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}}\right\rangle$, so
$\nabla f(2,8) = \langle 1, \frac{1}{4}\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8\rangle = \langle 3,-4\rangle$ is $\mathbf{u} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$, so
$D_{\mathbf{u}}f(2,8) = \nabla f(2,8)\cdot\mathbf{u} = \langle 1, \frac{1}{4}\rangle \cdot \langle \frac{3}{5}, -\frac{4}{5}\rangle = \frac{2}{5}$.

20. $f(x,y,z) = x^2 + y^2 + z^2 \quad \Rightarrow \quad \nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle$, so $\nabla f(2,1,3) = \langle 4,2,6\rangle$. The unit
vector in the direction of $\overrightarrow{PO} = \langle -2,-1,-3\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{14}}\langle -2,-1,-3\rangle$, so
$D_{\mathbf{u}}f(2,1,3) = \nabla f(2,1,3)\cdot\mathbf{u} = \langle 4,2,6\rangle \cdot \frac{1}{\sqrt{14}}\langle -2,-1,-3\rangle = -\frac{28}{\sqrt{14}} = -2\sqrt{14}$.

21. $f(x,y) = xe^{-y} + 3y \quad \Rightarrow \quad \nabla f(x,y) = \langle e^{-y}, 3 - xe^{-y} \rangle$, $\nabla f(1,0) = \langle 1, 2 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(1,0)| = \sqrt{5}$.

22. $f(x,y) = \ln(x^2 + y^2) \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2} \right\rangle$, $\nabla f(1,2) = \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$. Thus the maximum rate of change is $|\nabla f(1,2)| = \frac{2\sqrt{5}}{5}$ in the direction $\left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$ or $\langle 2, 4 \rangle$.

23. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

24. $f(x,y,z) = x^2y^3z^4 \quad \Rightarrow \quad \nabla f(x,y,z) = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle$, $\nabla f(1,1,1) = \langle 2, 3, 4 \rangle$. Thus the maximum rate of change is $|\nabla f(1,1,1)| = \sqrt{29}$ in the direction $\langle 2, 3, 4 \rangle$.

25. $f(x,y,z) = x + y/z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 1, \dfrac{1}{z}, -\dfrac{y}{z^2} \right\rangle$, so the maximum rate of change is $|\nabla f(4,3,-1)| = \sqrt{11}$ in the direction $\langle 1, -1, -3 \rangle$.

26. $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z} \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle \dfrac{1}{y}, \dfrac{1}{z} - \dfrac{x}{y^2}, -\dfrac{y}{z^2} \right\rangle$, so the maximum rate of change is $|\nabla f(4,2,1)| = \frac{\sqrt{17}}{2}$ in the direction $\left\langle \frac{1}{2}, 0, -2 \right\rangle$ or $\langle 1, 0, -4 \rangle$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}} f = |\nabla f| \cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}} f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4y - x^2y^3 \quad \Rightarrow \quad \nabla f(x,y) = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2 \rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$.

28. $f(x,y) = x^2 + \sin xy \quad \Rightarrow \quad f_x(x,y) = 2x + y\cos xy$, $f_y(x,y) = x\cos xy$ and $f_x(1,0) = 2(1) + (0)\cos 0 = 2$, $f_y(1,0) = (1)\cos 0 = 1$. If $\mathbf{u}$ is a unit vector which makes an angle θ with the positive x-axis, then $D_{\mathbf{u}} f(1,0) = f_x(1,0)\cos\theta + f_y(1,0)\sin\theta = 2\cos\theta + \sin\theta$. We want $D_{\mathbf{u}} f(1,0) = 1$, so $2\cos\theta + \sin\theta = 1 \quad \Rightarrow \quad \sin\theta = 1 - 2\cos\theta \quad \Rightarrow \quad \sin^2\theta = (1 - 2\cos\theta)^2 \quad \Rightarrow$ $1 - \cos^2\theta = 1 - 4\cos\theta + 4\cos^2\theta \quad \Rightarrow \quad 5\cos^2\theta - 4\cos\theta = 0 \quad \Rightarrow \quad \cos\theta(5\cos\theta - 4) = 0 \quad \Rightarrow \quad \cos\theta = 0$ or $\cos\theta = \frac{4}{5} \quad \Rightarrow \quad \theta = \frac{\pi}{2}$ or $\theta = 2\pi - \cos^{-1}\frac{4}{5} \approx 5.64$.

29. $T = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$ and $120 = T(1,2,2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1, -1, 1 \rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}} T(1,2,2) = \nabla T(1,2,2) \cdot \mathbf{u} = \left[-360\left(x^2+y^2+z^2\right)^{-3/2} \langle x, y, z \rangle\right]_{(1,2,2)} \cdot \mathbf{u}$$

$$= -\tfrac{40}{3}\langle 1, 2, 2 \rangle \cdot \tfrac{1}{\sqrt{3}}\langle 1, -1, 1 \rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2+y^2+z^2\right)^{-3/2}\langle x, y, z \rangle$, and since $\langle x, y, z \rangle$ is the position vector of the point (x,y,z), the vector $-\langle x, y, z \rangle$, and thus ∇T, always points toward the origin.

30. $\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z \rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1, -2, 1 \rangle$, $\nabla T(2,-1,2) = -400e^{-43}\langle 2, -3, 18 \rangle$ and

$$D_{\mathbf{u}} T(2,-1,2) = \left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26) = -\frac{5200\sqrt{6}}{3e^{43}}\ {}^\circ\mathrm{C/m}.$$

(b) $\nabla T(2,-1,2) = 400e^{-43}\langle -2, 3, -18 \rangle$ or equivalently $\langle -2, 3, -18 \rangle$.

(c) $|\nabla T| = 400e^{-x^2 - 3y^2 - 9z^2}\sqrt{x^2 + 9y^2 + 8z^2}\,{}^\circ\text{C}/\text{m}$ is the maximum rate of increase. At $(2, -1, 2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}\,{}^\circ\text{C}/\text{m}$.

31. $\nabla V(x, y, z) = \langle 10x - 3y + yz, xz - 3x, xy\rangle$, $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$

(a) $D_{\mathbf{u}}V(3, 4, 5) = \langle 38, 6, 12\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$ or equivalently $\langle 19, 3, 6\rangle$.

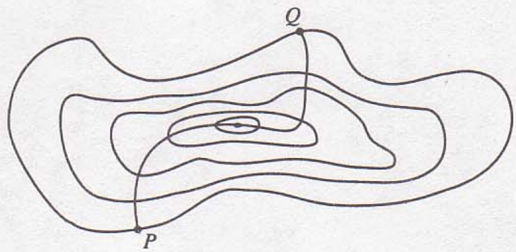
(c) $|\nabla V(3, 4, 5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

32. (a) Let $z = f(x, y) = 1000 - 0.01x^2 - 0.02y^2$. Then $\nabla f(x, y) = \langle -0.02x, -0.04y\rangle$. Proceed in the direction $\nabla f(60, 100) = \langle -1.2, -4\rangle$.

(b) The maximum slope is equal to the maximum directional derivative, which is $|\langle -1.2, -4\rangle| = \sqrt{17.44}$ and $\theta = \tan^{-1}\sqrt{17.44} \approx 76.5^\circ$.

33. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}f(1, 3) = f_x(1, 3) = 3$ and $D_{\overrightarrow{AC}}f(1, 3) = f_y(1, 3) = 26$. Therefore $\nabla f(1, 3) = \langle f_x(1, 3), f_y(1, 3)\rangle = \langle 3, 26\rangle$, and by definition, $D_{\overrightarrow{AD}}f(1, 3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$. Therefore, $D_{\overrightarrow{AD}}f(1, 3) = \langle 3, 26\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

34. The curve of steepest ascent is perpendicular to all of the contour lines.



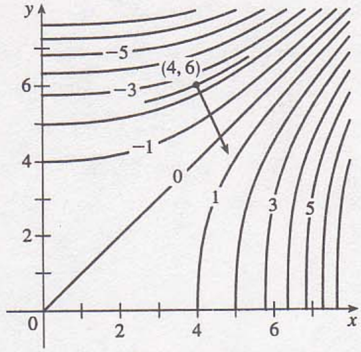
35. (a) $$\nabla(au + bv) = \left\langle \frac{\partial(au + bv)}{\partial x}, \frac{\partial(au + bv)}{\partial y}\right\rangle = \left\langle a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}, a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y}\right\rangle$$
$$= a\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle + b\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle = a\nabla u + b\nabla v$$

(b) $$\nabla(uv) = \left\langle v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}, v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}\right\rangle = v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle + u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle = v\nabla u + u\nabla v$$

(c) $$\nabla\left(\frac{u}{v}\right) = \left\langle \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}, \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}\right\rangle = \frac{v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle - u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle}{v^2} = \frac{v\nabla u - u\nabla v}{v^2}$$

(d) $$\nabla u^n = \left\langle \frac{\partial(u^n)}{\partial x}, \frac{\partial(u^n)}{\partial y}\right\rangle = \left\langle nu^{n-1}\frac{\partial u}{\partial x}, nu^{n-1}\frac{\partial u}{\partial y}\right\rangle = nu^{n-1}\nabla u.$$

36. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



37. Let $F(x,y,z) = x^2 + 2y^2 + 3z^2$. Then $x^2 + 2y^2 + 3z^2 = 21$ is a level surface of F.
$F_x(x,y,z) = 2x \quad\Rightarrow\quad F_x(4,-1,1) = 8$, $F_y(x,y,z) = 4y \quad\Rightarrow\quad F_y(4,-1,1) = -4$, and
$F_z(x,y,z) = 6z \quad\Rightarrow\quad F_z(4,-1,1) = 6$.

(a) Equation 19 gives an equation of the tangent plane at $(4,-1,1)$ as $8(x-4) - 4[y-(-1)] + 6(z-1) = 0$ or $4x - 2y + 3z = 21$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-4}{8} = \dfrac{y+1}{-4} = \dfrac{z-1}{6}$ or $\dfrac{x-4}{4} = \dfrac{y+1}{-2} = \dfrac{z-1}{3}$.

38. Let $F(x,y,z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 - 2$ is the level surface $F(x,y,z) = 2$.
$F_x(x,y,z) = -1 \quad\Rightarrow\quad F_x(-1,1,0) = -1$, $F_y(x,y,z) = 2y \quad\Rightarrow\quad F_y(-1,1,0) = 2$, and
$F_z(x,y,z) = 2z \quad\Rightarrow\quad F_z(-1,1,0) = 0$.

(a) An equation of the tangent plane is $-1(x+1) + 2(y-1) + 0(z-0) = 0$ or $-x + 2y = 3$.

(b) The normal line has symmetric equations $\dfrac{x+1}{-1} = \dfrac{y-1}{2}$, $z = 0$.

39. $F(x,y,z) = x^2 + y^2 - z^2 - 2xy + 4xz \quad\Rightarrow\quad \nabla F(x,y,z) = \langle 2x - 2y + 4z, 2y - 2x, -2z + 4x\rangle$,
$\nabla F(1,0,1) = \langle 6,-2,2\rangle$

(a) $6(x-1) - 2(y-0) + 2(z-1) = 0$ or $3x - y + z = 4$ (b) $\dfrac{x-1}{3} = -y = z-1$

40. $F(x,y,z) = x^2 - 2y^2 - 3z^2 + xyz \quad\Rightarrow\quad \nabla F(x,y,z) = \langle 2x + yz, -4y + xz, -6z + xy\rangle$,
$\nabla F(3,-2,-1) = \langle 8,5,0\rangle$

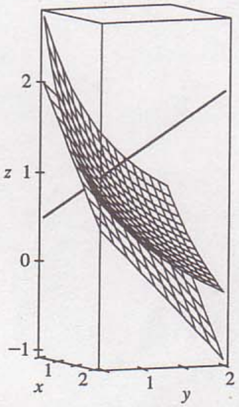
(a) $8(x-3) + 5(y+2) + 0(z+1) = 0$ or $8x + 5y = 14$ (b) $\dfrac{x-3}{8} = \dfrac{y+2}{5}$, $z = -1$

41. $F(x,y,z) = -z + xe^y\cos z \quad\Rightarrow\quad \nabla F(x,y,z) = \langle e^y\cos z, xe^y\cos z, -1 - xe^y\sin z\rangle$,
$\nabla F(1,0,0) = \langle 1,1,-1\rangle$
(a) $1(x-1) + 1(y-0) - 1(z-0) = 0$ or $x + y - z = 1$ (b) $x - 1 = y = -z$

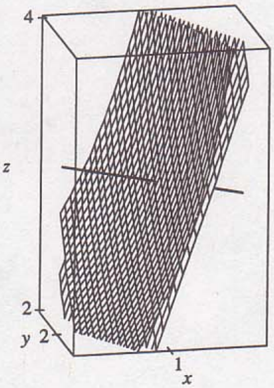
42. $F(x,y,z) = xe^{yz} \quad\Rightarrow\quad \nabla F(x,y,z) = \langle e^{yz}, xze^{yz}, xye^{yz}\rangle$, $\nabla F(1,0,5) = \langle 1,5,0\rangle$

(a) $1(x-1) + 5(y-0) + 0(z-5) = 0$ or $x + 5y = 1$ (b) $x - 1 = \dfrac{y}{5}$, $z = 5$

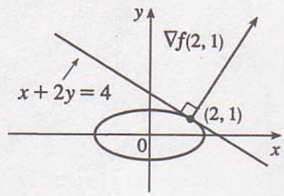
43. $F(x, y, z) = xy + yz + zx$,
$\nabla F(x, y, z) = \langle y + z, x + z, y + x \rangle$,
$\nabla F(1, 1, 1) = \langle 2, 2, 2 \rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$.



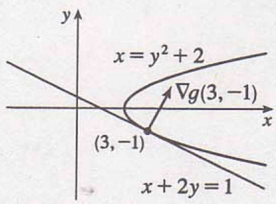
44. $F(x, y, z) = xyz$, $\nabla F(x, y, z) = \langle yz, xz, yx \rangle$,
$\nabla F(1, 2, 3) = \langle 6, 3, 2 \rangle$, so an equation of the tangent plane is $6x + 3y + 2z = 18$, and the normal line is given by $\dfrac{x - 1}{6} = \dfrac{y - 2}{3} = \dfrac{z - 3}{2}$ or $x = 1 + 6t$, $y = 2 + 3t$, $z = 3 + 2t$.



45. $\nabla f(x, y) = \langle 2x, 8y \rangle$, $\nabla f(2, 1) = \langle 4, 8 \rangle$. The tangent line has equation $\nabla f(2, 1) \cdot \langle x - 2, y - 1 \rangle = 0 \Rightarrow$
$4(x - 2) + 8(y - 1) = 0$, which simplifies to $x + 2y = 4$.



46. $\nabla g(x, y) = \langle 1, -2y \rangle$, $\nabla g(3, -1) = \langle 1, 2 \rangle$. The tangent line has equation $\nabla g(3, -1) \cdot \langle x - 3, y + 1 \rangle = 0 \Rightarrow$
$1(x - 3) + 2(y + 1) = 0$, which simplifies to $x + 2y = 1$.



47. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y + \dfrac{2z_0}{c^2}z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence $\dfrac{x_0}{a^2}x + \dfrac{y_0}{b^2}y + \dfrac{z_0}{c^2}z = 1$ is an equation of the tangent plane.

48. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-2z_0}{c^2} \right\rangle$, so an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y - \dfrac{2z_0}{c^2}z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} - \dfrac{z_0^2}{c^2}\right) = 2$ or $\dfrac{x_0}{a^2}x + \dfrac{y_0}{b^2}y - \dfrac{z_0}{c^2}z = 1$.

49. $\nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{-1}{c} \right\rangle$, so an equation of the tangent plane is

$\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y - \frac{1}{c}z = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} - \frac{z_0}{c}$ or $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y = \frac{z}{c} + 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right) - \frac{z_0}{c}$. But $\frac{z_0}{c} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$, so the equation can be written as $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y = \frac{z + z_0}{c}$.

50. Since $\nabla f(x_0, y_0, z_0) = \langle 2x_0, 4y_0, 6z_0 \rangle$ and $\langle 3, -1, 3 \rangle$ are both normal vectors to the surface at (x_0, y_0, z_0), we need $\langle 2x_0, 4y_0, 6z_0 \rangle = c\,\langle 3, -1, 3 \rangle$ or $\langle x_0, 2y_0, 3z_0 \rangle = k\,\langle 3, -1, 3 \rangle$. Thus $x_0 = 3k$, $y_0 = -\frac{1}{2}k$ and $z_0 = k$. But $x_0^2 + 2y_0^2 + 3z_0^2 = 1$ or $\left(9 + \frac{1}{2} + 3\right)k^2 = 1$, so $k = \pm\frac{\sqrt{2}}{5}$ and there are two such points: $\left(\pm\frac{3\sqrt{2}}{5}, \mp\frac{1}{5\sqrt{2}}, \pm\frac{\sqrt{2}}{5}\right)$.

51. $\nabla f(x_0, y_0, z_0) = \langle 2x_0, -2y_0, 4z_0 \rangle$ and the given line has direction numbers 2, 4, 6, so $\langle 2x_0, -2y_0, 4z_0 \rangle = k\,\langle 2, 4, 6 \rangle$ or $x_0 = k$, $y_0 = -2k$ and $z_0 = \frac{3}{2}k$. But $x_0^2 - y_0^2 + 2z_0^2 = 1$ or $\left(1 - 4 + \frac{9}{2}\right)k^2 = 1$, so $k = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$ and there are two such points: $\left(\pm\frac{\sqrt{6}}{3}, \mp\frac{2\sqrt{6}}{3}, \pm\frac{\sqrt{6}}{2}\right)$.

52. First note that the point $(1, 1, 2)$ is on both surfaces. For the ellipsoid, an equation of the tangent plane at $(1, 1, 2)$ is $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$, and for the sphere, an equation of the tangent plane at $(1, 1, 2)$ is $(2 - 8)\,x + (2 - 6)\,y + (4 - 8)\,z = -18$ or $-6x - 4y - 4z = -18$ or $3x + 2y + 2z = 9$. Since these tangent planes are the same, the surfaces are tangent to each other at the point $(1, 1, 2)$.

53. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. Then an equation of the tangent plane to the cone at this point is $2x_0x + 2y_0y - 2z_0z = 2\left(x_0^2 + y_0^2 - z_0^2\right)$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

54. Let (x_0, y_0, z_0) be a point on the sphere. Then the normal line is given by $\frac{x - x_0}{2x_0} = \frac{y - y_0}{2y_0} = \frac{z - z_0}{2z_0}$. For the center $(0, 0, 0)$ to be on the line, we need $-\frac{x_0}{2x_0} = -\frac{y_0}{2y_0} = -\frac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

55. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is

$\frac{x}{2\sqrt{x_0}} + \frac{y}{2\sqrt{y_0}} + \frac{z}{2\sqrt{z_0}} = \frac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is $\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

56. Here the equation of the tangent plane to the point (x_0, y_0, z_0) is $y_0z_0x + x_0z_0y + x_0y_0z = 3x_0y_0z_0$ or $\frac{x}{3x_0} + \frac{y}{3y_0} + \frac{z}{3z_0} = 1$. Then the x-, y-, and z-intercepts are $3x_0$, $3y_0$ and $3z_0$ respectively, and their product is $27x_0y_0z_0 = 27c^3$, a constant.

57. If $f(x, y, z) = z - x^2 - y^2$ and $g(x, y, z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1, 1, 2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line. We have: $\nabla f(x, y, z) = \langle -2x, -2y, 1 \rangle \quad\Rightarrow\quad \nabla f(-1, 1, 2) = \langle 2, -2, 1 \rangle$, and $\nabla g(x, y, z) = \langle 8x, 2y, 2z \rangle \quad\Rightarrow$

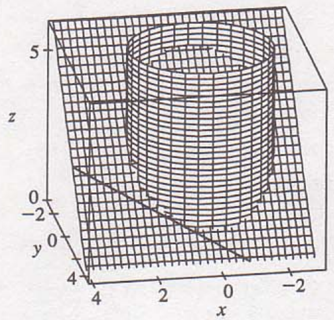
$\nabla g(-1, 1, 2) = \langle -8, 2, 4 \rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$. Parametric equations are: $x = -1 - 10t$, $y = 1 - 16t$, $z = 2 - 12t$.

58. (a) Let $f(x,y,z) = y + z$ and $g(x,y,z) = x^2 + y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1,2,1)$ and the vector $\mathbf{v} = \nabla f \times \nabla g$ is parallel to the tangent line. We have $\nabla f(x,y,z) = \langle 0,1,1\rangle \;\Rightarrow\; \nabla f(1,2,1) = \langle 0,1,1\rangle$, and $\nabla g(x,y,z) = \langle 2x, 2y, 0\rangle \;\Rightarrow\; \nabla g(1,2,1) = \langle 2,4,0\rangle$. Hence

$$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}.$$

So parametric equations of the desired tangent line are $x = 1 - 4t$, $y = 2 + 2t$, $z = 1 - 2t$.

(b)



59. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at P $\Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z\rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z\rangle = 0$ at P $\Leftrightarrow$ $F_x G_x + F_y G_y + F_z G_z = 0$ at P.

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so $\nabla F \cdot \nabla G = \langle 2x, 2y, -2z\rangle \cdot \langle 2x, 2y, 2z\rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point $\langle x,y,z\rangle$ lies on the graph of $F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

60. (a) The function $f(x,y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 15.2.8 [ET 14.2.8].)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{(h\cdot 0)^{1/3} - 0}{h} = 0,$$

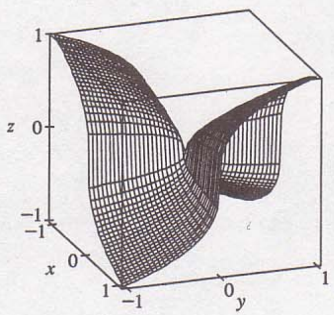
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{(0{\cdot}h)^{1/3} - 0}{h} = 0.$$

Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ where $a \neq 0$ and $b \neq 0$. Thus

$$D_{\mathbf{u}} f(0,0) = \lim_{h\to 0} \frac{f(0+ha, 0+hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{(ha)(hb)}}{h} = \lim_{h\to 0} \frac{\sqrt[3]{ab}}{h^{1/3}}$$

and this limit does not exist, so $D_{\mathbf{u}} f(0,0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

61. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$. Then we know that at the given point, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = a f_x + b f_y$ and $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = c f_x + d f_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y \rangle$ at the given point. In fact,
$$\nabla f = \left\langle \frac{d D_{\mathbf{u}} f - b D_{\mathbf{v}} f}{ad - bc}, \frac{a D_{\mathbf{v}} f - c D_{\mathbf{u}} f}{ad - bc} \right\rangle.$$

62. Since $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 15.4.7 [ET 14.4.7] we have $\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0)$, $\langle \Delta x, \Delta y \rangle = \mathbf{x} - \mathbf{x}_0$ so $(\Delta x, \Delta y) \to (0, 0)$ is equivalent to $\mathbf{x} \to \mathbf{x}_0$ and $\langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle = \nabla f(\mathbf{x}_0)$. Substituting into (15.4.7 [ET 14.4.7]) gives $f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \epsilon_1, \epsilon_2 \rangle \cdot \langle \Delta x, \Delta y \rangle$ or $\langle \epsilon_1, \epsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$, and so
$\dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \dfrac{\langle \epsilon_1, \epsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}$. But $\dfrac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}$ is a unit vector so $\displaystyle\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\langle \epsilon_1, \epsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$ since $\epsilon_1, \epsilon_2 \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. Hence
$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0.$$

15.7 Maximum and Minimum Values ET 14.7

1. (a) First we compute $D(1, 1) = f_{xx}(1, 1)\, f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$ by the Second Derivatives Test.

(b) $D(1, 1) = f_{xx}(1, 1)\, f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1, 1) < 0$, f has a saddle point at $(1, 1)$ by the Second Derivatives Test.

2. (a) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0, 2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0, 2) < 0$, g has a local maximum at $(0, 2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0, 2)$.

3. In the figure, a point at approximately $(1, 1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1, 1)$. The level curves near $(0, 0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there. To verify our predictions, we have $f(x, y) = 4 + x^3 + y^3 - 3xy \;\Rightarrow\; f_x(x, y) = 3x^2 - 3y$, $f_y(x, y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3\left(x^2\right)^2 - 3x = 0 \;\Rightarrow\; 3x\left(x^3 - 1\right) = 0 \;\Rightarrow\; x = 0$ or $x = 1$. Then we have two critical points, $(0, 0)$ and $(1, 1)$. The second partial derivatives are $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -3$, and $f_{yy}(x, y) = 6y$, so

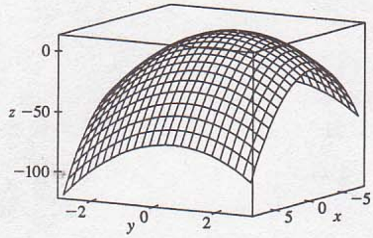
$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm 1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

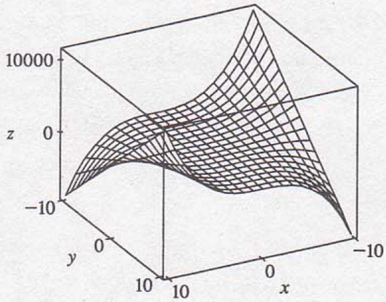
To confirm our predictions, we have $f(x,y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x,y) = 3 - 3x^2$, $f_y(x,y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \Rightarrow x = \pm 1$ and $-4y + 4y^3 = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, \pm 1)$. The second partial derivatives are $f_{xx}(x,y) = -6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2 - 4$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D > 0,\ f_{xx} < 0 \Rightarrow f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D < 0 \Rightarrow f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D > 0,\ f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D > 0,\ f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1,-1)$

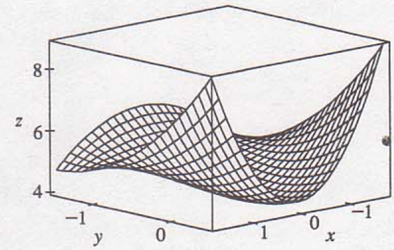
5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \Rightarrow f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$. $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1, \frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1, \frac{1}{2}\right) = -2 < 0$, $f\left(-1, \frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.



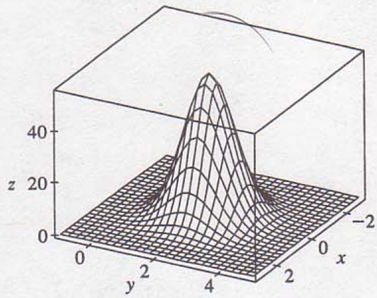
6. $f(x,y) = x^3 y + 12x^2 - 8y \Rightarrow f_x = 3x^2 y + 24x$, $f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$. Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives $12y + 48 = 0 \Rightarrow y = -4$. Thus, the only critical point is $(2,-4)$. $D(2,-4) = (-24)(0) - 12^2 = -144 < 0$, so $(2,-4)$ is a saddle point.



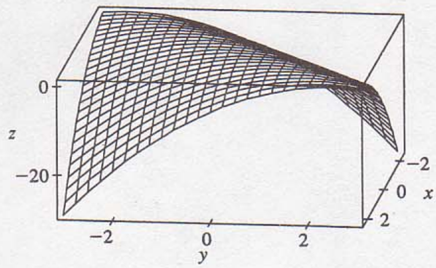
7. $f(x,y) = x^2 + y^2 + x^2y + 4 \quad\Rightarrow\quad f_x = 2x + 2xy$, $f_y = 2y + x^2$, $f_{xx} = 2 + 2y$, $f_{yy} = 2$, $f_{xy} = 2x$. Then $f_y = 0$ implies $y = -\frac{1}{2}x^2$, substituting into $f_x = 0$ gives $2x - x^3 = 0$ so $x = 0$ or $x = \pm\sqrt{2}$. Thus the critical points are $(0,0)$, $(\sqrt{2},-1)$ and $(-\sqrt{2},-1)$. Now $D(0,0) = 4$, $D(\sqrt{2},-1) = -8 = D(-\sqrt{2},-1)$, $f_{xx}(0,0) = 2$, $f_{xx}(\pm\sqrt{2},-1) = 0$.
Thus $f(0,0) = 4$ is a local minimum and $(\pm\sqrt{2},-1)$ are saddle points.



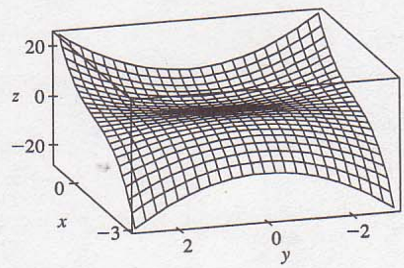
8. $f(x,y) = e^{4y-x^2-y^2} \quad\Rightarrow\quad f_x = -2xe^{4y-x^2-y^2}$, $f_y = (4-2y)\,e^{4y-x^2-y^2}$, $f_{xx} = (4x^2-2)\,e^{4y-x^2-y^2}$, $f_{xy} = -2x(4-2y)\,e^{4y-x^2-y^2}$, $f_{yy} = (4y^2-16y+14)\,e^{4y-x^2-y^2}$. Then $f_x = 0$ and $f_y = 0$ implies $x = 0$ and $y = 2$, so the only critical point is $(0,2)$. $D(0,2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$ and $f_{xx}(0,2) = -2e^4 < 0$, so $f(0,2) = e^4$ is a local maximum.



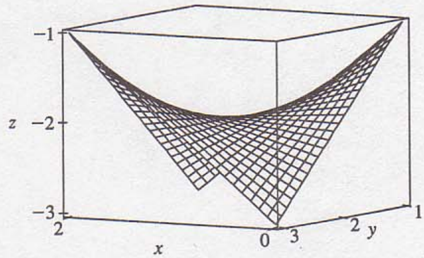
9. $f(x,y) = 1 + 2xy - x^2 - y^2 \quad\Rightarrow\quad f_x = 2y - 2x$, $f_y = 2x - 2y$, $f_{xx} = f_{yy} = -2$, $f_{xy} = 2$. Then $f_x = 0$ and $f_y = 0$ implies $x = y$ so the critical points are all points of the form (x_0,x_0). But $D(x_0,x_0) = 4 - 4 = 0$ so the Second Derivatives Test gives no information. However $1 + 2xy - x^2 - y^2 = 1 - (x-y)^2$ and $1 - (x-y)^2 \le 1$ for all (x,y), with equality if and only if $x = y$. Thus $f(x_0,x_0) = 1$ are local maxima.



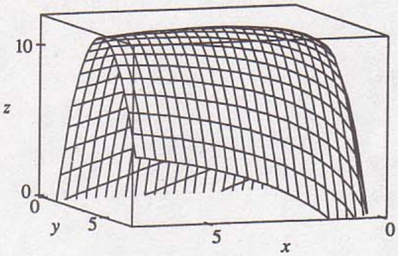
10. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow\quad f_x = 6x^2 + y^2 + 10x$, $f_y = 2xy + 2y$, $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$. Then $f_y = 0$ implies $y = 0$ or $x = -1$. Substituting into $f_x = 0$ gives the critical points $(0,0)$, $(-\frac{5}{3},0)$, $(-1,\pm 2)$. Now $D(0,0) = 20 > 0$ and $f_{xx}(0,0) = 10 > 0$, so $f(0,0) = 0$ is a local minimum. Also $f_{xx}(-\frac{5}{3},0) < 0$, $D(-\frac{5}{3},0) > 0$, and $D(-1,\pm 2) < 0$. Hence $f(-\frac{5}{3},0) = \frac{125}{27}$ is a local maximum while $(-1,\pm 2)$ are saddle points.



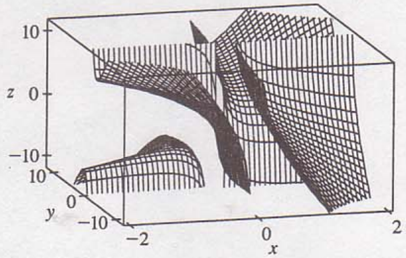
11. $f(x,y) = xy - 2x - y \quad\Rightarrow\quad f_x = y - 2$, $f_y = x - 1$, $f_{xx} = f_{yy} = 0$, $f_{xy} = 1$ and the only critical point is $(1,2)$. Now $D(1,2) = -1$, so $(1,2)$ is a saddle point and f has no local maximum or minimum.

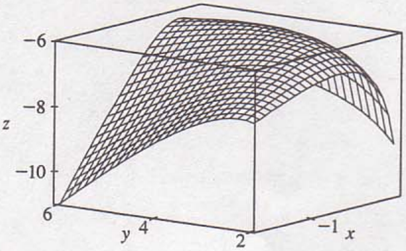


12. $f(x,y) = y\sqrt{x} - y^2 - x + 6y \;\Rightarrow\; f_x = y/(2\sqrt{x}) - 1$,
$f_y = \sqrt{x} - 2y + 6$, $f_{yy} = -2$, $f_{xx} = -\frac{1}{4}yx^{-3/2}$, $f_{xy} = 1/(2\sqrt{x})$.
Then $f_x = 0$ implies $y = 2\sqrt{x}$ and substituting into $f_y = 0$ gives
$-3\sqrt{x} + 6 = 0$ or $x = 4$. Thus the only critical point is $(4,4)$.
$D(4,4) = -\frac{1}{8}(-2) - \left(\frac{1}{4}\right)^2 > 0$ and $f_{xx}(4,4) = -\frac{1}{8}$, so
$f(4,4) = 12$ is a local maximum.

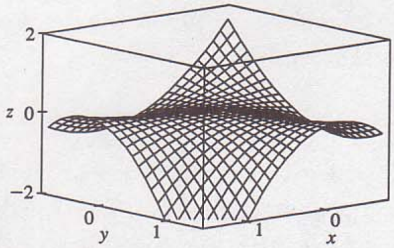


13. $f(x,y) = \dfrac{x^2y^2 - 8x + y}{xy} \;\Rightarrow\; f_x = y - x^{-2}$, $f_y = x + 8y^{-2}$, $f_{xx} = 2x^{-3}$, $f_{yy} = -16y^{-3}$ and $f_{xy} = 1$.
Then $f_x = 0$ implies $y = x^{-2}$, substituting into $f_y = 0$ gives $x + 8x^4 = 0$ so $x = 0$ or $x = -\frac{1}{2}$ but $(0,y)$ is not in the domain of f. Thus the only critical point is $\left(-\frac{1}{2}, 4\right)$. Then $f_{xx}\left(-\frac{1}{2}, 4\right) = -16$ and $D\left(-\frac{1}{2}, 4\right) = 4 - 1 > 0$ so $f\left(-\frac{1}{2}, 4\right) = -6$ is a local maximum.



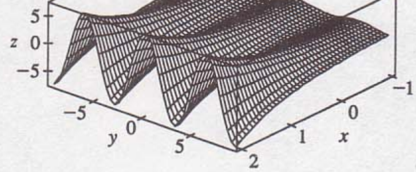


14. $f(x,y) = xy(1 - x - y) \;\Rightarrow\; f_x = y - 2xy - y^2$,
$f_y = x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 1 - 2x - 2y$.
Then $f_x = 0$ implies $y = 0$ or $y = 1 - 2x$. Substituting $y = 0$ into
$f_y = 0$ gives $x = 0$ or $x = 1$ and substituting $y = 1 - 2x$ into
$f_y = 0$ gives $3x^2 - x = 0$ so $x = 0$ or $\frac{1}{3}$. Thus the critical points are
$(0,0)$, $(1,0)$, $(0,1)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.

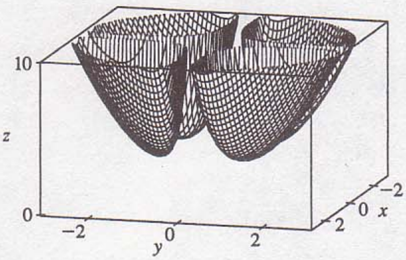


$D(0,0) = D(1,0) = D(0,1) = -1$ while $D\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0$. Thus $(0,0)$, $(1,0)$ and $(0,1)$ are saddle points, and $f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$ is a local maximum.

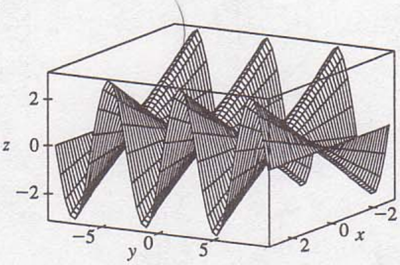
15. $f(x,y) = e^x \cos y \;\Rightarrow\; f_x = e^x \cos y$, $f_y = -e^x \sin y$. Now
$f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer. But
$\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



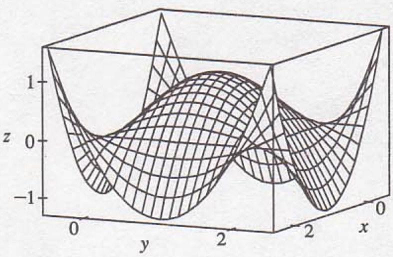
16. $f(x,y) = x^2 + y^2 + \dfrac{1}{x^2y^2} \;\Rightarrow\; f_x = 2x - 2x^{-3}y^{-2}$, $f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$, $f_{xy} = 4x^{-3}y^{-3}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. Now $D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(\pm 1, \pm 1) = f(\pm 1, \mp 1) = 3$ are local minima.



17. $f(x,y) = x \sin y \;\Rightarrow\; f_x = \sin y$, $f_y = x \cos y$, $f_{xx} = 0$, $f_{yy} = -x \sin y$ and $f_{xy} = \cos y$. Then $f_x = 0$ if and only if $y = n\pi$, n an integer, and substituting into $f_y = 0$ requires $x = 0$ for each of these y-values. Thus the critical points are $(0, n\pi)$, n an integer. But $D(0, n\pi) = -\cos^2(n\pi) < 0$ so each critical point is a saddle point.

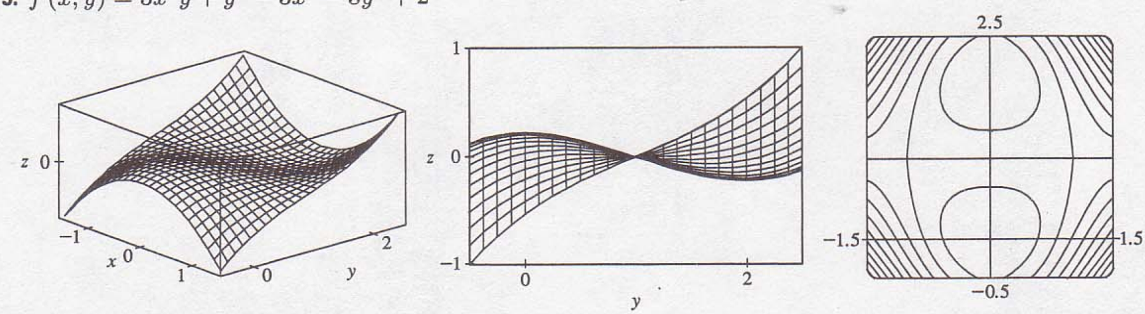


18. $f(x,y) = (2x - x^2)(2y - y^2) \;\Rightarrow\; f_x = (2 - 2x)(2y - y^2)$, $f_y = (2x - x^2)(2 - 2y)$, $f_{xx} = -2(2y - y^2)$, $f_{yy} = -2(2x - x^2)$ and $f_{xy} = (2 - 2x)(2 - 2y)$. Then $f_x = 0$ implies $x = 1$ or $y = 0$ or $y = 2$ and when $x = 1$, $f_y = 0$ implies $y = 1$, when $y = 0$, $f_y = 0$ implies $x = 0$ or $x = 2$ and when $y = 2$, $f_y = 0$ implies $x = 0$ or $x = 2$. Thus the critical points are $(1,1)$, $(0,0)$, $(2,0)$, $(0,2)$ and $(2,2)$.



Now $D(0,0) = D(2,0) = D(0,2) = D(2,2) = -16$ so these critical points are saddle points, and $D(1,1) = 4$ with $f_{xx}(1,1) = -2$, so $f(1,1) = 1$ is a local maximum.

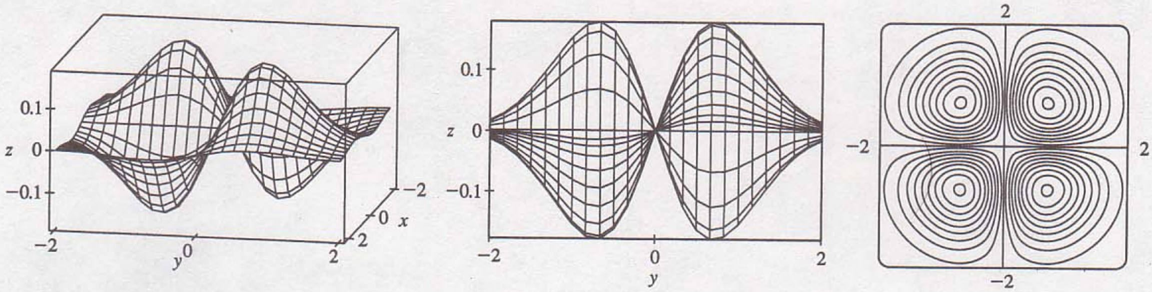
19. $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$



From the graphs, it appears that f has a local maximum $f(0,0) \approx 2$ and a local minimum $f(0,2) \approx -2$. There appear to be saddle points near $(\pm 1, 1)$.

$f_x = 6xy - 6x$, $f_y = 3x^2 + 3y^2 - 6y$. Then $f_x = 0$ implies $x = 0$ or $y = 1$ and when $x = 0$, $f_y = 0$ implies $y = 0$ or $y = 2$; when $y = 1$, $f_y = 0$ implies $x^2 = 1$ or $x = \pm 1$. Thus the critical points are $(0,0)$, $(0,2)$, $(\pm 1, 1)$. Now $f_{xx} = 6y - 6$, $f_{yy} = 6y - 6$ and $f_{xy} = 6x$, so $D(0,0) = D(0,2) = 36 > 0$ while $D(\pm 1, 1) = -36 < 0$ and $f_{xx}(0,0) = -6$, $f_{xx}(0,2) = 6$. Hence $(\pm 1, 1)$ are saddle points while $f(0,0) = 2$ is a local maximum and $f(0,2) = -2$ is a local minimum.

20. $f(x,y) = xye^{-x^2-y^2}$



There appear to be local maxima of about $f(\pm 0.7, \pm 0.7) \approx 0.18$ and local minima of about $f(\pm 0.7, \mp 0.7) \approx -0.18$. Also, there seems to be a saddle point at the origin.

$f_x = ye^{-x^2-y^2}(1 - 2x^2)$, $f_y = xe^{-x^2-y^2}(1 - 2y^2)$, $f_{xx} = 2xye^{-x^2-y^2}(2x^2 - 3)$,

$f_{yy} = 2xye^{-x^2-y^2}(2y^2 - 3)$, $f_{xy} = (1 - 2x^2)\,e^{-x^2-y^2}(1 - 2y^2)$. Then $f_x = 0$ implies $y = 0$ or $x = \pm\frac{1}{\sqrt{2}}$.

Substituting these values into $f_y = 0$ gives the critical points $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Then

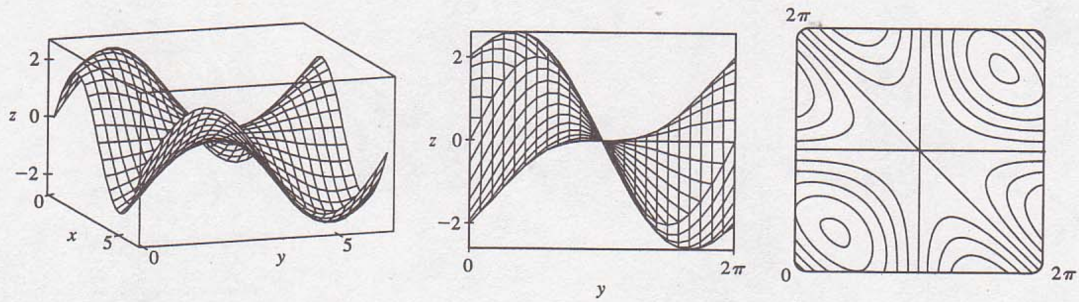
$D(x,y) = e^{2(-x^2-y^2)}\left[4x^2y^2(2x^2-3)(2y^2-3) - (1-2x^2)^2(1-2y^2)^2\right]$, so $D(0,0) = -1$, while

$D\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$ and $D\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$. But $f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$, $f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) > 0$,

$f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > 0$ and $f_{xx}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$. Hence $(0,0)$ is a saddle point;

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$ are local minima and $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$ are local maxima.

21. $f(x,y) = \sin x + \sin y + \sin(x+y)$, $0 \le x \le 2\pi$, $0 \le y \le 2\pi$

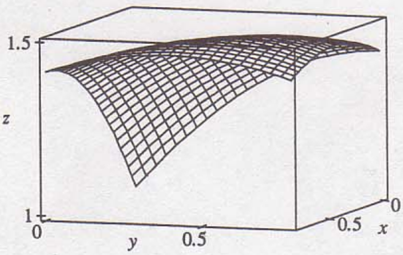


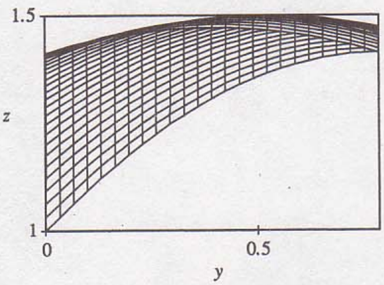
From the graphs it appears that f has a local maximum at about $(1,1)$ with value approximately 2.6, a local minimum at about $(5,5)$ with value approximately -2.6, and a saddle point at about $(3,3)$.

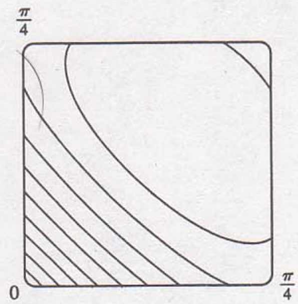
$f_x = \cos x + \cos(x+y)$, $f_y = \cos y + \cos(x+y)$, $f_{xx} = -\sin x - \sin(x+y)$, $f_{yy} = -\sin y - \sin(x+y)$,

$f_{xy} = -\sin(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, yielding the critical points (π, π), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π, π). Now $D(x,y) = \sin x \sin y + \sin x \sin(x+y) + \sin y \sin(x+y)$. So $D(\pi, \pi) = 0$ and the Second Derivatives Test doesn't apply. $D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

22. $f(x,y) = \sin x + \sin y + \cos(x+y)$, $0 \le x \le \frac{\pi}{4}$, $0 \le y \le \frac{\pi}{4}$

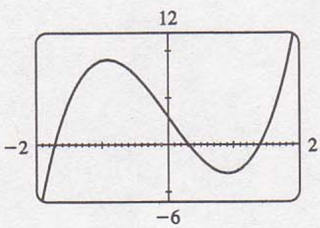


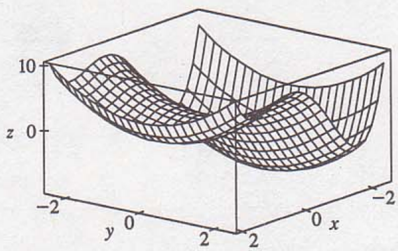




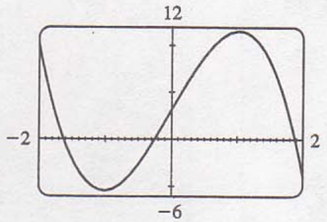
From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$.
$f_x = \cos x - \sin(x+y)$, $f_y = \cos y - \sin(x+y)$, $f_{xx} = -\sin x - \cos(x+y)$, $f_{yy} = -\sin y - \cos(x+y)$, $f_{xy} = -\cos(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \neq 0$ for $0 \le x \le \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

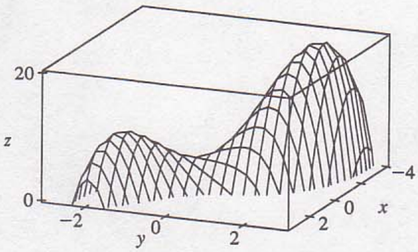
23. $f(x,y) = x^4 - 5x^2 + y^2 + 3x + 2 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 10x + 3$ and $f_y(x,y) = 2y$. $f_y = 0 \quad \Rightarrow \quad y = 0$, and the graph of f_x shows that the roots of $f_x = 0$ are approximately $x = -1.714$, 0.312 and 1.402. (Alternatively, we could have used a calculator or a CAS to find these roots.) So to three decimal places, the critical points are $(-1.714, 0)$, $(1.402, 0)$, and $(0.312, 0)$. Now since $f_{xx} = 12x^2 - 10$, $f_{xy} = 0$, $f_{yy} = 2$, and $D = 24x^2 - 20$, we have $D(-1.714, 0) > 0$, $f_{xx}(-1.714, 0) > 0$, $D(1.402, 0) > 0$, $f_{xx}(1.402, 0) > 0$, and $D(0.312, 0) < 0$. Therefore $f(-1.714, 0) \approx -9.200$ and $f(1.402, 0) \approx 0.242$ are local minima, and $(0.312, 0)$ is a saddle point. The lowest point on the graph is approximately $(-1.714, 0, -9.200)$.



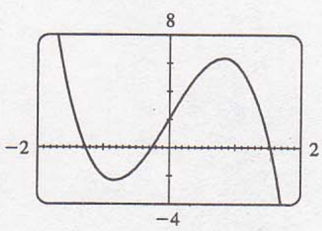


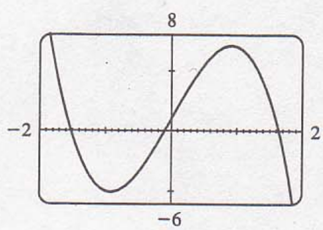
24. $f(x,y) = 5 - 10xy - 4x^2 + 3y - y^4 \Rightarrow f_x(x,y) = -10y - 8x$, $f_y(x,y) = -10x + 3 - 4y^3$. Now $f_x = 0 \Rightarrow x = -\frac{5}{4}y$, so using a graph, we find solutions to $0 = f_y\left(-\frac{5}{4}y, y\right) = -10\left(-\frac{5}{4}y\right) + 3 - 4y^3 = -4y^3 + \frac{25}{2}y + 3$. (Alternatively, we could have found the roots of $f_x = f_y = 0$ directly, using a calculator or a CAS.) To three decimal places, the solutions are $y \approx 1.877$, -0.245 and -1.633, so f has critical points at approximately $(-2.347, 1.877)$, $(0.306, -0.245)$, and $(2.041, -1.633)$. Now since $f_{xx} = -8$, $f_{xy} = -10$, $f_{yy} = -12y^2$, and $D = 96y^2 - 100$, we have $D(-2.347, 1.877) > 0$, $D(0.306, -0.245) < 0$, and $D(2.041, -1.633) > 0$. Therefore, since $f_{xx} < 0$ everywhere, $f(-2.347, 1.877) \approx 20.238$ and $f(2.041, -1.633) \approx 9.657$ are local maxima, and $(0.306, -0.245)$ is a saddle point. The highest point on the graph is approximately $(-2.347, 1.877, 20.238)$.

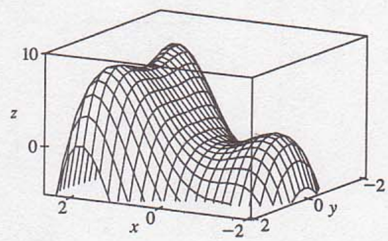




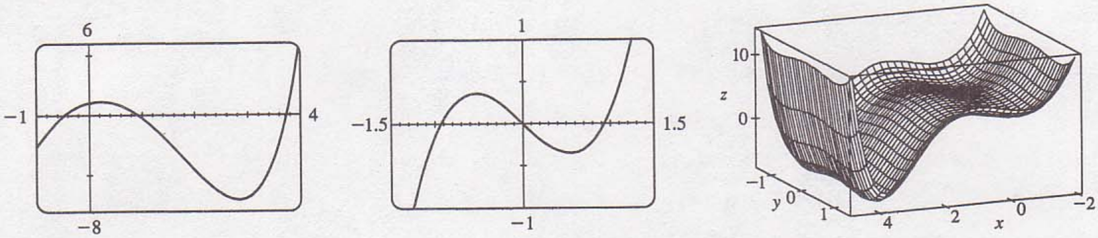
25. $f(x,y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4 \Rightarrow f_x(x,y) = 2 + 8x + 2y^2 - 4x^3$, $f_y(x,y) = -2y + 4xy - 4y^3$. Now $f_y = 0 \Leftrightarrow 2y\left(2y^2 - 2x + 1\right) = 0 \Leftrightarrow y = 0$ or $y^2 = x - \frac{1}{2}$. The first of these implies that $f_x = -4x^3 + 8x + 2$, and the second implies that $f_x = 2 + 8x + 2\left(x - \frac{1}{2}\right) - 4x^3 = -4x^3 + 10x + 1$. From the graphs, we see that the first possibility for f_x has roots at approximately -1.267, -0.259, and 1.526, and the second has a root at approximately 1.629 (the negative roots do not give critical points, since $y^2 = x - \frac{1}{2}$ must be positive). So to three decimal places, f has critical points at $(-1.267, 0)$, $(-0.259, 0)$, $(1.526, 0)$, and $(1.629, \pm 1.063)$. Now since $f_{xx} = 8 - 12x^2$, $f_{xy} = 4y$, $f_{yy} = 4x - 12y^2$, and $D = \left(8 - 12x^2\right)\left(4x - 12y^2\right) - 16y^2$, we have $D(-1.267, 0) > 0$, $f_{xx}(-1.267, 0) > 0$, $D(-0.259, 0) < 0$, $D(1.526, 0) < 0$, $D(1.629, \pm 1.063) > 0$, and $f_{xx}(1.629, \pm 1.063) < 0$. Therefore, to three decimal places, $f(-1.267, 0) \approx 1.310$ and $f(1.629, \pm 1.063) \approx 8.105$ are local maxima, and $(-0.259, 0)$ and $(1.526, 0)$ are saddle points. The highest points on the graph are approximately $(1.629, \pm 1.063, 8.105)$.



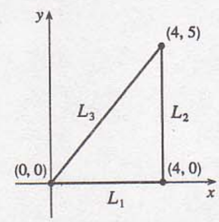




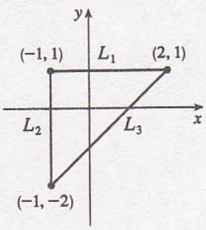
26. $f(x,y) = e^x + y^4 - x^3 + 4\cos y \quad \Rightarrow \quad f_x(x,y) = e^x - 3x^2$ and $f_y(x,y) = 4y^3 - 4\sin y$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -0.459, 0.910$, or 3.733, and $f_y = 0$ when $y \approx 0$ or ± 0.929. (Alternatively, we could have used a calculator or a CAS to find the roots of $f_x = 0$ and $f_y = 0$.) So, to three decimal places, f has critical points at $(-0.459, 0)$, $(-0.459, \pm 0.929)$, $(0.910, 0)$, $(0.910, \pm 0.929)$, $(3.733, 0)$, and $(3.733, \pm 0.929)$. Now $f_{xx} = e^x - 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4\cos y$, and $D = (e^x - 6x)(12y^2 - 4\cos y)$. Therefore $D(-0.459, 0) < 0$, $D(-0.459, \pm 0.929) > 0$, $f_{xx}(-0.459, \pm 0.929) > 0$, $D(0.910, 0) > 0$, $f_{xx}(0.910, 0) < 0$, $D(0.910, \pm 0.929) < 0$, $D(3.733, 0) < 0$, $D(3.733, \pm 0.929) > 0$, and $f_{xx}(3.733, \pm 0.929) > 0$. So $f(-0.459, \pm 0.929) \approx 3.868$ and $f(3.733, \pm 0.929) \approx -7.077$ are local minima, $f(0.910, 0) \approx 5.731$ is a local maximum, and $(-0.459, 0)$, $(0.910, \pm 0.929)$, and $(3.733, 0)$ are saddle points. The lowest points on the graph are approximately $(3.733, \pm 0.929, -7.077)$.



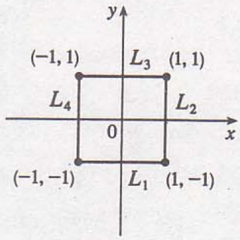
27. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = -3$, $f_y = 4$ so there are no critical points inside D. Thus the absolute extrema must both occur on the boundary. Along L_1, $y = 0$ and $f(x, 0) = 5 - 3x$, a decreasing function in x, so the maximum value is $f(0,0) = 5$ and the minimum value is $f(4,0) = -7$. Along L_2, $x = 4$ and $f(4, y) = -7 + 4y$, an increasing function in y, so the minimum value is $f(4,0) = -7$ and the maximum value is $f(4,5) = 13$. Along L_3, $y = \frac{5}{4}x$ and $f\left(x, \frac{5}{4}x\right) = 5 + 2x$, an increasing function in x, so the minimum value is $f(0,0) = 5$ and the maximum value is $f(4,5) = 13$. Thus the absolute minimum of f on D is $f(4,0) = -7$ and the absolute maximum is $f(4,5) = 13$.



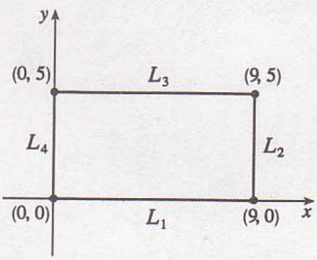
28. $f_x = 2x + 2y$ and $f_y = 2x + 6y$. Setting $f_x = f_y = 0$ gives $x = y = 0$ which yields the critical point $(0,0)$ where $f(0,0) = 0$. Along L_1: $y = 1$ and $f(x, 1) = x^2 + 2x + 3$, $-1 \le x \le 2$, which has a maximum at $x = 2$ where $f(2,1) = 11$, and a minimum at $x = -1$ where $f(-1,1) = 2$. Along L_2: $x = -1$ and $f(-1, y) = 1 - 2y + 3y^2$, $-2 \le y \le 1$, which has a maximum at $y = -2$ where $f(-1,-2) = 17$ and a minimum at $y = \frac{1}{3}$ where $f\left(-1, \frac{1}{3}\right) = \frac{2}{3}$. Along L_3: $y = x - 1$ and $f(x, x-1) = 6x^2 - 8x + 3$, $-1 \le x \le 2$, which has a maximum at $x = -1$ where $f(-1,-2) = 17$ and a minimum at $x = \frac{2}{3}$ where $f\left(\frac{2}{3}, -\frac{1}{3}\right) = \frac{1}{3}$. As a result, the absolute maximum value of f on D is $f(-1,-2) = 17$ and the minimum value is $f(0,0) = 0$.



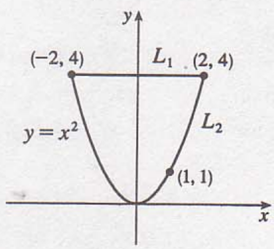
29. In Exercise 7, we found the critical points of f; only $(0,0)$ with $f(0,0)=4$ is in D. On L_1: $y=-1$, $f(x,-1)=5$, a constant. On L_2: $x=1$, $f(1,y)=y^2+y+5$, a quadratic in y which attains its maximum at $(1,1)$, $f(1,1)=7$ and its minimum at $\left(1,-\frac{1}{2}\right)$, $f\left(1,-\frac{1}{2}\right)=\frac{17}{4}$. On L_3: $f(x,1)=2x^2+5$ which attains its maximum at $(-1,1)$ and $(1,1)$ with $f(\pm1,1)=7$ and its minimum at $(0,1)$, $f(0,1)=5$. On L_4: $f(-1,y)=y^2+y+5$ with maximum at $(-1,1)$, $f(-1,1)=7$ and minimum at $\left(-1,-\frac{1}{2}\right)$, $f\left(-1,-\frac{1}{2}\right)=\frac{17}{4}$. Thus the absolute maximum is attained at both $(\pm1,1)$ with $f(\pm1,1)=7$ and the absolute minimum on D is attained at $(0,0)$ with $f(0,0)=4$.



30. Since $x \ge 0$ in D, f is continuous on D. In Exercise 12 we found that the only critical point of f is $(4,4)$ and $f(4,4)=-\frac{1}{8}$. [Note that $(4,4)$ is in D.]
On L_1: $f(x,0)=-x$, so the maximum value is $f(0,0)=0$ and the minimum value is $f(9,0)=-9$.
On L_2: $f(9,y)=9y-y^2-9$, a quadratic in y which attains its maximum at $y=\frac{9}{2}$, $f\left(9,\frac{9}{2}\right)=\frac{45}{4}$ and its minimum at $y=0$, $f(9,0)=-9$.
On L_3: $f(x,5)=5\sqrt{x}-x+5$, a function whose maximum is attained at $x=\frac{25}{4}$, $f\left(\frac{25}{4},5\right)=\frac{45}{4}$ and its minimum at $x=0$, $f(0,5)=5$.
On L_4: $f(0,y)=-y^2+6y$, a quadratic in y which attains its maximum at $y=3$, $f(0,3)=9$ and its minimum at $y=0$, $f(0,0)=0$. Thus the absolute maximum of f on D is $f\left(\frac{25}{4},5\right)=f\left(9,\frac{9}{2}\right)=\frac{45}{4}$ and the absolute minimum is $f(9,0)=-9$.



31. $f_x(x,y)=y-1$ and $f_y(x,y)=x-1$ and so the critical point is $(1,1)$ (in D), where $f(1,1)=0$. Along L_1: $y=4$, so $f(x,4)=1+4x-x-4=3x-3$, $-2\le x\le 2$, which is an increasing function and has a maximum value when $x=2$ where $f(2,4)=3$ and a minimum of $f(-2,4)=-9$. Along L_2: $y=x^2$, so let $g(x)=f\left(x,x^2\right)=x^3-x^2-x+1$. Then
$g'(x)=3x^2-2x-1=0 \quad\Leftrightarrow\quad x=-\frac{1}{3}$ or $x=1$.
$f\left(-\frac{1}{3},\frac{1}{9}\right)=\frac{32}{27}$ and $f(1,1)=0$. As a result, the absolute maximum and minimum values of f on D are $f(2,4)=3$ and $f(-2,4)=-9$.

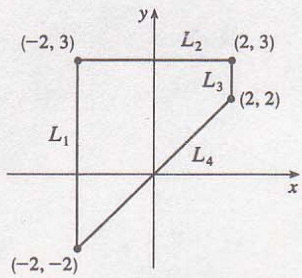


32. $f_x=4x+1$, $f_y=2y$ and the only critical point is $\left(-\frac{1}{4},0\right)$ (and this point is in D) and $f\left(-\frac{1}{4},0\right)=-\frac{17}{8}$. On the circle $x^2+y^2=4$, $f(x,y)=x^2+x+2$, a quadratic in x which attains its minimum at $\left(-\frac{1}{2},\pm\frac{\sqrt{15}}{2}\right)$, $f\left(-\frac{1}{2},\pm\frac{\sqrt{15}}{2}\right)=\frac{7}{4}$ and its maximum at $(2,0)$, $f(2,0)=8$. Thus the absolute maximum of f on D is $f(2,0)=8$ while the absolute minimum is $f\left(-\frac{1}{4},0\right)=-\frac{17}{8}$.

33. $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1-x^2)^2 = x^4 + 2x^3 - 2x^2 + 1, -1 \leq x \leq 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \quad \Rightarrow \quad x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2,-3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta, 0 \leq \theta \leq 2\pi$.

34. $f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2) = 14$, $f(-1,2) = 18$. Along L_1: $x = -2$ and $f(-2,y) = -2 - y^3 + 12y, -2 \leq y \leq 3$, which has a maximum at $y = 2$ where $f(-2,2) = 14$ and a minimum at $y = -2$ where $f(-2,-2) = -18$. Along L_2: $x = 2$ and $f(2,y) = 2 - y^3 + 12y, 2 \leq y \leq 3$, which has a maximum at $y = 2$ where $f(2,2) = 18$ and a minimum at $y = 3$ where $f(2,3) = 11$. Along L_3: $y = 3$ and $f(x,3) = x^3 - 3x + 9, -2 \leq x \leq 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1,3) = f(2,3) = 11$ and a minimum at $x = 1$ and $x = -2$ where $f(1,3) = f(-2,3) = 7$. Along L_4: $y = x$ and $f(x,x) = 9x, -2 \leq x \leq 2$, which has a maximum at $x = 2$ where $f(2,2) = 18$ and a minimum at $x = -2$ where $f(-2,-2) = -18$. So the absolute maximum value of f on D is $f(2,2) = 18$ and the minimum is $f(-2,-2) = -18$.



35. $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 \quad \Rightarrow \quad f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y - x - 1)(2xy - 1)$ and $f_y(x,y) = -2(x^2y - x - 1)x^2$. Setting $f_y(x,y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$. There are no critical points for $x = 0$, since $f_x(0,y) = -2$, so we set $x^2y - x - 1 = 0 \quad \Leftrightarrow \quad y = \dfrac{x+1}{x^2}$ $(x \neq 0)$, so

$$f_x\left(x, \frac{x+1}{x^2}\right) = -2(x^2-1)(2x) - 2\left(x^2\frac{x+1}{x^2} - x - 1\right)\left(2x\frac{x+1}{x^2} - 1\right) = -4x(x^2-1).$$ Therefore $f_x(x,y) = f_y(x,y) = 0$ at the points $(1,2)$ and $(-1,0)$.

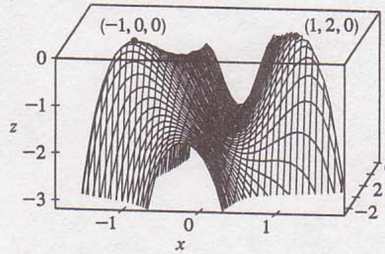
To classify these critical points, we calculate

$f_{xx}(x,y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$,

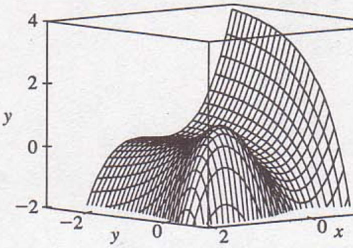
$f_{yy}(x,y) = -2x^4$, and $f_{xy}(x,y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$$\begin{aligned} D(-1,0) &= f_{xx}(-1,0)\, f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 \\ &= 16 > 0, \end{aligned}$$

$f_{xx}(-1,0) = -10 < 0$, $D(1,2) = 16 > 0$, and $f_{xx}(1,2) = -26 < 0$, so both $(-1,0)$ and $(1,2)$ give local maxima.



36. $f(x,y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that (1) $f_x = 3e^y - 3x^2 = 0$ and (2) $f_y = 3xe^y - 3e^{3y} = 0$. From (1) we obtain $e^y = x^2$, and then (2) gives $3x^3 - 3x^6 = 0 \Rightarrow x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes (1) impossible. So substituting $x = 1$ into (1) gives $y = 0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since $D(1,0) = \left[-6x\left(3xe^y - 9e^{3y}\right) - \left(3e^y\right)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

37. $d = \sqrt{(x-2)^2 + (y+2)^2 + (z-3)^2}$, where $z = \frac{1}{3}(6x + 4y - 2)$, so we minimize $d^2 = f(x,y) = (x-2)^2 + (y+2)^2 + \left(2x + \frac{4}{3}y - \frac{11}{3}\right)^2$. Then $f_x = 10x + \frac{16}{3}y - \frac{56}{3}$ and $f_y = \frac{50}{9}y + \frac{16}{3}x - \frac{52}{9}$. Solving $50y + 48x = 52$ and $16y + 30x = 56$ simultaneously gives $x = \frac{164}{61}$, $y = -\frac{94}{61}$. The absolute minimum must occur at a critical point. Thus $d^2 = \left(\frac{42}{61}\right)^2 + \left(\frac{28}{61}\right)^2 + \left(-\frac{21}{61}\right)^2$ or $d = \frac{7}{\sqrt{61}}$.

38. Here $d = \sqrt{(x+4)^2 + (y-1)^2 + (z-3)^2}$, where $z = 1 + y - 2x$. So we minimize $d^2 = f(x,y) = (x+4)^2 + (y-1)^2 + (-2 - 2x + y)^2$. Then $f_x = 2(x+4) - 4(-2 - 2x + y) = 10x - 4y + 16 = 0$ implies $y = \frac{5}{2}x + 4$ and $f_y = 4y - 4x - 6 = 0$, so the only critical point is $\left(-\frac{5}{3}, -\frac{1}{6}\right)$. Thus the closest point to $(-4,1,3)$ is $\left(-\frac{5}{3}, -\frac{1}{6}, \frac{25}{6}\right)$.

39. Minimize $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1$. Then $f_x = 2x + y$, $f_y = 2y + x$ so the critical point is $(0,0)$ and $D(0,0) = 4 - 1 > 0$ with $f_{xx}(0,0) = 2$ so this is a minimum. Thus $z^2 = 1$ or $z = \pm 1$ and the points on the surface are $(0,0,\pm 1)$.

40. Since $z = 1/\left(x^2y^2\right)$ on the surface, we minimize $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + x^{-4}y^{-4} = f(x,y)$. $f_x = 2x - \dfrac{4}{x^5y^4}$, $f_y = 2y - \dfrac{4}{x^4y^5}$, so the critical points occur when $2x = \dfrac{4}{x^5y^4}$ and $2y = \dfrac{4}{x^4y^5}$ or $x^6y^4 = 2 = x^4y^6$, so $x^2 = y^2 \Rightarrow x = \pm y$ and $x^{10} = 2 \Rightarrow x = \pm 2^{1/10}$, $y = \pm 2^{1/10}$. The four critical points are $\left(\pm 2^{1/10}, \pm 2^{1/10}\right)$. The absolute minimum must occur at these points (there is no maximum since the surface is infinite in extent). Thus the points on the surface closest to the origin are $\left(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5}\right)$.

41. $x + y + z = 100$, so maximize $f(x, y) = xy(100 - x - y)$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y = 0$ or $y = 100 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 100$ and substituting $y = 100 - 2x$ into $f_y = 0$ gives $3x^2 - 100x = 0$ so $x = 0$ or $\frac{100}{3}$. Thus the critical points are $(0, 0)$, $(100, 0)$, $(0, 100)$ and $\left(\frac{100}{3}, \frac{100}{3}\right)$.

$D(0,0) = D(100,0) = D(0,100) = -10{,}000$ while $D\left(\frac{100}{3}, \frac{100}{3}\right) = \frac{10{,}000}{3}$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $(0,0)$, $(100,0)$ and $(0,100)$ are saddle points whereas $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$.

42. Maximize $f(x, y) = x^a y^b (100 - x - y)^c$.

$f_x = ax^{a-1}y^b(100 - x - y)^c - cx^a y^b (100 - x - y)^{c-1} = x^{a-1}y^b(100 - x - y)^{c-1}[a(100 - x - y) - cx]$ and $f_y = x^a y^{b-1}(100 - x - y)^{c-1}[b(100 - x - y) - cy]$. Since x, y and z are all positive, the only critical point occurs when $x = a\dfrac{100 - y}{a + c}$ and $y = \dfrac{100b}{a + b + c}$. Thus the point is $\left(\dfrac{100a}{a + b + c}, \dfrac{100b}{a + b + c}\right)$ and the numbers are $x = \dfrac{100a}{a + b + c}$, $y = \dfrac{100b}{a + b + c}$, $z = \dfrac{100c}{a + b + c}$.

43. Maximize $f(x, y) = xy\left(36 - 9x^2 - 36y^2\right)^{1/2}/2$ with (x, y, z) in first octant. Then

$$f_x = \frac{y\left(36 - 9x^2 - 36y^2\right)^{1/2}}{2} + \frac{-9x^2 y\left(36 - 9x^2 - 36y^2\right)^{-1/2}}{2} = \frac{\left(36y - 18x^2 y - 36y^3\right)}{2\left(36 - 9x^2 - 36y^2\right)^{1/2}} \text{ and}$$

$f_y = \dfrac{36x - 9x^3 - 72xy^2}{2\left(36 - 9x^2 - 36y^2\right)^{1/2}}$. Setting $f_x = 0$ gives $y = 0$ or $y^2 = \dfrac{2 - x^2}{2}$ but $y > 0$, so only the latter solution applies. Substituting this y into $f_y = 0$ gives $x^2 = \frac{4}{3}$ or $x = \frac{2}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}}$ and then $z^2 = (36 - 12 - 12)/4 = 3$. The fact that this gives a maximum volume follows from the geometry. This maximum volume is $V = (2x)(2y)(2z) = 8\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\sqrt{3}\right) = \frac{16}{\sqrt{3}}$.

44. Here maximize $f(x, y) = xy\dfrac{\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}{a^2b^2}$. Then

$f_x = yc^2\dfrac{a^2b^2 - 2b^2x^2 - a^2y^2}{a^2b^2\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}$ and $f_y = xc^2\dfrac{a^2b^2 - 2a^2y^2 - b^2x^2}{a^2b^2\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}$. Then $f_x = 0$ (with $x, y > 0$) implies $y^2 = \dfrac{a^2b^2 - 2b^2x^2}{a^2}$ and substituting into $f_y = 0$ implies $3b^2x^2 = a^2b^2$ or $x = \frac{1}{\sqrt{3}}a$, $y = \frac{1}{\sqrt{3}}b$ and then $z = \frac{1}{\sqrt{3}}c$. Thus the maximum volume of such a rectangle is $V = (2x)(2y)(2z) = \frac{8}{3\sqrt{3}}abc$.

45. Maximize $f(x, y) = \dfrac{xy}{3}(6 - x - 2y)$, then the maximum volume is $V = xyz$.

$f_x = \frac{1}{3}\left(6y - 2xy - y^2\right) = \frac{1}{3}y(6 - 2x - 2y)$ and $f_y = \frac{1}{3}x(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2, 1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

46. Surface area $= 2(xy + xz + yz) = 64\text{ cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume $f(x,y) = xy\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x+y)^2} = y^2\dfrac{32 - 2xy - x^2}{(x+y)^2}$ and $f_y = x^2\dfrac{32 - 2xy - y^2}{(x+y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or $3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

47. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is $V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

48. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x,y) = 5xy + 2V(x+y)/(xy) = 5xy + 2V\left(x^{-1} + y^{-1}\right)$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2Vy^{-2}$, $f_x = 0$ implies $y = 2V/(5x^2)$, $f_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the box which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

49. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\text{ m}^3$. Then $f(x,y) = xy + [64{,}000(x+y)/xy] = xy + 64{,}000\left(x^{-1} + y^{-1}\right)$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x,y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40,40)$ and $f_{xx}(40,40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

50. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving $P = P(q,r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \geq 0$, $r \geq 0$, and $q + r \leq 1$. Thus, we want to find the absolute maximum of the continuous function $P(q,r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q,r) = 2 - 4q - 2r = 0$ and $P_r(q,r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have $2 - 4(1 - 2q) - 2q = 0 \Rightarrow q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0$, $0 \leq q \leq 1$, $P(q,r) = P(q,0) = 2q - 2q^2$, $0 \leq q \leq 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0$, $0 \leq r \leq 1$ we have $P(0,r) = 2r - 2r^2$, $0 \leq r \leq 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1$, $0 \leq q \leq 1$, $P(q,r) = P(q, 1-q) = 2q - 2q^2$, $0 \leq q \leq 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q,r)$ on D is $\frac{2}{3}$.

51. Note that here the variables are m and b, and $f(m,b) = \sum\limits_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then

$f_m = \sum\limits_{i=1}^{n} -2x_i\,[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} \left(x_i y_i - mx_i^2 - bx_i\right) = 0$ or $\sum\limits_{i=1}^{n} x_i y_i = m \sum\limits_{i=1}^{n} x_i^2 + b \sum\limits_{i=1}^{n} x_i$

and $f_b = \sum\limits_{i=1}^{n} -2\,[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} y_i = m \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} b = m\left(\sum\limits_{i=1}^{n} x_i\right) + nb$. Thus we have

the two desired equations. Now $f_{mm} = \sum\limits_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum\limits_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum\limits_{i=1}^{n} 2x_i$. And $f_{mm}(m,b) > 0$

always and $D(m,b) = 4n\left(\sum\limits_{i=1}^{n} x_i^2\right) - 4\left(\sum\limits_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum\limits_{i=1}^{n} x_i^2\right) - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right] > 0$ always so the

solutions of these two equations do indeed minimize $\sum\limits_{i=1}^{n} d_i^2$.

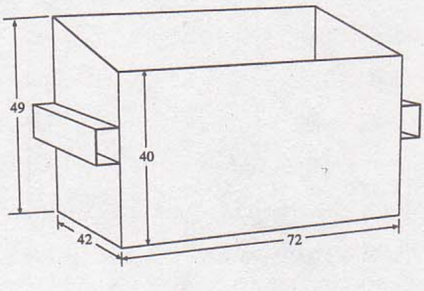
52. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1,2,3)$. Writing the equation of the plane as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the volume of the tetrahedron is given by $V = \frac{abc}{6}$. But $(1,2,3)$ must lie on the plane, so we need $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$ (★) and thus can think of c as a function of a and b. Then $V_a = \frac{b}{6}\left(c + a\frac{\partial c}{\partial a}\right)$ and $V_b = \frac{a}{6}\left(c + b\frac{\partial c}{\partial b}\right)$. Differentiating (★) with respect to a we get $-a^{-2} - 3c^{-2}\frac{\partial c}{\partial a} = 0 \;\Rightarrow\; \frac{\partial c}{\partial a} = \frac{-c^2}{3a^2}$, and differentiating (★) with respect to b gives $-2b^{-2} - 3c^{-2}\frac{\partial c}{\partial b} = 0 \;\Rightarrow\; \frac{\partial c}{\partial b} = \frac{-2c^2}{3b^2}$. Then $V_a = \frac{b}{6}\left(c + a\frac{-c^2}{3a^2}\right) = 0 \;\Rightarrow\; c = 3a$, and $V_b = \frac{a}{6}\left(c + b\frac{-2c^2}{3b^2}\right) = 0 \;\Rightarrow\; c = \frac{3}{2}b$. Thus $3a = \frac{3}{2}b$ or $b = 2a$. Putting these into (★) gives $\frac{3}{a} = 1$ or $a = 3$ and then $b = 6$, $c = 9$. Thus the equation of the required plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$ or $6x + 3y + 2z = 18$.

Applied Project □ Designing a Dumpster

Note: The difficulty and results of this project vary widely with the type of container studied. In addition to the variation of basic shapes of containers, containers may include additional constructed parts such as supports, lift pockets, wheels, etc. Also, a CAS or graphing utility may be needed to solve the resulting equations.

Here we present a typical solution for one particular trash dumpster.

1. The basic shape and dimensions (in inches) of an actual trash dumpster are as shown in the figure.



The front and back, as well as both sides, have an extra one-inch-wide flap that is folded under and welded to the base. In addition, the side panels each fold over one inch onto the front and back pieces where they are welded. Each side has a rectangular lift pocket, with cross-section 5 by 8 inches, made of the same material. These are attached with an extra one-inch width of steel on both top and bottom where each pocket is welded to the side sheet. All four sides have a "lip" at the top; the front and back panels have an extra 5 inches of steel at the top which is folded outward in three creases to form a rectangular tube. The edge is then welded back to the main sheet. The two sides form a top lip with separate sheets of steel 5 inches wide, similarly bent into three sides and welded to the main sheets (requiring two welds each). These extend beyond the main side sheets by 1.5 inches at each end in order to join with the lips on the front and back panels. The container has a hinged lid, extra steel supports on the base at each corner, metal "fins" serving as extra support for the side lift pockets, and wheels underneath. The volume of the container is $V = \frac{1}{2}(40 + 49) \times 42 \times 72 = 134{,}568 \text{ in}^3$ or 77.875 ft^3.

2. First, we assume that some aspects of the construction do not change with different dimensions, so they may be considered fixed costs. This includes the lid (with hinges), wheels, and extra steel supports. Also, the upper "lip" we previously described extends beyond the side width to connect to the other pieces. We can safely assume that this extra portion, including any associated welds, costs the same regardless of the container's dimensions, so we will consider just the portion matching the measurement of the side panels in our calculations. We will further assume that the angle of the top of the container should be preserved. Then to compute the variable costs, let x be the width, y the length, and z the height of the front of the container. The back of the container is 9 inches, or $\frac{3}{4}$ ft, taller than the front, so using similar triangles we can say the back panel has height $z + \frac{3}{14}x$. Measuring in feet, we want the volume to remain constant, so $V = \frac{1}{2}\left(z + z + \frac{3}{14}x\right)(x)(y) = xyz + \frac{3}{28}x^2y = 77.875$. To determine a function for the variable cost, we first find the area of each sheet of metal needed. The base has area xy ft^2. The front panel has visible area yz plus $\frac{1}{12}y$ for the portion folded onto the base and $\frac{5}{12}y$ for the steel at the top used to form the lip, so $\left(yz + \frac{1}{2}y\right)$ ft^2 in total. Similarly, the back sheet has area $y\left(z + \frac{3}{14}x\right) + \frac{1}{12}y + \frac{5}{12}y = yz + \frac{3}{14}xy + \frac{1}{2}y$. Each side has visible area $\frac{1}{2}\left[z + \left(z + \frac{3}{14}x\right)\right](x)$, and the sheet includes one-inch flaps folding onto the front and back panels, so with area $\frac{1}{12}z$ and $\frac{1}{12}\left(z + \frac{3}{14}x\right)$, and a one-inch flap to fold onto the base with area $\frac{1}{12}x$. The lift pocket is constructed of a piece of steel 20 inches by x ft (including the 2 extra inches used by the welds). The additional metal used to make the lip at the top of the panel has width 5 inches and length that we can determine using the Pythagorean Theorem: $x^2 + \left(\frac{3}{14}x\right)^2 = \text{length}^2$, so length $= \frac{\sqrt{205}}{14}x \approx 1.0227x$. Thus the area of steel needed for each side panel is approximately

$$\frac{1}{2}\left[z + \left(z + \tfrac{3}{14}x\right)\right](x) + \tfrac{1}{12}z + \tfrac{1}{12}\left(z + \tfrac{3}{14}x\right) + \tfrac{1}{12}x + \tfrac{5}{3}x + \tfrac{5}{12}(1.0227x) \approx xz + \tfrac{3}{28}x^2 + \tfrac{1}{6}z + 2.194x$$

We also have the following welds:

Weld	Length
Front, back welded to base	$2y$
Sides welded to base	$2x$
Sides welded to front	$2z$
Sides welded to back	$2\left(z + \frac{3}{14}x\right)$
Weld on front and back lip	$2y$
Two welds on each side lip	$4(1.0227x)$
Two welds for each lift pocket	$4x$

Thus the total length of welds needed is

$$2y + 2x + 2z + 2\left(z + \tfrac{3}{14}x\right) + 2y + 4\left(1.0227x\right) + 4x \approx 10.519x + 4y + 4z$$

Finally, the total variable cost is approximately

$$0.90\,(xy) + 0.70\left[\left(yz + \tfrac{1}{2}y\right) + \left(yz + \tfrac{3}{14}xy + \tfrac{1}{2}y\right) + 2\left(xz + \tfrac{3}{28}x^2 + \tfrac{1}{6}z + 2.194x\right)\right]$$
$$+0.18\left(10.519x + 4y + 4z\right)$$
$$\approx 1.05xy + 1.4yz + 1.42y + 1.4xz + 0.15x^2 + 0.953z + 4.965x$$

We would like to minimize this function while keeping volume constant, so since $xyz + \frac{3}{28}x^2y = 77.875$ we can substitute $z = \dfrac{77.875}{xy} - \frac{3}{28}x$ giving variable cost as a function of x and y:

$C\left(x,y\right) \approx 0.9xy + \dfrac{109.0}{x} + 1.42y + \dfrac{109.0}{y} + \dfrac{74.2}{xy} + 4.86x$. Using a CAS, we solve the system of equations $C_x\left(x,y\right) = 0$ and $C_y\left(x,y\right) = 0$; the only critical point within an appropriate domain is approximately $(3.58, 5.29)$. From the nature of the function C (or from a graph) we can determine that C has an absolute minimum at $(3.58, 5.29)$, and so the minimum cost is attained for $x \approx 3.58$ ft (or 43.0 in), $y \approx 5.29$ ft (or 63.5 in), and $z \approx \frac{77.875}{3.58(5.29)} - \frac{3}{28}\left(3.58\right) \approx 3.73$ ft (or 44.8 in).

3. The fixed cost aspects of the container which we did not include in our calculations, such as the wheels and lid, don't affect the validity of our results. Some of our other assumptions, however, may influence the accuracy of our findings. We simplified the price of the steel sheets to include cuts and bends, and we simplified the price of welding to include the labor and materials. This may not be accurate for areas of the container, such as the lip and lift pockets, that require several cuts, bends, and welds in a relatively small surface area. Consequently, increasing some dimensions of the container may not increase the cost in the same manner as our computations predict. If we do not assume that the angle of the sloped top of the container must be preserved, it is likely that we could farther improve our cost. Finally, our results show that the length of the container should be changed to minimize cost; this may not be possible if the two lift pockets must remain a fixed distance apart for handling by machinery.

4. The minimum variable cost using our values found in Problem 2 is $C\left(3.58, 5.29\right) \approx \$96.95$, while the current dimensions give an estimated variable cost of $C\left(3.5, 6.0\right) \approx \$97.30$. If we determine that our assumptions and simplifications are acceptable, our work shows that a slight savings can be gained by adjusting the dimensions of the container. However, the difference in cost is modest, and may not justify changes in the manufacturing process.

Discovery Project □ Quadratic Approximations and Critical Points

1. $Q\left(x,y\right) = f\left(a,b\right) + f_x\left(a,b\right)\left(x-a\right) + f_y\left(a,b\right)\left(y-b\right) + \frac{1}{2}f_{xx}\left(a,b\right)\left(x-a\right)^2$
$$+f_{xy}\left(a,b\right)\left(x-a\right)\left(y-b\right) + \tfrac{1}{2}f_{yy}\left(a,b\right)\left(y-b\right)^2,$$

so

$$Q_x\left(x,y\right) = f_x\left(a,b\right) + \tfrac{1}{2}f_{xx}\left(a,b\right)(2)\left(x-a\right) + f_{xy}\left(a,b\right)\left(y-b\right)$$
$$= f_x\left(a,b\right) + f_{xx}\left(a,b\right)\left(x-a\right) + f_{xy}\left(a,b\right)\left(y-b\right)$$

At (a,b) we have $Q_x(a,b) = f_x(a,b) + f_{xx}(a,b)(a-a) + f_{xy}(a,b)(b-b) = f_x(a,b)$.
Similarly, $Q_y(x,y) = f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b) \quad \Rightarrow$
$Q_y(a,b) = f_y(a,b) + f_{xy}(a,b)(a-a) + f_{yy}(a,b)(b-b) = f_y(a,b)$. For the second-order partial derivatives we have

$$\begin{aligned}
Q_{xx}(x,y) &= \frac{\partial}{\partial x}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xx}(a,b) \\
&\Rightarrow Q_{xx}(a,b) = f_{xx}(a,b) \\
Q_{xy}(x,y) &= \frac{\partial}{\partial y}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xy}(a,b) \\
&\Rightarrow Q_{xy}(a,b) = f_{xy}(a,b) \\
Q_{yy}(x,y) &= \frac{\partial}{\partial y}\left[f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b)\right] = f_{yy}(a,b) \\
&\Rightarrow Q_{yy}(a,b) = f_{yy}(a,b)
\end{aligned}$$

2. (a) First we find the partial derivatives and values that will be needed:

$$\begin{aligned}
f(x,y) &= e^{-x^2-y^2} & f(0,0) &= 1 \\
f_x(x,y) &= -2xe^{-x^2-y^2} & f_x(0,0) &= 0 \\
f_y(x,y) &= -2ye^{-x^2-y^2} & f_y(0,0) &= 0 \\
f_{xx}(x,y) &= \left(4x^2-2\right)e^{-x^2-y^2} & f_{xx}(0,0) &= -2 \\
f_{xy}(x,y) &= 4xye^{-x^2-y^2} & f_{xy}(0,0) &= 0 \\
f_{yy}(x,y) &= \left(4y^2-2\right)e^{-x^2-y^2} & f_{yy}(0,0) &= -2
\end{aligned}$$

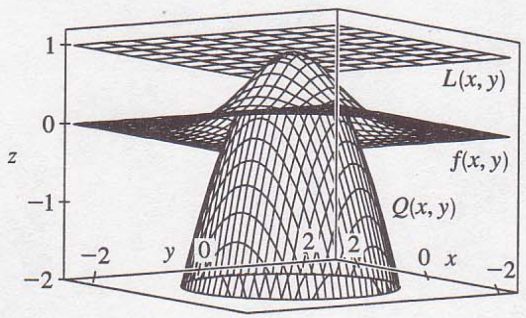
Then the first-degree Taylor polynomial of f at $(0,0)$ is

$$\begin{aligned}
L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + (0)(x-0) + (0)(y-0) \\
&= 1
\end{aligned}$$

The second-degree Taylor polynomial is given by

$$\begin{aligned}
Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2}f_{xx}(0,0)(x-0)^2 \\
&\qquad + f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2}f_{yy}(0,0)(y-0)^2 \\
&= 1 - x^2 - y^2
\end{aligned}$$

(b)



As we see from the graph, L approximates f well only for points (x,y) extremely close to the origin. Q is a much better approximation; the shape of its graph looks similar to that of the graph of f near the origin, and the values of Q appear to be good estimates for the values of f within a significant radius of the origin.

3. (a) First we find the partial derivatives and values that will be needed:

$$\begin{array}{llll} f(x,y) = xe^y & f(1,0) = 1 & f_{xx}(x,y) = 0 & f_{xx}(1,0) = 0 \\ f_x(x,y) = e^y & f_x(1,0) = 1 & f_{xy}(x,y) = e^y & f_{xy}(1,0) = 1 \\ f_y(x,y) = xe^y & f_y(1,0) = 1 & f_{yy}(x,y) = xe^y & f_{yy}(1,0) = 1 \end{array}$$

Then the first-degree Taylor polynomial of f at $(1,0)$ is

$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 1 + (1)(x-1) + (1)(y-0) \\ &= x + y \end{aligned}$$

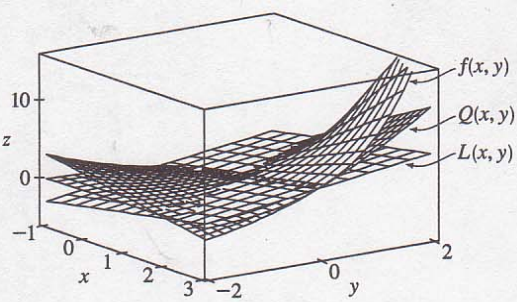
The second-degree Taylor polynomial is given by

$$\begin{aligned} Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \tfrac{1}{2} f_{xx}(1,0)(x-1)^2 \\ &\qquad + f_{xy}(1,0)(x-1)(y-0) + \tfrac{1}{2} f_{yy}(1,0)(y-0)^2 \\ &= \tfrac{1}{2}y^2 + x + xy \end{aligned}$$

(b)

$$\begin{aligned} L(0.9, 0.1) &= 0.9 + 0.1 = 1.0 \\ Q(0.9, 0.1) &= \tfrac{1}{2}(0.1)^2 + 0.9 + (0.9)(0.1) = 0.995 \\ f(0.9, 0.1) &= 0.9e^{0.1} \approx 0.9947 \end{aligned}$$

(c)



As we see from the graph, L and Q both approximate f reasonably well near the point $(1,0)$. As we venture farther from the point, the graph of Q follows the shape of the graph of f more closely than L.

4. (a) $f(x,y) = ax^2 + bxy + cy^2 = a\left[x^2 + \dfrac{b}{a}xy + \dfrac{c}{a}y^2\right]$

$$= a\left[x^2 + \frac{b}{a}xy + \left(\frac{b}{2a}y\right)^2 - \left(\frac{b}{2a}y\right)^2 + \frac{c}{a}y^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}y\right)^2 - \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2\right] = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

(b) For $D = 4ac - b^2$, from part (a) we have $f(x,y) = a\left[\left(x + \dfrac{b}{2a}y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right]$. If $D > 0$,

$\left(\dfrac{D}{4a^2}\right)y^2 \geq 0$ and $\left(x + \dfrac{b}{2a}y\right)^2 \geq 0$, so $\left[\left(x + \dfrac{b}{2a}y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \geq 0$. Here $a > 0$, thus

$f(x,y) = a\left[\left(x + \dfrac{b}{2a}y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \geq 0$. We know $f(0,0) = 0$, so $f(0,0) \leq f(x,y)$ for all (x,y), and by definition f has a local minimum at $(0,0)$.

(c) As in part (b), $\left[\left(x + \dfrac{b}{2a}y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \geq 0$, and since $a < 0$ we have

$f(x,y) = a\left[\left(x + \dfrac{b}{2a}y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \leq 0$. Since $f(0,0) = 0$, we must have $f(0,0) \geq f(x,y)$ for all (x,y), so by definition f has a local maximum at $(0,0)$.

(d) $f(x,y) = ax^2 + bxy + cy^2$, so $f_x(x,y) = 2ax + by \Rightarrow f_x(0,0) = 0$ and $f_y(x,y) = bx + 2cy \Rightarrow f_y(0,0) = 0$. Since $f(0,0) = 0$ and f and its partial derivatives are continuous, we know from Equation 15.4.2 [ET 14.4.2] that the tangent plane to the graph of f at $(0,0)$ is the plane $z = 0$. Then f has a saddle point at $(0,0)$ if the graph of f crosses the tangent plane at $(0,0)$, or equivalently, if some paths to the origin have positive function values while other paths have negative function values. Suppose we approach the origin along the x-axis; then we have $y = 0 \Rightarrow f(x,0) = ax^2$ which has the same sign as a. We must now find at least one path to the origin where $f(x,y)$ gives values with sign opposite that of a. Since

$f(x,y) = a\left[\left(x + \dfrac{b}{2a}y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right]$, if we approach the origin along the line $x = -\dfrac{b}{2a}y$, we have

$f\left(-\dfrac{b}{2a}y, y\right) = a\left[\left(-\dfrac{b}{2a}y + \dfrac{b}{2a}y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] = \dfrac{D}{4a}y^2$. Since $D < 0$, these values have signs opposite that of a. Thus, f has a saddle point at $(0,0)$.

5. (a) Since the partial derivatives of f exist at $(0,0)$ and $(0,0)$ is a critical point, we know $f_x(0,0) = 0$ and $f_y(0,0) = 0$. Then the second-degree Taylor polynomial of f at $(0,0)$ can be expressed as

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2}f_{xx}(0,0)(x-0)^2 \\ &\qquad + f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2}f_{yy}(0,0)(y-0)^2 \\ &= \tfrac{1}{2}f_{xx}(0,0)\,x^2 + f_{xy}(0,0)\,xy + \tfrac{1}{2}f_{yy}(0,0)\,y^2. \end{aligned}$$

(b) $Q(x,y) = \frac{1}{2}f_{xx}(0,0)\,x^2 + f_{xy}(0,0)\,xy + \frac{1}{2}f_{yy}(0,0)\,y^2$ fits the form of the polynomial function in Problem 4 with $a = \frac{1}{2}f_{xx}(0,0)$, $b = f_{xy}(0,0)$, and $c = \frac{1}{2}f_{yy}(0,0)$. Then we know Q is a paraboloid, and that Q has a local maximum, local minimum, or saddle point at $(0,0)$. Here, $D = 4ac - b^2 = 4\left(\frac{1}{2}\right)f_{xx}(0,0)\left(\frac{1}{2}\right)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = f_{xx}(0,0)\,f_{yy}(0,0) - [f_{xy}(0,0)]^2$, and if $D > 0$ with $a = \frac{1}{2}f_{xx}(0,0) > 0 \Rightarrow f_{xx}(0,0) > 0$, we know from Problem 4 that Q has a local minimum at $(0,0)$. Similarly, if $D > 0$ and $a < 0 \Rightarrow f_{xx}(0,0) < 0$, Q has a local maximum at $(0,0)$, and if $D < 0$, Q has a saddle point at $(0,0)$.

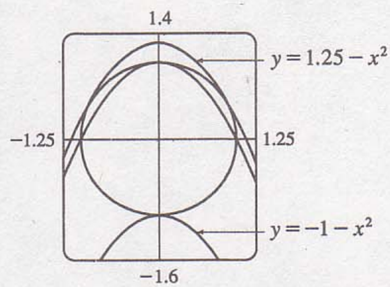
(c) Since $f(x,y) \approx Q(x,y)$ near $(0,0)$, part (b) suggests that for $D = f_{xx}(0,0)\,f_{yy}(0,0) - [f_{xy}(0,0)]^2$, if $D > 0$ and $f_{xx}(0,0) > 0$, f has a local minimum at $(0,0)$. If $D > 0$ and $f_{xx}(0,0) < 0$, f has a local maximum at $(0,0)$, and if $D < 0$, f has a saddle point at $(0,0)$. Together with the conditions given in part (a), this is precisely the Second Derivatives Test from Section 15.7 [ET 14.7].

15.8 Lagrange Multipliers

ET 14.8

1. At the extreme values of f, the level curves of f just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of c which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.

2. (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



(b) $\nabla f = \langle 2x, 1 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. So $2x = 2\lambda x \;\Rightarrow$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{5}{4}$ (the maximum value), $f(0, 1) = 1$, and $f(0, -1) = -1$ (the minimum value). These are our answers from (a).

3. $f(x, y) = x^2 - y^2$, $g(x, y) = x^2 + y^2 = 1 \;\Rightarrow\; \nabla f = \langle 2x, -2y \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2x = 2\lambda x$ implies $x = 0$ or $\lambda = 1$. If $x = 0$, then $x^2 + y^2 = 1$ implies $y = \pm 1$ and if $\lambda = 1$, then $-2y = 2\lambda y$ implies $y = 0$ and thus $x = \pm 1$. Thus the possible points for the extreme values of f are $(\pm 1, 0)$, $(0, \pm 1)$. But $f(\pm 1, 0) = 1$ while $f(0, \pm 1) = -1$ so the maximum value of f on $x^2 + y^2 = 1$ is $f(\pm 1, 0) = 1$ and the minimum value is $f(0, \pm 1) = -1$.

4. $f(x, y) = 4x + 6y$, $g(x, y) = x^2 + y^2 = 13 \;\Rightarrow\; \nabla f = \langle 4, 6 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \dfrac{2}{\lambda}$ and $y = \dfrac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\dfrac{2}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 \;\Rightarrow\; 13 = \dfrac{13}{\lambda^2} \;\Rightarrow\; \lambda = \pm 1$, so f has possible extreme values at the points $(2, 3)$, $(-2, -3)$. We compute $f(2, 3) = 26$ and $f(-2, -3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2, 3) = 26$ and the minimum value is $f(-2, -3) = -26$.

5. $f(x, y) = x^2 y$, $g(x, y) = x^2 + 2y^2 = 6 \;\Rightarrow\; \nabla f = \langle 2xy, x^2 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 4\lambda y \rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $\lambda = y$. If $x = 0$, then $x^2 = 4\lambda y$ implies $\lambda = 0$ or $y = 0$. However, if $y = 0$ then $g(x, y) = 0$, a contradiction. So $\lambda = 0$ and then $g(x, y) = 6 \;\Rightarrow\; y = \pm\sqrt{3}$. If $\lambda = y$, then $x^2 = 4\lambda y$ implies $x^2 = 4y^2$, and so $g(x, y) = 6 \;\Rightarrow\; 4y^2 + 2y^2 = 6 \;\Rightarrow\; y^2 = 1 \;\Rightarrow\; y = \pm 1$. Thus f has possible extreme values at the points $(0, \pm\sqrt{3})$, $(\pm 2, 1)$, and $(\pm 2, -1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2, 1) = 4$ and the minimum to be $f(\pm 2, -1) = -4$.

6. $f(x,y) = x^2 + y^2$, $g(x,y) = x^4 + y^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y\rangle$, $\lambda\nabla g = \langle 4\lambda x^3, 4\lambda y^3\rangle$. Then $x = 2\lambda x^3$ implies $x = 0$ or $\lambda = \dfrac{1}{2x^2}$. If $x = 0$, then $x^4 + y^4 = 1$ implies $y = \pm 1$. But $y = 2\lambda y^3$ implies $y = 0$ so $x = \pm 1$ or $\lambda = \dfrac{1}{2y^2}$ and $x^2 = y^2$ and $2x^4 = 1$ so $x = \pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0, \pm 1)$, $(\pm 1, 0)$, $\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4 + y^4 = 1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ and the minimum value being $f(0, \pm 1) = f(\pm 1, 0) = 1$.

7. $f(x,y,z) = 2x + 6y + 10z$, $g(x,y,z) = x^2 + y^2 + z^2 = 35 \Rightarrow \nabla f = \langle 2, 6, 10\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 2$, $2\lambda y = 6$, $2\lambda z = 10$ imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{3}{\lambda}$, and $z = \dfrac{5}{\lambda}$. But

$$35 = x^2 + y^2 + z^2 = \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 \Rightarrow 35 = \frac{35}{\lambda^2} \Rightarrow \lambda = \pm 1,$$ so f has possible extreme values at the points $(1, 3, 5)$, $(-1, -3, -5)$. The maximum value of f on $x^2 + y^2 + z^2 = 35$ is $f(1, 3, 5) = 70$, and the minimum is $f(-1, -3, -5) = -70$.

8. $f(x,y,z) = 8x - 4z$, $g(x,y,z) = x^2 + 10y^2 + z^2 = 5 \Rightarrow \nabla f = \langle 8, 0, -4\rangle$, $\lambda\nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \dfrac{4}{\lambda}$, $y = 0$, and $z = -\dfrac{2}{\lambda}$. But

$$5 = x^2 + 10y^2 + z^2 = \left(\frac{4}{\lambda}\right)^2 + 10(0)^2 + \left(-\frac{2}{\lambda}\right)^2 \Rightarrow 5 = \frac{20}{\lambda^2} \Rightarrow \lambda = \pm 2,$$ so f has possible extreme values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

9. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6 \Rightarrow \nabla f = \langle yz, xz, xy\rangle$, $\lambda\nabla g = \langle 2\lambda x, 4\lambda y, 6\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies $\lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z)$ or $x^2 = 2y^2$ and $z^2 = \frac{2}{3}y^2$. Thus $x^2 + 2y^2 + 3z^2 = 6$ implies $6y^2 = 6$ or $y = \pm 1$. Then the possible points are $\left(\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$. And the maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z) = x^2y^2z^2$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies (1) $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or (2) $\lambda = 0$ and one or two (but not three) of the coordinates are 0. If (1) then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

11. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^4 + y^4 + z^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$, then $\nabla f = \lambda\nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$.

Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1. Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

12. $f(x,y,z) = x^4 + y^4 + z^4$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 4x^3, 4y^3, 4z^3 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$.
Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ yielding 8 points each with an f-value of $\frac{1}{3}$.
Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.
Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1. Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

13. $f(x,y,z,t) = x + y + z + t$, $g(x,y,z,t) = x^2 + y^2 + z^2 + t^2 = 1 \quad \Rightarrow \quad \langle 1,1,1,1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t \rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ and the minimum value is $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$.

14. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \quad \Rightarrow$ $\langle 1, 1, \ldots, 1 \rangle = \langle 2\lambda x_1, 2\lambda x_2, \ldots, 2\lambda x_n \rangle$, so $\lambda = 1/(2x_1) = 1/(2x_2) = \cdots = 1/(2x_n)$ and $x_1 = x_2 = \cdots = x_n$. But $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, so $x_i = \pm 1/\sqrt{n}$ for $i = 1, \ldots, n$. Thus the maximum value of f is $f(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n}$ and the minimum value is $f(-1/\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}$.

15. $f(x,y,z) = x + 2y$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = y^2 + z^2 = 4 \quad \Rightarrow \quad \nabla f = \langle 1, 2, 0 \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$ and $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $1 = \lambda$, $2 = \lambda + 2\mu y$ and $0 = \lambda + 2\mu z$ so $\mu y = \frac{1}{2} = -\mu z$ or $y = 1/(2\mu)$, $z = -1/(2\mu)$. Thus $x + y + z = 1$ implies $x = 1$ and $y^2 + z^2 = 4$ implies $\mu = \pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1, \pm\sqrt{2}, \mp\sqrt{2}\right)$ and the maximum value is $f\left(1, \sqrt{2}, -\sqrt{2}\right) = 1 + 2\sqrt{2}$ and the minimum value is $f\left(1, -\sqrt{2}, \sqrt{2}\right) = 1 - 2\sqrt{2}$.

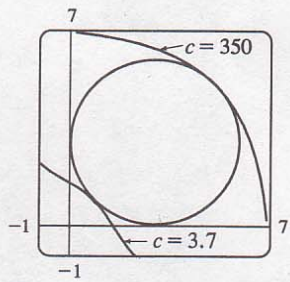
16. $f(x,y,z) = 3x - y - 3z$, $g(x,y,z) = x + y - z = 0$, $h(x,y,z) = x^2 + 2z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 3, -1, -3 \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, -\lambda \rangle$, $\mu \nabla h = (2\mu x, 0, 4\mu z)$. Then $3 = \lambda + 2\mu x$, $-1 = \lambda$ and $-3 = -\lambda + 4\mu z$, so $\lambda = -1$, $\mu z = -1$, $\mu x = 2$. Thus $h(x,y,z) = 1$ implies $\dfrac{4}{\mu^2} + 2\left(\dfrac{1}{\mu^2}\right) = 1$ or $\mu = \pm\sqrt{6}$, so $z = \mp\frac{1}{\sqrt{6}}$; $x = \pm\frac{2}{\sqrt{6}}$; and $g(x,y,z) = 0$ implies $y = \mp\frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is $f\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) = 2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) = -2\sqrt{6}$.

17. $f(x,y,z) = yz + xy$, $g(x,y,z) = xy = 1$, $h(x,y,z) = y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle y, x + z, y \rangle$, $\lambda \nabla g = \langle \lambda y, \lambda x, 0 \rangle$, $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $y = \lambda y$ implies $\lambda = 1$ [$y \neq 0$ since $g(x,y,z) = 1$], $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2z)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.
Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = yz + 1$ subject to $y^2 + z^2 = 1$.

18. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \Rightarrow x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f(-2, \pm 2\sqrt{3}) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1,0) = -7$.

19. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 8\lambda y \rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \Rightarrow x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

20. (a) The graphs of $f(x,y) = 3.7$ and $f(x,y) = 350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x,y)$ subject to the constraint $(x-3)^2 + (y-3)^2 = 9$.



(b) Let $g(x,y) = (x-3)^2 + (y-3)^2$. We calculate $f_x(x,y) = 3x^2 + 3y$, $f_y(x,y) = 3y^2 + 3x$, $g_x(x,y) = 2x - 6$, and $g_y(x,y) = 2y - 6$, and use a CAS to search for solutions to the equations

$g(x,y) = (x-3)^2 + (y-3)^2 = 9$, $f_x = \lambda g_x$, and $f_y = \lambda g_y$.

The solutions are $(x,y) = \left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) \approx (0.879, 0.879)$ and $(x,y) = \left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) \approx (5.121, 5.121)$. These give $f\left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} - \frac{243}{2}\sqrt{2} \approx 3.673$ and $f\left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} + \frac{243}{2}\sqrt{2} \approx 347.33$, in accordance with part (a).

21. $P(L,K) = bL^\alpha K^{1-\alpha}$, $g(L,K) = mL + nK = p \Rightarrow \nabla P = \langle \alpha b L^{\alpha-1} K^{1-\alpha}, (1-\alpha) b L^\alpha K^{-\alpha} \rangle$, $\lambda \nabla g = \langle \lambda m, \lambda n \rangle$. Then $\alpha b (K/L)^{1-\alpha} = \lambda m$ and $(1-\alpha) b (L/K)^\alpha = \lambda n$ and $mL + nK = p$, so $\alpha b (K/L)^{1-\alpha}/m = (1-\alpha) b (L/K)^\alpha / n$ or $n\alpha/[m(1-\alpha)] = (L/K)^\alpha (L/K)^{1-\alpha}$ or $L = Kn\alpha/[m(1-\alpha)]$. Substituting into $mL + nK = p$ gives $K = (1-\alpha)p/n$ and $L = \alpha p/m$ for the maximum production.

22. $C(L,K) = mL + nK$, $g(L,K) = bL^\alpha K^{1-\alpha} = Q \Rightarrow \nabla C = \langle m, n \rangle$,

$\lambda \nabla g = \langle \lambda \alpha b L^{\alpha-1} K^{1-\alpha}, \lambda(1-\alpha) b L^\alpha K^{-\alpha} \rangle$. Then $\dfrac{m}{\alpha b}\left(\dfrac{L}{K}\right)^{1-\alpha} = \dfrac{n}{(1-\alpha)b}\left(\dfrac{K}{L}\right)^\alpha$ and $bL^\alpha K^{1-\alpha} = Q$

$\Rightarrow \dfrac{n\alpha}{m(1-\alpha)} = \left(\dfrac{L}{K}\right)^{1-\alpha}\left(\dfrac{L}{K}\right)^\alpha \Rightarrow L = \dfrac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1-\alpha)}\right]^\alpha K^{1-\alpha} = Q$. Hence

$K = \dfrac{Q}{b(n\alpha/[m(1-\alpha)])^\alpha} = \dfrac{Qm^\alpha(1-\alpha)^\alpha}{bn^\alpha\alpha^\alpha}$ and $L = \dfrac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}} = \dfrac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$ minimizes cost.

23. Let the sides of the rectangle be x and y. Then $f(x,y) = xy$, $g(x,y) = 2x + 2y = p \Rightarrow \nabla f(x,y) = \langle y, x \rangle$, $\lambda \nabla g = \langle 2\lambda, 2\lambda \rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

24. Let $f(x,y,z) = s(s-x)(s-y)(s-z)$, $g(x,y,z) = x + y + z$. Then $\nabla f = \langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y) \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Thus (1) $(s-y)(s-z) = (s-x)(s-z)$ and (2) $(s-x)(s-z) = (s-x)(s-y)$. (1) implies $x = y$ while (2) implies $y = z$, so $x = y = z = p/3$ and the triangle with maximum area is equilateral.

25. $f(x,y,z) = (x-2)^2 + (y+2)^2 + (z-3)^2$, $g(x,y,z) = 6x + 4y - 3z = 2 \Rightarrow$ $\nabla f = \langle 2(x-2), 2(y+2), 2(z-3) \rangle = \lambda \nabla g = \langle 6\lambda, 4\lambda, -3\lambda \rangle$, so $x = 3\lambda + 2$, $y = 2\lambda - 2$, $z = -\frac{3}{2}\lambda + 3$ and $(18\lambda + 12) + (8\lambda - 8) + \frac{9}{2}\lambda - 9 = 2$ implies $\lambda = \frac{14}{61}$. Thus the shortest distance is $\sqrt{\left(\frac{42}{61}\right)^2 + \left(\frac{28}{61}\right)^2 + \left(-\frac{21}{61}\right)^2} = \frac{7}{\sqrt{61}}$.

26. $f(x,y,z) = (x+4)^2 + (y-1)^2 + (z-3)^2$, $g(x,y,z) = 2x - y + z = 1 \Rightarrow$ $\nabla f = \langle 2(x+4), 2(y-1), 2(z-3) \rangle = \lambda \nabla g = \langle 2\lambda, -\lambda, \lambda \rangle$, so $x = \lambda - 4$, $y = 1 - \frac{1}{2}\lambda$, $z = 3 + \frac{1}{2}\lambda$ and $2(\lambda - 4) - \left(1 - \frac{1}{2}\lambda\right) + \left(3 + \frac{1}{2}\lambda\right) = 1$ implies $\lambda = \frac{7}{3}$. Thus the point is $\left(-\frac{5}{3}, -\frac{1}{6}, \frac{25}{6}\right)$.

27. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = z^2 - xy - 1 = 0 \Rightarrow$ $\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle -\lambda y, -\lambda x, 2\lambda z \rangle$. Then $2z = 2\lambda z$ implies $z = 0$ or $\lambda = 1$. If $z = 0$ then $g(x,y,z) = 1$ implies $xy = -1$ or $x = -1/y$. Thus $2x = -\lambda y$ and $2y = -\lambda x$ imply $\lambda = 2/y^2 = 2y^2$ or $y = \pm 1$, $x = \pm 1$. If $\lambda = 1$, then $2x = -y$ and $2y = -x$ imply $x = y = 0$, so $z = \pm 1$. Hence the possible points are $(\pm 1, \mp 1, 0)$, $(0, 0, \pm 1)$ and the minimum value of f is $f(0,0,\pm 1) = 1$, so the points closest to the origin are $(0, 0, \pm 1)$.

28. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^2y^2z = 1 \Rightarrow$ $\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \left\langle 2\lambda xy^2z, 2\lambda x^2yz, \lambda x^2y^2 \right\rangle$. Then $\lambda y^2 z = 1$, $\lambda x^2 z = 1$ and $\lambda x^2 y^2 = 2z$ so $y^2 z = x^2 z$ and $x = \pm y$. Also $2z/1 = \lambda x^2 y^2 / \left(\lambda x^2 z\right)$ so $2z^2 = y^2$ and $y = \pm\sqrt{2}z$. But $x^2y^2z = 1$ implies $z > 0$ and $4z^5 = 1$. Thus the points are $\left(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5}\right)$, and the minimum distance is attained at each of these.

29. $f(x,y,z) = xyz$, $g(x,y,z) = x + y + z = 100 \Rightarrow \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Then $\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

30. $f(x,y,z) = x^a y^b z^c$, $g(x,y,z) = x + y + z = 100$ $\Rightarrow \nabla f = \left\langle ax^{a-1}y^bz^c, bx^ay^{b-1}z^c, cx^ay^bz^{c-1} \right\rangle = \lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Then $\lambda = ax^{a-1}y^bz^c = bx^ay^{b-1}z^c = cx^ay^bz^{c-1}$ or $ayz = bxz = cxy$. Thus $x = \dfrac{ay}{b}$, $z = \dfrac{cy}{b}$, and $\dfrac{ay}{b} + y + \dfrac{cy}{b} = 100$ implies that $y = \dfrac{100b}{a+b+c}$, $x = \dfrac{100a}{a+b+c}$ and $z = \dfrac{100c}{a+b+c}$ gives the maximum.

31. If the dimensions are $2x$, $2y$ and $2z$, then $f(x,y,z) = 8xyz$ and $g(x,y,z) = 9x^2 + 36y^2 + 4z^2 = 36 \Rightarrow$ $\nabla f = \langle 8yz, 8xz, 8xy \rangle = \lambda \nabla g = \langle 18\lambda x, 72\lambda y, 8\lambda z \rangle$. Thus $18\lambda x = 8yz$, $72\lambda y = 8xz$, $8\lambda z = 8xy$ so $x^2 = 4y^2$, $z^2 = 9y^2$ and $36y^2 + 36y^2 + 36y^2 = 36$ or $y = \frac{1}{\sqrt{3}}$ $(y > 0)$. Thus the volume of the largest such rectangle is $8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}}\right) = 16\sqrt{3}$.

32. $f(x,y,z) = 8xyz$, $g(x,y,z) = a^2b^2c^2 \Rightarrow \nabla f = \langle 8yz, 8xz, 8xy \rangle = \lambda \nabla g = \langle 2\lambda b^2c^2x, 2\lambda a^2c^2y, 2\lambda a^2b^2z \rangle$.
Then $4yz = \lambda b^2c^2x$, $4xz = \lambda a^2c^2y$, $4xy = \lambda a^2b^2z$ imply $\lambda = \dfrac{4yz}{b^2c^2x} = \dfrac{4xz}{a^2c^2y} = \dfrac{4xy}{a^2b^2z}$ or $\dfrac{y}{b^2x} = \dfrac{x}{a^2y}$ and $\dfrac{z}{c^2y} = \dfrac{y}{b^2z}$. Thus $x = \dfrac{ay}{b}$, $z = \dfrac{cy}{b}$, and $a^2c^2y^2 + c^2a^2y^2 + a^2c^2y^2 = a^2b^2c^2$, or $y = \dfrac{b}{\sqrt{3}}$, $x = \dfrac{a}{\sqrt{3}}$, $z = \dfrac{c}{\sqrt{3}}$ and the volume is $\frac{8}{3\sqrt{3}}abc$.

33. $f(x,y,z) = xyz$, $g(x,y,z) = x + 2y + 3z = 6 \Rightarrow \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, 2\lambda, 3\lambda \rangle$.
Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$ implies $x = 2y$, $z = \frac{2}{3}y$. But $2y + 2y + 2y = 6$ so $y = 1$, $x = 2$, $z = \frac{2}{3}$ and the volume is $V = \frac{4}{3}$.

34. $f(x,y,z) = xyz$, $g(x,y,z) = xy + yz + xz = 32 \Rightarrow$
$\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle$. Then (1) $\lambda(y+z) = yz$, (2) $\lambda(x+z) = xz$ and (3) $\lambda(x+y) = xy$. And (1) minus (2) implies $\lambda(y-x) = z(y-x)$ so $x = y$ or $\lambda = z$. If $\lambda = z$, then (1) implies $z(y+z) = yz$ or $z = 0$ which is false. Thus $x = y$. Similarly (2) minus (3) implies $\lambda(z-y) = x(z-y)$ so $y = z$ or $\lambda = x$. As above, $\lambda \neq x$, so $x = y = z$ and $3x^2 = 32$ or $x = y = z = \frac{8}{\sqrt{6}}$ cm.

35. $f(x,y,z) = xyz$, $g(x,y,z) = 4(x+y+z) = c \Rightarrow \nabla f = \langle yz, xz, xy \rangle$, $\lambda \nabla g = \langle 4\lambda, 4\lambda, 4\lambda \rangle$. Thus $4\lambda = yz = xz = xy$ or $x = y = z = \frac{1}{12}c$ are the dimensions giving the maximum volume.

36. $C(x,y,z) = 5xy + 2xz + 2yz$, $g(x,y,z) = xyz = V \Rightarrow$
$\nabla C = \langle 5y + 2z, 5x + 2z, 2x + 2y \rangle = \lambda \nabla g = \langle \lambda yz, \lambda xz, \lambda xy \rangle$. Then (1) $\lambda yz = 5y + 2z$, (2) $\lambda xz = 5x + 2z$, (3) $\lambda xy = 2(x+y)$ and (4) $xyz = V$. Now (1)–(2) implies $\lambda z(y-x) = 5(y-x)$, so $x = y$ or $\lambda = 5/z$, but z can't be 0, so $x = y$. Then twice (2) minus five times (3) together with $x = y$ implies $\lambda y(2x - 5y) = 2(2z - 5y)$ which gives $z = \frac{5}{2}y$ [again $\lambda \neq 2/y$ or else (3) implies $y = 0$]. Hence $\frac{5}{2}y^3 = V$ and the dimensions which minimize cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

37. $f(x,y,z) = xy + 2xz + 2yz$, $g(x,y,z) = xyz = 32{,}000 \text{ cm}^3 \Rightarrow$
$\nabla f = \langle 2z + y, 2z + x, 2(x+y) \rangle = \lambda \nabla g = \langle \lambda yz, \lambda xz, \lambda xy \rangle$. Then (1) $\lambda yz = 2z + y$, (2) $\lambda xz = 2z + x$, and (3) $\lambda xy = 2(x+y)$. Now (1)–(2) implies $\lambda z(y-x) = y - x$, so $x = y$ or $\lambda = 1/z$. If $\lambda = 1/z$ then (1) implies $z = 0$ which can't be, so $x = y$. But twice (2) minus (3) together with $x = y$ implies $\lambda y(2x - y) = (4z + 2y) - 4y$ or $\lambda y(2z - y) = 2(2z - y)$ so $z = y/2$ or $\lambda = 2/y$. If $\lambda = 2/y$ then (3) implies $y = 0$ which can't be. Thus $x = y = 2z$ and $\frac{1}{2}y^3 = 32{,}000$ or $y = 40$ and the dimensions which minimize the volume are $x = y = 40$ cm, $z = 20$ cm.

38. Let the dimensions of the box be x, y, and z, so its volume is $f(x,y,z) = xyz$, its surface area is $g(x,y,z) = xy + yz + xz = 750$ and its total edge length is $h(x,y,z) = x + y + z = 50$. Then
$\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle + \langle \mu, \mu, \mu \rangle$. So (1) $yz = \lambda(y+z) + \mu$, (2) $xz = \lambda(x+z) + \mu$, and (3) $xy = \lambda(x+y) + \mu$. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then (1) minus (2) implies $z(y-x) = \lambda(y-x)$ or $\lambda = z$, and (1) minus (3) implies $y(z-x) = \lambda(z-x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence $50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points

$\left(\frac{1}{3}\left(50 \mp 10\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right)\right)$. Thus the minimum of f is
$f\left(\frac{1}{3}\left(50 - 10\sqrt{3}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 - 2500\sqrt{10}\right)$, and its maximum is
$f\left(\frac{1}{3}\left(50 + 10\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 + 2500\sqrt{10}\right)$.
Note: If either y or z is the distinct side, then symmetry gives the same result.

39. We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda\rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu\rangle$. Thus we need (1) $2x = \lambda + 2\mu x$, (2) $2y = \lambda + 2\mu y$, (3) $2z = 2\lambda - \mu$, (4) $x + y + 2z = 2$, and (5) $x^2 + y^2 - z = 0$. From (1) and (2), $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in (3) gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into (1) says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then (4) and (5) become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then (4) and (5) become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

40. To find the highest and lowest points on the ellipse of intersection, we can find the maximum and minimum values of $f(x, y, z) = z$ subject to the constraints $g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$. Now $\nabla f = \langle 0, 0, 1\rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda\rangle$, $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu\rangle$, and $\nabla f = \lambda \nabla g + \mu \nabla h$ implies (1) $0 = \lambda + 2\mu x$, (2) $0 = \lambda + 2\mu y$, and (3) $1 = 2\lambda - \mu$. From (1) and (2) we have $2\mu x = 2\mu y$, so $x = y$. [$\mu \neq 0$, otherwise (3) gives $\lambda = \frac{1}{2}$ which contradicts (1).] The constraints become (4) $2x + 2z = 2$ and (5) $2x^2 - z = 0$. Substituting $z = 2x^2$ from (5) into (4) gives $2x + 4x^2 = 2 \quad \Rightarrow \quad (2x - 1)(x + 1) = 0 \quad \Rightarrow \quad x = \frac{1}{2}, -1$. Thus f has possible extreme values at the points $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$. The highest point on the ellipse is $f(-1, -1, 2) = 2$, and the lowest point is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$.

41. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to $g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$$\nabla f = \left\langle \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n), \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n), \ldots, \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$$

and $\lambda \nabla g = \langle \lambda, \lambda, \ldots, \lambda\rangle$, so we need to solve the system of equations

$$\frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n) = \lambda \quad \Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_1$$
$$\frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n) = \lambda \quad \Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_2$$
$$\vdots$$
$$\frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) = \lambda \quad \Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_n$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus

$x_1 = x_2 = \cdots = x_n$. But $x_1 + x_2 + \cdots + x_n = c \quad \Rightarrow \quad nx_1 = c \quad \Rightarrow \quad x_1 = \dfrac{c}{n} = x_2 = x_3 = \cdots = x_n$.

Then the only point where f can have an extreme value is $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is $f\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right) = \sqrt[n]{\dfrac{c}{n} \cdot \dfrac{c}{n} \cdot \cdots \cdot \dfrac{c}{n}} = \dfrac{c}{n}$.

(b) From part (a), $\dfrac{c}{n}$ is the maximum value of f. Thus $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \le \dfrac{c}{n}$. But $x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \le \dfrac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\dfrac{c}{n}$, but this can occur only at the point $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \dfrac{c}{n}$.

42. (a) Let $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum\limits_{i=1}^{n} x_i y_i$, $g(x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} x_i^2$, and $h(x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} y_i^2$. Then $\nabla f = \nabla \sum\limits_{i=1}^{n} x_i y_i = \langle y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n\rangle$, $\nabla g = \nabla \sum\limits_{i=1}^{n} x_i^2 = \langle 2x_1, 2x_2, \ldots, 2x_n, 0, 0, \ldots, 0\rangle$ and $\nabla h = \nabla \sum\limits_{i=1}^{n} y_i^2 = \langle 0, 0, \ldots, 0, 2y_1, 2y_2, \ldots, 2y_n\rangle$. So $\nabla f = \lambda \nabla g + \mu \nabla h \quad \Leftrightarrow \quad y_i = 2\lambda x_i$ and $x_i = 2\mu y_i$, $1 \le i \le n$. Then $1 = \sum\limits_{i=1}^{n} y_i^2 = \sum\limits_{i=1}^{n} 4\lambda^2 x_i^2 = 4\lambda^2 \sum\limits_{i=1}^{n} x_i^2 = 4\lambda^2 \quad \Rightarrow \quad \lambda = \pm\frac{1}{2}$.

If $\lambda = \frac{1}{2}$ then $y_i = 2\left(\frac{1}{2}\right) x_i = x_i$, $1 \le i \le n$. Thus $\sum\limits_{i=1}^{n} x_i y_i = \sum\limits_{i=1}^{n} x_i^2 = 1$. Similarly if $\lambda = -\frac{1}{2}$ we get $y_i = -x_i$ and $\sum\limits_{i=1}^{n} x_i y_i = -1$. Similarly we get $\mu = \pm\frac{1}{2}$ giving $y_i = \pm x_i$, $1 \le i \le n$, and $\sum\limits_{i=1}^{n} x_i y_i = \pm 1$. Thus the maximum value of $\sum\limits_{i=1}^{n} x_i y_i$ is 1.

(b) Here we assume $\sum\limits_{i=1}^{n} a_i^2 \ne 0$ and $\sum\limits_{i=1}^{n} b_i^2 \ne 0$. (If $\sum\limits_{i=1}^{n} a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially true.) $x_i = \dfrac{a_i}{\sqrt{\sum a_i^2}} \quad \Rightarrow \quad \sum x_i^2 = \dfrac{\sum a_i^2}{\sum a_i^2} = 1$, and $y_i = \dfrac{b_i}{\sqrt{\sum b_i^2}} \quad \Rightarrow \quad \sum y_i^2 = \dfrac{\sum b_i^2}{\sum b_i^2} = 1$. Therefore, from (a), $\sum x_i y_i = \sum \dfrac{a_i b_i}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}} \le 1 \quad \Leftrightarrow \quad \sum a_i b_i \le \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$.

Applied Project □ Rocket Science

1. Initially the rocket engine has mass $M_r = M_1$ and payload mass $P = M_2 + M_3 + A$. Then the change in velocity resulting from the first stage is $\Delta V_1 = -c \ln\left(1 - \dfrac{(1-S)\,M_1}{M_2 + M_3 + A + M_1}\right)$. After the first stage is jettisoned we can consider the rocket engine to have mass $M_r = M_2$ and the payload to have mass $P = M_3 + A$. The resulting change in velocity from the second stage is $\Delta V_2 = -c \ln\left(1 - \dfrac{(1-S)\,M_2}{M_3 + A + M_2}\right)$. When only the third stage remains, we have $M_r = M_3$ and $P = A$, so the resulting change in velocity is $\Delta V_3 = -c \ln\left(1 - \dfrac{(1-S)\,M_3}{A + M_3}\right)$.

Since the rocket started from rest, the final velocity attained is

$$
\begin{aligned}
v_f &= \Delta V_1 + \Delta V_2 + \Delta V_3 \\
&= -c \ln\left(1 - \frac{(1-S)\,M_1}{M_2 + M_3 + A + M_1}\right) + (-c)\ln\left(1 - \frac{(1-S)\,M_2}{M_3 + A + M_2}\right) \\
&\qquad + (-c)\ln\left(1 - \frac{(1-S)\,M_3}{A + M_3}\right) \\
&= -c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A - (1-S)\,M_1}{M_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A - (1-S)\,M_2}{M_2 + M_3 + A}\right)\right. \\
&\qquad \left. + \ln\left(\frac{M_3 + A - (1-S)\,M_3}{M_3 + A}\right)\right] \\
&= c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]
\end{aligned}
$$

2. Define $N_1 = \dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}$, $N_2 = \dfrac{M_2 + M_3 + A}{SM_2 + M_3 + A}$, and $N_3 = \dfrac{M_3 + A}{SM_3 + A}$. Then

$$
\begin{aligned}
\frac{(1-S)\,N_1}{1 - SN_1} &= \frac{(1-S)\,\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}}{1 - S\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}} \\
&= \frac{(1-S)\,(M_1 + M_2 + M_3 + A)}{SM_1 + M_2 + M_3 + A - S\,(M_1 + M_2 + M_3 + A)} \\
&= \frac{(1-S)\,(M_1 + M_2 + M_3 + A)}{(1-S)\,(M_2 + M_3 + A)} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A}
\end{aligned}
$$

as desired.

Similarly,

$$
\frac{(1-S)\,N_2}{1 - SN_2} = \frac{(1-S)\,(M_2 + M_3 + A)}{SM_2 + M_3 + A - S\,(M_2 + M_3 + A)} = \frac{(1-S)\,(M_2 + M_3 + A)}{(1-S)\,(M_3 + A)} = \frac{M_2 + M_3 + A}{M_3 + A}
$$

and

$$
\frac{(1-S)\,N_3}{1 - SN_3} = \frac{(1-S)\,(M_3 + A)}{SM_3 + A - S\,(M_3 + A)} = \frac{(1-S)\,(M_3 + A)}{(1-S)\,(A)} = \frac{M_3 + A}{A}
$$

Then

$$
\begin{aligned}
\frac{M + A}{A} &= \frac{M_1 + M_2 + M_3 + A}{A} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} \cdot \frac{M_2 + M_3 + A}{M_3 + A} \cdot \frac{M_3 + A}{A} \\
&= \frac{(1-S)\,N_1}{1 - SN_1} \cdot \frac{(1-S)\,N_2}{1 - SN_2} \cdot \frac{(1-S)\,N_3}{1 - SN_3} = \frac{(1-S)^3\,N_1 N_2 N_3}{(1 - SN_1)\,(1 - SN_2)\,(1 - SN_3)}
\end{aligned}
$$

3. Since $A > 0$, $M + A$ and consequently $\dfrac{M+A}{A}$ is minimized for the same values as M. $\ln x$ is a strictly increasing function, so $\ln\left(\dfrac{M+A}{A}\right)$ must give a minimum for the same values as $\dfrac{M+A}{A}$ and hence M. We then wish to minimize $\ln\left(\dfrac{M+A}{A}\right)$ subject to the constraint $c\,(\ln N_1 + \ln N_2 + \ln N_3) = v_f$. From Problem 2,

$$\ln\left(\frac{M+A}{A}\right) = \ln\left(\frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}\right)$$
$$= 3\ln(1-S) + \ln N_1 + \ln N_2 + \ln N_3 - \ln(1-SN_1) - \ln(1-SN_2) - \ln(1-SN_3)$$

Using the method of Lagrange multipliers, we need to solve
$\nabla\left[\ln\left(\dfrac{M+A}{A}\right)\right] = \lambda\nabla\left[c\,(\ln N_1 + \ln N_2 + \ln N_3)\right]$ with $c\,(\ln N_1 + \ln N_2 + \ln N_3) = v_f$ in terms of N_1, N_2, and N_3. The resulting system is

$$\frac{1}{N_1} + \frac{S}{1-SN_1} = \lambda\frac{c}{N_1} \qquad \frac{1}{N_2} + \frac{S}{1-SN_2} = \lambda\frac{c}{N_2} \qquad \frac{1}{N_3} + \frac{S}{1-SN_3} = \lambda\frac{c}{N_3}$$
$$c\,(\ln N_1 + \ln N_2 + \ln N_3) = v_f$$

One approach to solving the system is isolating $c\lambda$ in the first three equations which gives
$1 + \dfrac{SN_1}{1-SN_1} = c\lambda = 1 + \dfrac{SN_2}{1-SN_2} = 1 + \dfrac{SN_3}{1-SN_3} \;\Rightarrow\; \dfrac{N_1}{1-SN_1} = \dfrac{N_2}{1-SN_2} = \dfrac{N_3}{1-SN_3} \;\Rightarrow$
$N_1 = N_2 = N_3$ (verify!). This says the fourth equation can be expressed as $c\,(\ln N_1 + \ln N_1 + \ln N_1) = v_f \;\Rightarrow$
$3c\ln N_1 = v_f \;\Rightarrow\; \ln N_1 = \dfrac{v_f}{3c}$. Thus the minimum mass M of the rocket engine is attained for
$N_1 = N_2 = N_3 = e^{v_f/(3c)}$.

4. Using the previous results,

$\dfrac{M+A}{A} = \dfrac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)} = \dfrac{(1-S)^3\left[e^{v_f/(3c)}\right]^3}{\left[1-Se^{v_f/(3c)}\right]^3} = \dfrac{(1-S)^3 e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3}$. Then
$M = \dfrac{A\,(1-S)^3 e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3} - A$.

5. (a) From Problem 4, $M = \dfrac{A\,(1-0.2)^3 e^{(17{,}500/6000)}}{\left(1-0.2e^{[17{,}500/(3\cdot 6000)]}\right)^3} - A \approx 90.4A - A = 89.4A$.

(b) First, $N_3 = \dfrac{M_3 + A}{SM_3 + A} \;\Rightarrow\; e^{[17{,}500/(3\cdot 6000)]} = \dfrac{M_3 + A}{0.2M_3 + A} \;\Rightarrow\; M_3 = \dfrac{A\left(1 - e^{35/36}\right)}{0.2e^{35/36} - 1} \approx 3.49A$. Then

$N_2 = \dfrac{M_2 + M_3 + A}{SM_2 + M_3 + A} = \dfrac{M_2 + 3.49A + A}{0.2M_2 + 3.49A + A} \;\Rightarrow\; M_2 = \dfrac{4.49A\left(1 - e^{35/36}\right)}{0.2e^{35/36} - 1} \approx 15.67A$
and $N_3 = \dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} = \dfrac{M_1 + 15.67A + 3.49A + A}{0.2M_1 + 15.67A + 3.49A + A} \;\Rightarrow$
$M_1 = \dfrac{20.16A\left(1 - e^{35/36}\right)}{0.2e^{35/36} - 1} \approx 70.36A$.

6. As in Problem 5, $N_3 = \dfrac{M_3 + A}{SM_3 + A} \;\Rightarrow\; e^{24{,}700/(3\cdot 6000)} = \dfrac{M_3 + A}{0.2M_3 + A} \;\Rightarrow$

$M_3 = \dfrac{A\left(1 - e^{247/180}\right)}{0.2e^{247/180} - 1} \approx 13.9A$, $N_2 = \dfrac{M_2 + M_3 + A}{SM_2 + M_3 + A} = \dfrac{M_2 + 13.9A + A}{0.2M_2 + 13.9A + A} \;\Rightarrow$

$M_2 = \dfrac{14.9A\left(1 - e^{247/180}\right)}{0.2e^{247/180} - 1} \approx 208A$, and $N_3 = \dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} = \dfrac{M_1 + 208A + 13.9A + A}{0.2M_1 + 208A + 13.9A + A}$

$\Rightarrow \quad M_1 = \dfrac{222.9A\left(1 - e^{247/180}\right)}{0.2e^{247/180} - 1} \approx 3110A$. Here $A = 500$, so the mass of each stage of the rocket engine is approximately $M_1 = 3110\,(500) = 1{,}550{,}000$ lb, $M_2 = 208\,(500) = 104{,}000$ lb, and $M_3 = 13.9\,(500) = 6950$ lb.

Applied Project □ Hydro-Turbine Optimization

1. We wish to maximize the total energy production for a given total flow, so we can say Q_T is fixed and we want to maximize $KW_1 + KW_2 + KW_3$. Notice each KW_i has a constant factor $\left(170 - 1.6 \cdot 10^{-6} Q_T^2\right)$, so to simplify the computations we can equivalently maximize

$$\begin{aligned}
f\left(Q_1, Q_2, Q_3\right) &= \frac{KW_1 + KW_2 + KW_3}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\
&= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) \\
&\qquad + \left(-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2\right) \\
&\qquad\qquad + \left(-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5} Q_3^2\right)
\end{aligned}$$

subject to the constraint $g\left(Q_1, Q_2, Q_3\right) = Q_1 + Q_2 + Q_3 = Q_T$. So first we find the values of Q_1, Q_2, Q_3 where $\nabla f\left(Q_1, Q_2, Q_3\right) = \lambda \nabla g\left(Q_1, Q_2, Q_3\right)$ and $Q_1 + Q_2 + Q_3 = Q_T$ which is equivalent to solving the system

$$\begin{aligned}
0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 &= \lambda \\
0.1358 - 2\left(4.69 \cdot 10^{-5}\right) Q_2 &= \lambda \\
0.1380 - 2\left(3.84 \cdot 10^{-5}\right) Q_3 &= \lambda \\
Q_1 + Q_2 + Q_3 &= Q_T
\end{aligned}$$

Comparing the first and third equations, we have $0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 = 0.1380 - 2\left(3.84 \cdot 10^{-5}\right) Q_3$ $\Rightarrow \quad Q_1 = -126.2255 + 0.9412Q_3$. From the second and third equations, $0.1358 - 2\left(4.69 \cdot 10^{-5}\right) Q_2 = 0.1380 - 2\left(3.84 \cdot 10^{-5}\right) Q_3 \;\Rightarrow\; Q_2 = -23.4542 + 0.8188Q_3$. Substituting into $Q_1 + Q_2 + Q_3 = Q_T$ gives $\left(-126.2255 + 0.9412Q_3\right) + \left(-23.4542 + 0.8188Q_3\right) + Q_3 = Q_T \;\Rightarrow$ $2.76Q_3 = Q_T + 149.6797 \;\Rightarrow\; Q_3 = 0.3623Q_T + 54.23$. Then $Q_1 = -126.2255 + 0.9412Q_3 = -126.2255 + 0.9412\left(0.3623Q_T + 54.23\right) = 0.3410Q_T - 75.18$ and $Q_2 = -23.4542 + 0.8188\left(0.3623Q_T + 54.23\right) = 0.2967Q_T + 20.95$. As long as we maintain $250 \le Q_1 \le 1110$, $250 \le Q_2 \le 1110$, and $250 \le Q_3 \le 1225$, we can reason from the nature of the functions KW_i that these values give a maximum of f, and hence a maximum energy production, and not a minimum.

2. From Problem 1, the value of Q_1 that maximizes energy production is $0.3410Q_T - 75.18$, but since $250 \le Q_1 \le 1110$, we must have $250 \le 0.3410Q_T - 75.18 \le 1110 \Rightarrow 325.18 \le 0.3410Q_T \le 1185.18 \Rightarrow 953.6 \le Q_T \le 3475.6$. Similarly, $250 \le Q_2 \le 1110 \Rightarrow 250 \le 0.2967Q_T + 20.95 \le 1110 \Rightarrow 772.0 \le Q_T \le 3670.5$, and $250 \le Q_3 \le 1225 \Rightarrow 250 \le 0.3623Q_T + 54.23 \le 1225 \Rightarrow 540.4 \le Q_T \le 3231.5$. Consolidating these results, we see that the values from Problem 1 are applicable only for $953.6 \le Q_T \le 3231.5$.

3. If $Q_T = 2500$, the results from Problem 1 show that the maximum energy production occurs for

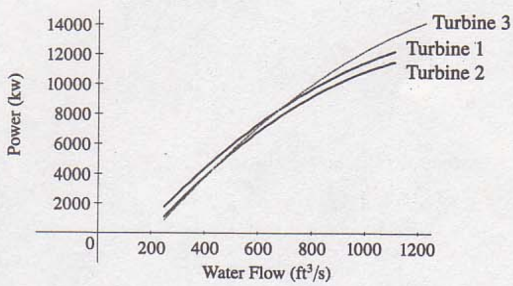
$$Q_1 = 0.3410Q_T - 75.18 = 0.3410\,(2500) - 75.18 = 777.3$$

$$Q_2 = 0.2967Q_T + 20.95 = 0.2967\,(2500) + 20.95 = 762.7$$

$$Q_3 = 0.3623Q_T + 54.23 = 0.3623\,(2500) + 54.23 = 960.0$$

The energy produced for these values is $KW_1 + KW_2 + KW_3 \approx 8915.2 + 8285.1 + 11{,}211.3 \approx 28{,}411.6$. We compute the energy production for a nearby distribution, $Q_1 = 770$, $Q_2 = 760$, and $Q_3 = 970$: $KW_1 + KW_2 + KW_3 \approx 8839.8 + 8257.4 + 11{,}313.5 = 28{,}410.7$. For another example, we take $Q_1 = 780$, $Q_2 = 765$, and $Q_3 = 955$: $KW_1 + KW_2 + KW_3 \approx 8942.9 + 8308.8 + 11{,}159.7 = 28{,}411.4$. These distributions are both close to the distribution from Problem 1 and both give slightly lower energy productions, suggesting that $Q_1 = 777.3$, $Q_2 = 762.7$, and $Q_3 = 960.0$ is indeed the optimal distribution.

4. First we graph each power function in its domain if all of the flow is directed to that turbine (so $Q_i = Q_T$).



If we use only one turbine, the graph indicates that for a water flow of 1000 ft^3/s, Turbine 3 produces the most power, approximately 12,200 kW. In comparison, if we use all three turbines, the results of Problem 1 with $Q_T = 1000$ give $Q_1 = 265.8$, $Q_2 = 317.7$, and $Q_3 = 416.5$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx 8397.4$ kW. Here, using only one turbine produces significantly more energy! If the flow is only 600 ft^3/s, we do not have the option of using all three turbines, as the domain restrictions require a minimum of 250 ft^3/s in each turbine. We can use just one turbine, then, and from the graph Turbine 1 produces the most energy for a water flow of 600 ft^3.

5. If we examine the graph from Problem 4, we see that for water flows above approximately 450 ft^3/s, Turbine 2 produces the least amount of power. Therefore it seems reasonable to assume that we should distribute the incoming flow of 1500 ft^3/s between Turbines 1 and 3. (This can be verified by computing the power produced with the other pairs of turbines for comparison.) So now we wish to maximize $KW_1 + KW_3$ subject to the constraint $Q_1 + Q_3 = Q_T$ where $Q_T = 1500$.

As in Problem 1, we can equivalently maximize

$$f(Q_1, Q_3) = \frac{KW_1 + KW_3}{170 - 1.6 \cdot 10^{-6} Q_T^2}$$

$$= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5} Q_3^2\right)$$

subject to the constraint $g(Q_1, Q_3) = Q_1 + Q_3 = Q_T$.

Then we solve $\nabla f(Q_1, Q_3) = \lambda \nabla g(Q_1, Q_3) \quad \Rightarrow \quad 0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 = \lambda$ and $0.1380 - 2\left(3.84 \cdot 10^{-5}\right) Q_3 = \lambda$, thus $0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 = 0.1380 - 2\left(3.84 \cdot 10^{-5}\right) Q_3 \quad \Rightarrow$ $Q_1 = -126.2255 + 0.9412Q_3$. Substituting into $Q_1 + Q_3 = Q_T$ gives $-126.2255 + 0.9412Q_3 + Q_3 = 1500$ $\Rightarrow \quad Q_3 \approx 837.7$, and then $Q_1 = Q_T - Q_3 \approx 1500 - 837.7 = 662.3$. So we should apportion approximately 662.3 ft^3/s to Turbine 1 and the remaining 837.7 ft^3/s to Turbine 3. The resulting energy production is $KW_1 + KW_3 \approx 7952.1 + 10{,}256.2 = 18{,}208.3$ kW. (We can verify that this is indeed a maximum energy production by checking nearby distributions.) In comparison, if we use all three turbines with $Q_T = 1500$ we get $Q_1 = 436.3$, $Q_2 = 466.0$, and $Q_3 = 597.7$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx 16{,}538.7$ kW. Clearly, for this flow level it is beneficial to use only two turbines.

6. Note that an incoming flow of 3400 ft^3/s is not within the domain we established in Problem 2, so we cannot simply use our previous work to give the optimal distribution. We will need to use all three turbines, due to the capacity limitations of each individual turbine, but 3400 is less than the maximum combined capacity of 3445 ft^3/s, so we still must decide how to distribute the flows. From the graph in Problem 4, Turbine 3 produces the most power for the higher flows, so it seems reasonable to use Turbine 3 at its maximum capacity of 1225 and distribute the remaining 2175 ft^3/s flow between Turbines 1 and 2. We can again use the technique of Lagrange multipliers to determine the optimal distribution. Following the procedure we used in Problem 5, we wish to maximize $KW_1 + KW_2$ subject to the constraint $Q_1 + Q_2 = Q_T$ where $Q_T = 2175$. We can equivalently maximize

$$f(Q_1, Q_2) = \frac{KW_1 + KW_2}{170 - 1.6 \cdot 10^{-6} Q_T^2}$$

$$= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2\right)$$

subject to the constraint $g(Q_1, Q_2) = Q_1 + Q_2 = Q_T$. Then we solve $\nabla f(Q_1, Q_2) = \lambda \nabla g(Q_1, Q_2) \quad \Rightarrow$ $0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 = \lambda$ and $0.1358 - 2\left(4.69 \cdot 10^{-5}\right) Q_2 = \lambda$, thus $0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 = 0.1358 - 2\left(4.69 \cdot 10^{-5}\right) Q_2 \quad \Rightarrow \quad Q_1 = -99.2647 + 1.1495Q_2$. Substituting into $Q_1 + Q_2 = Q_T$ gives $-99.2647 + 1.1495Q_2 + Q_2 = 2175 \quad \Rightarrow \quad Q_2 \approx 1058.0$, and then $Q_1 \approx 1117.0$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, 1110 and 1225 ft^3/s, and the remaining 1065 ft^3/s to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.

15 Review ET 14

CONCEPT CHECK

1. (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

   (b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy-plane. Also, we can use an arrow diagram such as Figure 1 in Section 15.1 [ET 14.1].

2. A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.

3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f. We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.

4. (a) See Definition 15.2.4 [ET 14.2.4].

   (b) If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.

5. (a) See (2) and (3) in Section 15.3 [ET 14.3].

   (b) See the discussion preceding Example 2 in Section 15.3 [ET 14.3].

   (c) To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

6. See the statement of Clairaut's Theorem on page 936 [ET 902].

7. (a) See (2) in Section 15.4 [ET 14.4].

   (b) See (19) and the preceding discussion in Section 15.6 [ET 14.6].

8. See (3) and (4) and the accompanying discussion in Section 15.4 [ET 14.4]. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b). Thus it is the linear function which best approximates f near (a, b).

9. (a) See Definition 15.4.7 [ET 14.4.7].

   (b) Use Theorem 15.4.8 [ET 14.4.8].

10. See (10) and the associated discussion in Section 15.4 [ET 14.4].

11. See (2) and (3) in Section 15.5 [ET 14.5].

12. See (7) and the preceding discussion in Section 15.5 [ET 14.5].

13. (a) See Definition 15.6.2 [ET 14.6.2]. We can interpret it as the rate of change of f at (x_0, y_0) in the direction of $\mathbf{u}$. Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction $\mathbf{u}$, the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the slope of the tangent line to C at P. (See Figure 5 in Section 15.6 [ET 14.6].)

    (b) See Theorem 15.6.3 [ET 14.6.3].

14. (a) See (8) and (13) in Section 15.6 [ET 14.6].

    (b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ or $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

    (c) The gradient vector of a function points in the direction of maximum rate of increase of the function. On a graph of the function, the gradient points in the direction of steepest ascent.

15. (a) f has a local maximum at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b).

(b) f has an absolute maximum at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) in the domain of f.

(c) f has a local minimum at (a,b) if $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b).

(d) f has an absolute minimum at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) in the domain of f.

(e) f has a saddle point at (a,b) if $f(a,b)$ is a local maximum in one direction but a local minimum in another.

16. (a) By Theorem 15.7.2 [ET 14.7.2], if f has a local maximum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

(b) A critical point of f is a point (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or one of these partial derivatives does not exist.

17. See (3) in Section 15.7 [ET 14.7].

18. (a) See Figure 11 and the accompanying discussion in Section 15.7 [ET 14.7].

(b) See Theorem 15.7.8 [ET 14.7.8].

(c) See the procedure outlined in (9) in Section 15.7 [ET 14.7].

19. See the discussion beginning on page 985 [ET 951]; see the discussion preceding Example 5 in Section 15.8 [ET 14.8].

TRUE-FALSE QUIZ

1. True. $f_y(a,b) = \lim\limits_{h\to 0} \dfrac{f(a,b+h) - f(a,b)}{h}$ from Equation 15.3.3 [ET 14.3.3]. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a,b) = \lim\limits_{y\to b} \dfrac{f(a,y) - f(a,b)}{y - b}$.

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \ne f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$.

4. True. From Equation 15.6.14 [ET 14.6.14] we get $D_{\mathbf{k}} f(x,y,z) = \nabla f(x,y,z) \cdot \langle 0,0,1\rangle = f_z(x,y,z)$.

5. False. See Example 15.2.3 [ET 14.2.3].

6. False. See Exercise 15.4.42 [ET 14.4.42].

7. True. If f has a local minimum and f is differentiable at (a,b) then by Theorem 15.7.2 [ET 14.7.2], $f_x(a,b) = 0$ and $f_y(a,b) = 0$, so $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle = \langle 0,0\rangle = \mathbf{0}$.

8. False. If f is not continuous at $(2,5)$, then we can have $\lim\limits_{(x,y)\to(2,5)} f(x,y) \ne f(2,5)$. (See Example 15.2.7 [ET 14.2.7].)

9. False. $\nabla f(x,y) = \langle 0, 1/y\rangle$.

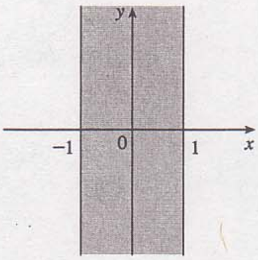
10. True. This is part (c) of the Second Derivatives Test (15.7.3 [ET 14.7.3]).

11. True. $\nabla f = \langle \cos x, \cos y\rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos\theta| \le 1$, so $|\nabla f| \le \sqrt{2}$. Now $D_{\mathbf{u}} f(x,y) = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}| \cos\theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}} f(x,y)| \le \sqrt{2}\cdot 1 \cdot 1 = \sqrt{2}$.

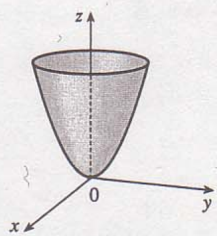
12. False. See Exercise 15.7.35 [14.7.35].

EXERCISES

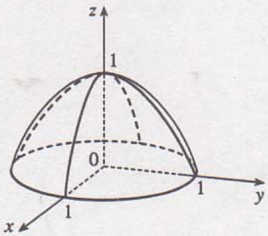
1. The domain of $\sin^{-1} x$ is $-1 \leq x \leq 1$ while the domain of $\tan^{-1} y$ is all real numbers, so the domain of $f(x, y) = \sin^{-1} x + \tan^{-1} y$ is $\{(x, y) \mid -1 \leq x \leq 1\}$.



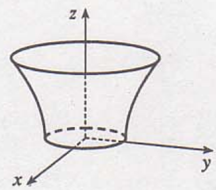
2. $D = \{(x, y, z) \mid z \geq x^2 + y^2\}$, the points on and above the paraboloid $z = x^2 + y^2$.



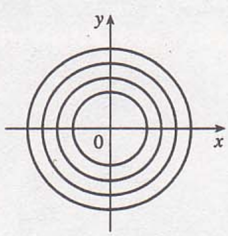
3. $z = f(x, y) = 1 - x^2 - y^2$, a paraboloid with vertex $(0, 0, 1)$.



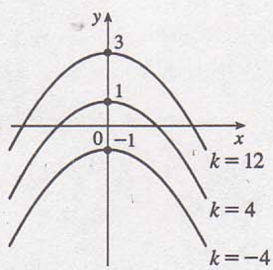
4. $z = f(x, y) = \sqrt{x^2 + y^2 - 1}$, so $z \geq 0$ and $1 = x^2 + y^2 - z^2$. Thus the graph is the upper half of a hyperboloid of one sheet.



5. Let $k = e^{-c} = e^{-(x^2+y^2)}$ be the level curves. Then $-\ln k = c = x^2 + y^2$, so we have a family of concentric circles.



6. $k = x^2 + 4y$ or $4(y - k/4) = -x^2$, a family of parabolas with vertex at $(0, k/4)$.



7. f is a rational function, so it is continuous on its domain. Since f is defined at $(1, 1)$, we use direct substitution to evaluate the limit: $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2 + 2y^2} = \frac{2(1)(1)}{1^2 + 2(1)^2} = \frac{2}{3}$.

8. As $(x,y) \to (0,0)$ along the x-axis, $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ along this line. But $f(x,x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x,y) \to (0,0)$ along the line $x = y$, $f(x,y) \to \frac{2}{3}$. Thus the limit doesn't exist.

9. (a) $T_x(6,4) = \lim\limits_{h\to 0} \dfrac{T(6+h,4) - T(6,4)}{h}$, so we can approximate $T_x(6,4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6,4) \approx \dfrac{T(8,4) - T(6,4)}{2} = \dfrac{86-80}{2} = 3$,

$T_x(6,4) \approx \dfrac{T(4,4) - T(6,4)}{-2} = \dfrac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately $3.5\,°\mathrm{C/m}$. Similarly, $T_y(6,4) = \lim\limits_{h\to 0} \dfrac{T(6,4+h) - T(6,4)}{h}$, which we can approximate with $h = \pm 2$: $T_y(6,4) \approx \dfrac{T(6,6) - T(6,4)}{2} = \dfrac{75-80}{2} = -2.5$,

$T_y(6,4) \approx \dfrac{T(6,2) - T(6,4)}{-2} = \dfrac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately $-3.0\,°\mathrm{C/m}$.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 15.6.9 [ET 14.6.9],

$D_{\mathbf{u}}T(6,4) = \nabla T(6,4) \cdot \mathbf{u} = T_x(6,4)\,\frac{1}{\sqrt{2}} + T_y(6,4)\,\frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}T(6,4) \approx (3.5)\,\frac{1}{\sqrt{2}} + (-3.0)\,\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35\,°\mathrm{C/m}$.

Alternatively, we can use Definition 15.6.2 [ET 14.6.2]:

$D_{\mathbf{u}}T(6,4) = \lim\limits_{h\to 0} \dfrac{T\left(6 + h\frac{1}{\sqrt{2}}, 4 + h\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$, which we can estimate with $h = \pm 2\sqrt{2}$. Then

$D_{\mathbf{u}}T(6,4) \approx \dfrac{T(8,6) - T(6,4)}{2\sqrt{2}} = \dfrac{80-80}{2\sqrt{2}} = 0$, $D_{\mathbf{u}}T(6,4) \approx \dfrac{T(4,2) - T(6,4)}{-2\sqrt{2}} = \dfrac{74-80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$.

Averaging these values, we have $D_{\mathbf{u}}T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1\,°\mathrm{C/m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[T_x(x,y)\right] = \lim\limits_{h\to 0} \dfrac{T_x(x,y+h) - T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim\limits_{h\to 0} \dfrac{T_x(6,4+h) - T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \dfrac{T(8,6) - T(6,6)}{2} = \dfrac{80-75}{2} = 2.5$, $T_x(6,6) \approx \dfrac{T(4,6) - T(6,6)}{-2} = \dfrac{68-75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$T_x(6,2) \approx \dfrac{T(8,2) - T(6,2)}{2} = \dfrac{90-87}{2} = 1.5$, $T_x(6,2) \approx \dfrac{T(4,2) - T(6,2)}{-2} = \dfrac{74-87}{-2} = 6.5$.

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$T_{xy}(6,4) \approx \dfrac{T_x(6,6) - T_x(6,4)}{2} = \dfrac{3.0-3.5}{2} = -0.25$,

$T_{xy}(6,4) \approx \dfrac{T_x(6,2) - T_x(6,4)}{-2} = \dfrac{4.0-3.5}{-2} = -0.25$. Averaging these values, we have $T_{xy}(6,4) \approx -0.25$.

10. From the table, $T(6,4) = 80$, and from Exercise 9 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$\begin{aligned} T(x,y) &\approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) \\ &= 3.5x - 3y + 71 \end{aligned}$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1\,^\circ\text{C}$.

11. $f(x,y) = 3x^4 - x\sqrt{y} \;\Rightarrow\; f_x = 12x^3 - \sqrt{y},\ f_y = -\frac{1}{2}xy^{-1/2}$

12. $g(x,y) = \dfrac{x}{\sqrt{x+2y}} \;\Rightarrow\; g_x = \dfrac{\sqrt{x+2y} - x(x+2y)^{-1/2}/2}{x+2y} = \dfrac{2y + x/2}{(x+2y)^{3/2}},\ g_y = -x(x+2y)^{-3/2}$

13. $f(s,t) = e^{2s}\cos\pi t \;\Rightarrow\; f_s = 2e^{2s}\cos\pi t,\ f_t = -\pi e^{2s}\sin\pi t$

14. $g(r,s) = r\sin\sqrt{r^2+s^2} \;\Rightarrow\; g_r = \sin\sqrt{r^2+s^2} + r^2\left(r^2+s^2\right)^{-1/2}\cos\sqrt{r^2+s^2}$, $g_s = rs\left(r^2+s^2\right)^{-1/2}\cos\sqrt{r^2+s^2}$

15. $f(x,y,z) = xy^z \;\Rightarrow\; f_x = y^z,\ f_y = xzy^{z-1},\ f_z = xy^z\ln y$

16. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D \;\Rightarrow$ $\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S-35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35-35) \approx 3.587$, thus in $10\,^\circ$C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

17. $f(x,y) = x^2y^3 - 2x^4 + y^2 \;\Rightarrow\; f_x = 2xy^3 - 8x^3,\ f_y = 3x^2y^2 + 2y,\ f_{xx} = 2y^3 - 24x^2,\ f_{yy} = 6x^2y + 2$, and $f_{xy} = f_{yx} = 6xy^2$.

18. $f(x,y) = x^3\ln(x-y) \;\Rightarrow\; f_x = x^2\left[3\ln(x-y) + x/(x-y)\right],\ f_y = -x^3/(x-y)$,

$$\begin{aligned} f_{xx} &= 6x\ln(x-y) + 3x^2(x-y)^{-1} + \left(2x^3 - 3x^2y\right)(x-y)^{-2} \\ &= 6x\ln(x-y) + x^2(5x-6y)/(x-y)^2, \end{aligned}$$

$f_{yy} = -x^3/(x-y)^2$, and $f_{xy} = f_{yx} = x^2(3y-2x)/(x-y)^2$.

19. $f(x,y,z) = xy^2z^3 \;\Rightarrow\; f_x = y^2z^3,\ f_y = 2xyz^3,\ f_z = 3xy^2z^2,\ f_{xx} = 0,\ f_{yy} = 2xz^3,\ f_{zz} = 6xy^2z$, $f_{xy} = f_{yx} = 2yz^3$, $f_{xz} = f_{zx} = 3y^2z^2$, and $f_{yz} = f_{zy} = 6xyz^2$.

20. $f(x,y,z) = xe^y\cos z \;\Rightarrow\; f_x = e^y\cos z,\ f_y = xe^y\cos z,\ f_z = -xe^y\sin z,\ f_{xx} = 0,\ f_{yy} = xe^y\cos z$, $f_{zz} = -xe^y\cos z$, $f_{xy} = f_{yx} = e^y\cos z$, $f_{xz} = f_{zx} = -e^y\sin z$, and $f_{yz} = f_{zy} = -xe^y\sin z$.

21. $u = x^y \;\Rightarrow\; u_x = yx^{y-1},\ u_y = x^y\ln x$ and $(x/y)\,u_x + (\ln x)^{-1}u_y = x^y + x^y = 2u$.

22. $\rho = \sqrt{x^2+y^2+z^2} \;\Rightarrow\; \rho_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}},\ \rho_{xx} = \dfrac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$.

By symmetry, $\rho_{yy} = \dfrac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$ and $\rho_{zz} = \dfrac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$. Thus

$$\rho_{xx} + \rho_{yy} + \rho_{zz} = 2\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+z^2)^{1/2}} = \frac{2}{\rho}.$$

23. $z_x(0,1) = 0$, $z_y(0,1) = 6$ and an equation of the tangent plane is $z - 5 = 6(y-1)$ or $z - 6y = -1$.

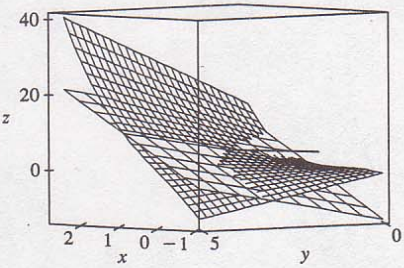
24. $z_x(1,0)=1$, $z_y(1,0)=1$, and an equation of the tangent plane is $z-1=x-1+y$ or $x+y-z=0$.

25. $F(x,y,z)=xy^2z^3$, $\nabla F=\langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ and $\nabla F(3,2,1)=\langle 4,12,36\rangle$. Thus an equation of the tangent plane is $4(x-3)+12(y-2)+36(z-1)=0$ or $x+3y+9z=18$.

26. $z^2=29-x^2-y^2$, so $z_x=-2x/(2z)=-x/z$, $z_y=-y/z$ and $z_x(2,3,4)=-\frac{1}{2}$, $z_y(2,3,4)=-\frac{3}{4}$. An equation of the tangent plane is $z-4=-\frac{1}{2}(x-2)+\left(-\frac{3}{4}\right)(y-3)$ or $2x+3y+4z=29$. (Alternately, use Exercise 15.6.49 [ET 14.6.49] with $a=b=c=\sqrt{29}$.)

27. $F(x,y,z)=x^2+2y^2-3z^2$, $F_x=2x$, $F_y=4y$, $F_z=-6z$; $F_x(3,2,-1)=6$, $F_y(3,2,-1)=8$, $F_z(3,2,-1)=6$. So an equation of the tangent plane is $6(x-3)+8(y-2)+6(z+1)=0$ or $3x+4y+3z=14$.

28. Let $f(x,y)=x^3+2xy$. Then $f_x(x,y)=3x^2+2y$ and $f_y(x,y)=2x$, so $f_x(1,2)=7$, $f_y(1,2)=2$ and an equation of the tangent plane is $z-5=7(x-1)+2(y-2)$ or $7x+2y-z=6$. The normal line is given by $\dfrac{x-1}{7}=\dfrac{y-2}{2}=\dfrac{z-5}{-1}$ or $x=7t+1$, $y=2t+2$, $z=-t+5$.



29. $F(x,y,z)=x^2+y^2+z^2$, $\nabla F(x_0,y_0,z_0)=\langle 2x_0,2y_0,2z_0\rangle=k\langle 2,1,-3\rangle$ or $x_0=k$, $y_0=\frac{1}{2}k$ and $z_0=-\frac{3}{2}k$. But $x_0^2+y_0^2+z_0^2=1$, so $\frac{7}{2}k^2=1$ and $k=\pm\sqrt{\frac{2}{7}}$. Hence there are two such points: $\left(\pm\sqrt{\frac{2}{7}},\pm\frac{1}{\sqrt{14}},\mp\frac{3}{\sqrt{14}}\right)$.

30. $z=x^2\tan^{-1}y \;\Rightarrow\; dz=\left(2x\tan^{-1}y\right)dx+\left[x^2/\left(y^2+1\right)\right]dy$

31. $f(x,y,z)=x^3\sqrt{y^2+z^2} \;\Rightarrow\; f_x(x,y,z)=3x^2\sqrt{y^2+z^2}$, $f_y(x,y,z)=\dfrac{yx^3}{\sqrt{y^2+z^2}}$, and $f_z(x,y,z)=\dfrac{zx^3}{\sqrt{y^2+z^2}}$, so $f(2,3,4)=8(5)=40$, $f_x(2,3,4)=3(4)\sqrt{25}=60$, $f_y(2,3,4)=\frac{3(8)}{\sqrt{25}}=\frac{24}{5}$, and $f_z(2,3,4)=\frac{4(8)}{\sqrt{25}}=\frac{32}{5}$. Then the linear approximation of f at $(2,3,4)$ is

$$\begin{aligned} f(x,y,z)&\approx f(2,3,4)+f_x(2,3,4)(x-2)+f_y(2,3,4)(y-3)+f_z(2,3,4)(z-4)\\ &=40+60(x-2)+\tfrac{24}{5}(y-3)+\tfrac{32}{5}(z-4)=60x+\tfrac{24}{5}y+\tfrac{32}{5}z-120\end{aligned}$$

Then

$$\begin{aligned}(1.98)^3\sqrt{(3.01)^2+(3.97)^2}&=f(1.98,3.01,3.97)\approx 60(1.98)+\tfrac{24}{5}(3.01)+\tfrac{32}{5}(3.97)-120\\ &=38.656\end{aligned}$$

32. (a) $dA=\dfrac{\partial A}{\partial x}dx+\dfrac{\partial A}{\partial y}dy=\frac{1}{2}y\,dx+\frac{1}{2}x\,dy$ and $|\Delta x|\le 0.002$, $|\Delta y|\le 0.002$. Thus the maximum error in the calculated area is about $dA=6(0.002)+\frac{5}{2}(0.002)=0.017\text{ m}^2$ or 170 cm^2.

(b) $z=\sqrt{x^2+y^2}$, $dz=\dfrac{x}{\sqrt{x^2+y^2}}dx+\dfrac{y}{\sqrt{x^2+y^2}}dy$ and $|\Delta x|\le 0.002$, $|\Delta y|\le 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz=\frac{5}{13}(0.002)+\frac{12}{13}(0.002)=\frac{0.17}{65}\approx 0.0026$ m or 0.26 cm.

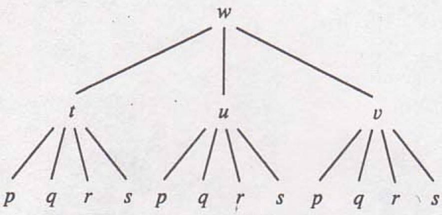
33. $\dfrac{dw}{dt} = \dfrac{1}{2\sqrt{x}}\left(2e^{2t}\right) + \dfrac{2y}{z}\left(3t^2+4\right) + \dfrac{-y^2}{z^2}(2t) = e^t + \dfrac{2y}{z}\left(3t^2+4\right) - 2t\dfrac{y^2}{z^2}$

34. $\dfrac{\partial z}{\partial u} = (-y\sin xy - y\sin x)(2u) + (-x\sin xy + \cos x) = \cos x - 2uy\sin x - (\sin xy)(x+2uy)$,

$\dfrac{\partial z}{\partial v} = (-y\sin xy - y\sin x)(1) + (-x\sin xy + \cos x)(-2v) = -2v\cos x + (\sin xy)(2vx - y) - y\sin x$

35. By the Chain Rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s=1$ and $t=2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so $\dfrac{\partial z}{\partial s} = f_x(3,6)\,g_s(1,2) + f_y(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so $\dfrac{\partial z}{\partial t} = f_x(3,6)\,g_t(1,2) + f_y(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108$.

36.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$
$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$$
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

37. $\dfrac{\partial z}{\partial x} = 2xf'\left(x^2-y^2\right)$, $\dfrac{\partial z}{\partial y} = 1 - 2yf'\left(x^2-y^2\right)$ $\left[\text{where } f' = \dfrac{df}{d\left(x^2-y^2\right)}\right]$. Then

$y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 2xyf'\left(x^2-y^2\right) + x - 2xyf'\left(x^2-y^2\right) = x$.

38. $A = \frac{1}{2}xy\sin\theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(y\sin\theta)\dfrac{dx}{dt} + (x\sin\theta)\dfrac{dy}{dt} + (xy\cos\theta)\dfrac{d\theta}{dt}\right]$. So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$,

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(25)(3) + (20)(-2) + \left(1000\sqrt{3}\right)(0.05)\right] = \dfrac{35 + 50\sqrt{3}}{2} \approx 60.8 \text{ in}^2/\text{s}$.

39. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}y + \dfrac{\partial z}{\partial v}\dfrac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}y + \frac{\partial^2 z}{\partial v\partial u}\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\partial v}y\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\dfrac{\partial z}{\partial y} = x\dfrac{\partial z}{\partial u} + \dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\partial u}\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{x} + \frac{\partial^2 z}{\partial u\partial v}x\right)\\ &= x^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Thus

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=\frac{2y}{x}\frac{\partial z}{\partial v}+x^2y^2\frac{\partial^2 z}{\partial u^2}-2y^2\frac{\partial^2 z}{\partial u\partial v}+\frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}-x^2y^2\frac{\partial^2 z}{\partial u^2}-2y^2\frac{\partial^2 z}{\partial u\partial v}-\frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}$$
$$=\frac{2y}{x}\frac{\partial z}{\partial v}-4y^2\frac{\partial^2 z}{\partial u\partial v}=2v\frac{\partial z}{\partial v}-4uv\frac{\partial^2 z}{\partial u\partial v}$$

since $y=xv=\dfrac{uv}{y}$ or $y^2=uv$.

40. $F(x,y,z)=e^{xyz}-yz^4-x^2z^3=0$, so $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=-\dfrac{yze^{xyz}-2xz^3}{xye^{xyz}-4yz^3-3x^2z^2}=\dfrac{2xz^3-yze^{xyz}}{xye^{xyz}-4yz^3-3x^2z^2}$

and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}=-\dfrac{xze^{xyz}-z^4}{xye^{xyz}-4yz^3-3x^2z^2}=\dfrac{z^4-xze^{xyz}}{xye^{xyz}-4yz^3-3x^2z^2}$.

41. $\nabla f=\left\langle z^2\sqrt{y}e^{x\sqrt{y}},\dfrac{xz^2e^{x\sqrt{y}}}{2\sqrt{y}},2ze^{x\sqrt{y}}\right\rangle=ze^{x\sqrt{y}}\left\langle z\sqrt{y},\dfrac{xz}{2\sqrt{y}},2\right\rangle$

42. (a) By Theorem 15.6.15 [ET 14.6.15], the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}}f=|\nabla f|\cos\theta$ (see the proof of Theorem 15.6.15 [ET 14.6.15]) has a minimum when $\theta=\pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}=0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}f=|\nabla f|\cos\theta=\frac{1}{2}|\nabla f|\;\Leftrightarrow\;\cos\theta=\frac{1}{2}$ $\Leftrightarrow\;\theta=\frac{\pi}{3}$.

43. $\nabla f=\langle 1/\sqrt{x},-2y\rangle$, $\nabla f(1,5)=\langle 1,-10\rangle$, $\mathbf{u}=\frac{1}{5}\langle 3,-4\rangle$. Then $D_{\mathbf{u}}f(1,5)=\frac{43}{5}$.

44. $\nabla f=\left\langle 2xy+\sqrt{1+z},x^2,x/\left(2\sqrt{1+z}\right)\right\rangle$, $\nabla f(1,2,3)=\left\langle 6,1,\frac{1}{4}\right\rangle$, and $\mathbf{u}=\left\langle\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right\rangle$. Then $D_{\mathbf{u}}f(1,2,3)=\frac{25}{6}$.

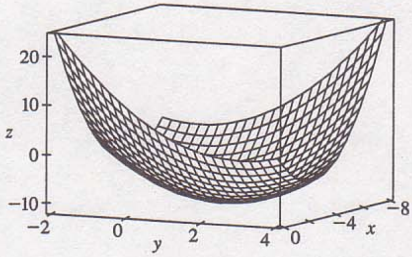
45. $\nabla f=\left\langle 2xy,x^2+1/\left(2\sqrt{y}\right)\right\rangle$, $|\nabla f(2,1)|=\left|\left\langle 4,\frac{9}{2}\right\rangle\right|$. Thus the maximum rate of change of f at $(2,1)$ is $\frac{\sqrt{145}}{2}$ in the direction $\left\langle 4,\frac{9}{2}\right\rangle$.

46. $\nabla f=\langle zye^{xy},zxe^{xy},e^{xy}\rangle$, $\nabla f(0,1,2)=\langle 2,0,1\rangle$ is the direction of most rapid increase while the rate is $|\langle 2,0,1\rangle|=\sqrt{5}$.

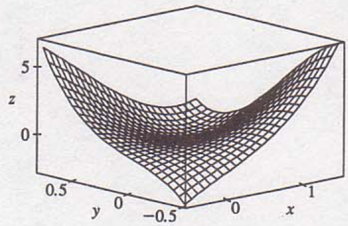
47. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8}=\frac{5}{8}=0.625$ knots/mi.

48. The surfaces are $f(x,y,z)=z-2x^2+y^2=0$ and $g(x,y,z)=z-4=0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2,2,4)$. The vector $\mathbf{v}=\nabla f\times\nabla g$ is therefore parallel to the line.
$\nabla f(x,y,z)=\langle -4x,2y,1\rangle\;\Rightarrow\;\nabla f(-2,2,4)=\langle 8,4,1\rangle$, $\nabla g(x,y,z)=\langle 0,0,1\rangle\;\Rightarrow$
$\nabla g\langle -2,2,4\rangle=\langle 0,0,1\rangle$. Hence $\mathbf{v}=\nabla f\times\nabla g=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\8&4&1\\0&0&1\end{vmatrix}=4\,\mathbf{i}-8\,\mathbf{j}$. Thus, parametric equations are:
$x=-2+4t$, $y=2-8t$, and $z=4$.

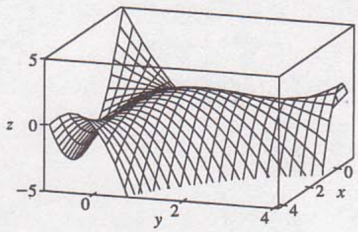
49. $f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10 \Rightarrow f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4,1)$ and $f_{xx}(-4,1) > 0$, $D(-4,1) = 3 > 0$, so $f(-4,1) = -11$ is a local minimum.



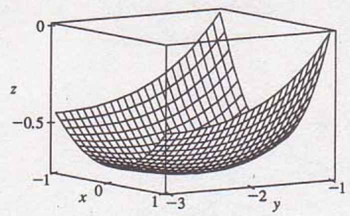
50. $f(x,y) = x^3 - 6xy + 8y^3 \Rightarrow f_x = 3x^2 - 6y$, $f_y = -6x + 24y^2$, $f_{xx} = 6x$, $f_{yy} = 48y$, $f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x(x^3 - 1) = 0$, so the critical points are $(0,0)$, $(1,\frac{1}{2})$. $D(0,0) = -36 < 0$ so $(0,0)$ is a saddle point while $f_{xx}(1,\frac{1}{2}) = 6 > 0$ and $D(1,\frac{1}{2}) = 108 > 0$ so $f(1,\frac{1}{2}) = -1$ is a local minimum.



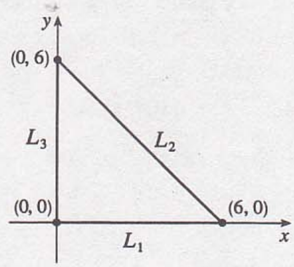
51. $f(x,y) = 3xy - x^2y - xy^2 \Rightarrow f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 3 - 2x - 2y$. $\frac{1}{n}$Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0,0)$, $(3,0)$, $(0,3)$ and $(1,1)$. $D(0,0) = D(3,0) = D(0,3) = -9 < 0$ so $(0,0)$, $(3,0)$, and $(0,3)$ are saddle points. $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = -2 < 0$, so $f(1,1) = 1$ is a local maximum.



52. $f(x,y) = (x^2 + y)\, e^{y/2} \Rightarrow f_x = 2xe^{y/2}$, $f_y = e^{y/2}(2 + x^2 + y)/2$, $f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2}(4 + x^2 + y)/4$, $f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0,-2) > 0$, $D(0,-2) = e^{-2} - 0 > 0$ so $f(0,-2) = -2/e$ is a local minimum.

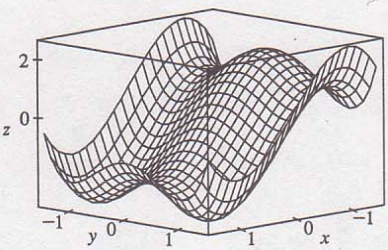


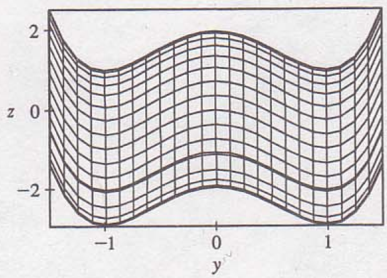
53. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2,0)$ isn't inside D. Thus the only critical point inside D is $(1,2)$ and $f(1,2) = 4$. Secondly we consider the boundary of D. On L_1, $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2, $x = -y + 6$ and $f(-y+6,y) = y^2(6-y)(-2) = -2(6y^2 - y^3)$ which has critical points at $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3, $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

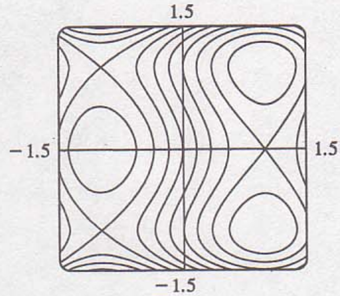


54. Inside D: $f_x = 2xe^{-x^2-y^2}\left(1 - x^2 - 2y^2\right) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}\left(2 - x^2 - 2y^2\right) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0,0)$, $(0,\pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f\,(0,0) = 0$, $f\,(\pm 1, 0) = e^{-1}$ and $f\,(0,\pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f\,(x,y) = e^{-4}\left(4 + y^2\right)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f\,(\pm 2, 0) = 4e^{-4}$, $f\,(0,\pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f\,(0,\pm 1) = 2e^{-1}$ and the absolute minimum is $f\,(0,0) = 0$.

55. $f\,(x,y) = x^3 - 3x + y^4 - 2y^2$



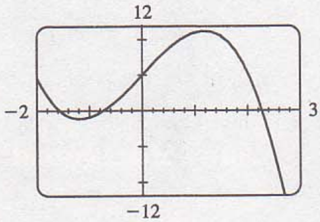


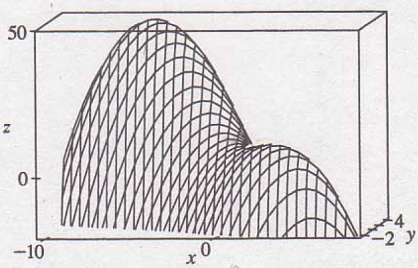


From the graphs, it appears that f has a local maximum $f\,(-1,0) \approx 2$, local minima $f\,(1,\pm 1) \approx -3$, and saddle points at $(-1,\pm 1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \quad \Leftrightarrow \quad x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \quad \Leftrightarrow$ $y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D\,(-1,0) = 24 > 0$ and $f_{xx}\,(-1,0) = -6 < 0$ indicating a local maximum $f\,(-1,0) = 2$; $D\,(1,\pm 1) = 48 > 0$ and $f_{xx}\,(1,\pm 1) = 6 > 0$ indicating local minima $f\,(1,\pm 1) = -3$; and $D\,(-1,\pm 1) = -48$ and $D\,(1,0) = -24$, indicating saddle points.

56. $f\,(x,y) = 12 + 10y - 2x^2 - 8xy - y^4 \quad \Rightarrow \quad f_x\,(x,y) = -4x - 8y$, $f_y\,(x,y) = 10 - 8x - 4y^3$. Now $f_x\,(x,y) = 0 \quad \Rightarrow \quad x = -2x$, and substituting this into $f_y\,(x,y) = 0$ gives $10 + 16y - 4y^3 = 0 \quad \Leftrightarrow$ $5 + 8y - 2y^3 = 0$.





From the first graph, we see that this is true when $y \approx -1.542, -0.717$, or 2.260. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D\,(3.085, -1.542) > 0$, $D\,(1.434, -0.717) < 0$, and $D\,(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f\,(x,y)$ has local maxima $f\,(-4.519, 2.260) \approx 49.373$ and $f\,(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

57. $f(x,y) = x^2y$, $g(x,y) = x^2 + y^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ imply $\lambda = x^2/(2y)$ and $\lambda = y$ if $x \neq 0$ and $y \neq 0$. Hence $x^2 = 2y^2$. Then $x^2 + y^2 = 1$ implies $3y^2 = 1$ so $y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm\sqrt{\frac{2}{3}}$. [Note if $x = 0$ then $x^2 = 2\lambda y$ implies $y = 0$ and $f(0,0) = 0$.] Thus the possible points are $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{3}}\right)$ and the absolute maxima are $f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ while the absolute minima are $f\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$.

58. $f(x,y) = 1/x + 1/y$, $g(x,y) = 1/x^2 + 1/y^2 = 1 \quad \Rightarrow$
$\nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

59. $f(x,y,z) = x + y + z$, $g(x,y,z) = 1/x + 1/y + 1/z = 1 \quad \Rightarrow$
$\nabla f = \langle 1,1,1 \rangle = \lambda \nabla g = \langle -\lambda x^{-2}, -\lambda y^{-2}, -\lambda z^{-2} \rangle$. Thus $\lambda = -x^2 = -y^2 = -z^2$ or $y = \pm x$, $z = \pm x$. Substituting into $1/x + 1/y + 1/z = 1$ gives (1) $3/x = 1$ so $x = 3$, or (2) $1/x = 1$ so $x = 1$, or (3) $-1/x = 1$ so $x = -1$ with the associated points (1) $(3,3,3)$, (2) $(1,1,-1)$ or $(1,-1,1)$, (3) $(-1,1,1)$. Thus the absolute maximum is $f(3,3,3) = 9$ and the absolute minimum is $f(1,1,-1) = f(1,-1,1) = f(-1,1,1) = 1$.

60. $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = x - y + 2z = 2 \quad \Rightarrow$
$\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and (1) $2x = \lambda + \mu$, (2) $4y = \lambda - \mu$, (3) $6z = \lambda + 2\mu$, (4) $x + y + z = 1$, (5) $x - y + 2z = 2$. Then six times (1) plus three times (2) plus two times (3) implies $12(x + y + z) = 11\lambda + 7\mu$, so (4) gives $11\lambda + 7\mu = 12$. Also six times (1) minus three times (2) plus four times (3) implies $12(x - y + 2z) = 7\lambda + 17\mu$, so (5) gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into (1), (2) and (3) implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$. Now since $(0,0,1)$ satisfies both constraints and $f(0,0,1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

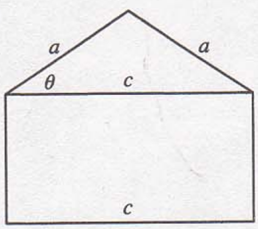
61. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = xy^2z^3 = 2 \quad \Rightarrow$
$\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle \lambda y^2z^3, 2\lambda xyz^3, 3\lambda xy^2z^2 \rangle$. Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so (1) $2x = \lambda y^2z^3$, (2) $1 = \lambda xz^3$, (3) $2 = 3\lambda xy^2z$. Then (2) and (3) imply $\dfrac{1}{xz^3} = \dfrac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so $y = \pm z\sqrt{\frac{2}{3}}$. Similarly (1) and (3) imply $\dfrac{2x}{y^2z^3} = \dfrac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm\frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x,y,z) = 2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3 = 2$ or $z = \pm 3^{1/4}$ and the possible points are $\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$, $\left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$. However at each of these points

f takes on the same value, $2\sqrt{3}$. But $(2,1,1)$ also satisfies $g(x,y,z)=2$ and $f(2,1,1)=6>2\sqrt{3}$. Thus f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3=2$.

Alternate Solution: $g(x,y,z)=xy^2z^3=2$ implies $y^2=\dfrac{2}{xz^3}$, so minimize $f(x,z)=x^2+\dfrac{2}{xz^3}+z^2$. Then $f_x=2x-\dfrac{2}{x^2z^3}$, $f_z=-\dfrac{6}{xz^4}+2z$, $f_{xx}=2+\dfrac{4}{x^3z^3}$, $f_{zz}=\dfrac{24}{xz^5}+2$ and $f_{xz}=\dfrac{6}{x^2z^4}$. Now $f_x=0$ implies $2x^3z^3-2=0$ or $z=1/x$. Substituting into $f_y=0$ implies $-6x^3+2x^{-1}=0$ or $x=\frac{1}{\sqrt[4]{3}}$, so the two critical points are $\left(\pm\frac{1}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)=(2+4)\left(2+\frac{24}{3}\right)-\left(\frac{6}{\sqrt{3}}\right)^2>0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)=6>0$, so each point is a minimum. Finally, $y^2=\dfrac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{\sqrt{2}}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{\sqrt{2}}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)$.

62. $V=xyz$, say x is the length and $x+2y+2z\le 84$, $x>0$, $y>0$, $z>0$. First maximize V subject to $x+2y+2z=84$ with x,y,z all positive. Then $\langle yz,xz,xy\rangle=\langle\lambda,2\lambda,2\lambda\rangle$ implies $2yz=xz$ or $x=2y$ and $xz=xy$ or $z=y$. Thus $g(x,y,z)=84$ implies $6y=84$ or $y=14=z$, $x=28$, so the volume is $V=5488$ cubic units. Since $(80,1,1)$ also satisfies $g(x,y,z)=84$ and $V(80,1,1)=80$ cubic units, $(28,14,14)$ gives an absolute maximum of V subject to $g(x,y,z)=84$. But if $x+2y+2z<84$, there exists $\alpha>0$ such that $x+2y+2z=84-\alpha$ and as above $6y=84-\alpha$ implies $y=(84-\alpha)/6=z$, $x=(84-\alpha)/3$ with $V=(84-\alpha)^3/6^2\cdot 3<(84)^3/6^2\cdot 3=5488$. Hence we have shown that the maximum of V subject to $g(x,y,z)\le 84$ is the maximum of V subject to $g(x,y,z)=84$ (an intuitively obvious fact).

63.



The area of the triangle is $\frac{1}{2}ca\sin\theta$ and the area of the rectangle is bc. Thus, the area of the whole object is $f(a,b,c)=\frac{1}{2}ca\sin\theta+bc$. The perimeter of the object is $g(a,b,c)=2a+2b+c=P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2\sin^2\theta+\left(\frac{1}{2}c\right)^2=a^2\ \Rightarrow\ \sin\theta=\dfrac{1}{2a}\sqrt{4a^2-c^2}$. Thus $f(a,b,c)=\dfrac{c}{4}\sqrt{4a^2-c^2}+bc$. (Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f=\lambda\nabla g$ which gives the following equations: (1) $ca\left(4a^2-c^2\right)^{-1/2}=2\lambda$, (2) $c=2\lambda$, (3) $\frac{1}{4}\left(4a^2-c^2\right)^{1/2}-\frac{1}{4}c^2\left(4a^2-c^2\right)^{-1/2}+b=\lambda$, and (4) $2a+2b+c=P$. From (2), $\lambda=\frac{1}{2}c$ and so (1) produces $ca\left(4a^2-c^2\right)^{-1/2}=c\ \Rightarrow\ \left(4a^2-c^2\right)^{1/2}=a\ \Rightarrow\ 4a^2-c^2=a^2\ \Rightarrow$ (5) $c=\sqrt{3}a$. Similarly, since $\left(4a^2-c^2\right)^{1/2}=a$ and $\lambda=\frac{1}{2}c$, (3) gives $\dfrac{a}{4}-\dfrac{c^2}{4a}+b=\dfrac{c}{2}$, so from (5), $\dfrac{a}{4}-\dfrac{3a}{4}+b=\dfrac{\sqrt{3}a}{2}$ $\Rightarrow\ -\dfrac{a}{2}-\dfrac{\sqrt{3}a}{2}=-b\ \Rightarrow$ (6) $b=\frac{a}{2}\left(1+\sqrt{3}\right)$. Substituting (5) and (6) into (4) we get:

$2a+a\left(1+\sqrt{3}\right)+\sqrt{3}a=P\ \Rightarrow\ 3a+2\sqrt{3}a=P\ \Rightarrow\ a=\dfrac{P}{3+2\sqrt{3}}=\dfrac{2\sqrt{3}-3}{3}P$ and thus

$b=\dfrac{(2\sqrt{3}-3)\left(1+\sqrt{3}\right)}{6}P=\dfrac{3-\sqrt{3}}{6}P$ and $c=\left(2-\sqrt{3}\right)P$.

64. (a) $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + f(x(t), y(t))\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j} + \left(f_x\dfrac{dx}{dt} + f_y\dfrac{dy}{dt}\right)\mathbf{k}$ (by the Chain Rule). Therefore

$$\begin{aligned} K = \tfrac{1}{2}m\,|\mathbf{v}|^2 &= \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt}\right)^2\right] \\ &= \frac{m}{2}\left[\left(1 + f_x^2\right)\left(\frac{dx}{dt}\right)^2 + 2f_xf_y\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \left(1 + f_y^2\right)\left(\frac{dy}{dt}\right)^2\right] \end{aligned}$$

(b) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2x}{dt^2}\,\mathbf{i} + \dfrac{d^2y}{dt^2}\,\mathbf{j} + \left[f_{xx}\left(\dfrac{dx}{dt}\right)^2 + 2f_{xy}\dfrac{dx}{dt}\dfrac{dy}{dt} + f_{yy}\left(\dfrac{dy}{dt}\right)^2 + f_x\dfrac{d^2x}{dt^2} + f_y\dfrac{d^2y}{dt^2}\right]\mathbf{k}$

(c) If $z = x^2 + y^2$, where $x = t\cos t$ and $y = t\sin t$, then $z = f(x, y) = t^2$.

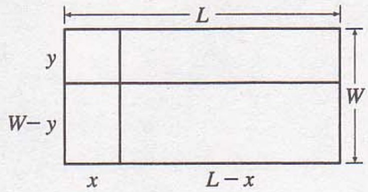
$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v} = (\cos t - t\sin t)\,\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j} + 2t\,\mathbf{k}$,

$K = \dfrac{m}{2}\left[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2\right] = \dfrac{m}{2}\left(1 + t^2 + 4t^2\right) = \dfrac{m}{2}\left(1 + 5t^2\right)$, and

$\mathbf{a} = (-2\sin t - t\cos t)\,\mathbf{i} + (2\cos t - t\sin t)\,\mathbf{j} + 2\,\mathbf{k}$. Notice that it is easier not to use the formulas in (a) and (b).

Problems Plus

1. The areas of the smaller rectangles are $A_1 = xy$, $A_2 = (L-x)\,y$, $A_3 = (L-x)\,(W-y)$, $A_4 = x\,(W-y)$. For $0 \le x \le L$, $0 \le y \le W$, let

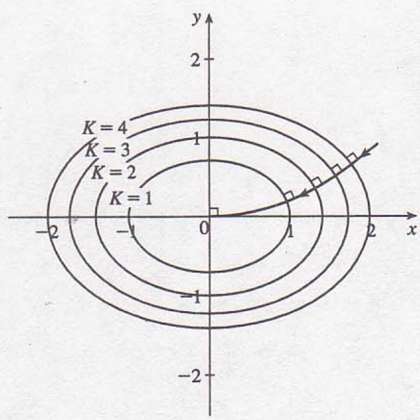


$$\begin{aligned} f(x,y) &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \\ &= x^2y^2 + (L-x)^2 y^2 + (L-x)^2 (W-y)^2 + x^2 (W-y)^2 \\ &= \left[x^2 + (L-x)^2\right]\left[y^2 + (W-y)^2\right] \end{aligned}$$

Then we need to find the maximum and minimum values of $f(x,y)$. Here $f_x(x,y) = [2x - 2(L-x)]\left[y^2 + (W-y)^2\right] = 0 \;\Rightarrow\; 4x - 2L = 0$ or $x = \frac{1}{2}L$, and $f_y(x,y) = \left[x^2 + (L-x)^2\right][2y - 2(W-y)] = 0 \;\Rightarrow\; 4y - 2W = 0$ or $y = W/2$. Also $f_{xx} = 4\left[y^2 + (W-y)^2\right]$, $f_{yy} = 4\left[x^2 + (L-x)^2\right]$, and $f_{xy} = (4x - 2L)(4y - 2W)$. Then $D = 16\left[y^2 + (W-y)^2\right]\left[x^2 + (L-x)^2\right] - (4x-2L)^2(4y-2W)^2$. Thus when $x = \frac{1}{2}L$ and $y = \frac{1}{2}W$, $D > 0$ and $f_{xx} = 2W^2 > 0$. Thus a minimum of f occurs at $\left(\frac{1}{2}L, \frac{1}{2}W\right)$ and this minimum value is $f\left(\frac{1}{2}L, \frac{1}{2}W\right) = \frac{1}{4}L^2W^2$. There are no other critical points, so the maximum must occur on the boundary. Now along the width of the rectangle let $g(y) = f(0,y) = f(L,y) = L^2\left[y^2 + (W-y)^2\right]$, $0 \le y \le W$. Then $g'(y) = L^2[2y - 2(W-y)] = 0 \;\Leftrightarrow\; y = \frac{1}{2}W$. And $g\left(\frac{1}{2}\right) = \frac{1}{2}L^2W^2$. Checking the endpoints, we get $g(0) = g(W) = L^2W^2$. Along the length of the rectangle let $h(x) = f(x,0) = f(x,W) = W^2\left[x^2 + (L-x)^2\right]$, $0 \le x \le L$. By symmetry $h'(x) = 0 \;\Leftrightarrow\; x = \frac{1}{2}L$ and $h\left(\frac{1}{2}L\right) = \frac{1}{2}L^2W^2$. At the endpoints we have $h(0) = h(L) = L^2W^2$. Therefore L^2W^2 is the maximum value of f. This maximum value of f occurs when the "cutting" lines correspond to sides of the rectangle.

2. (a) The level curves of the function $C(x,y) = e^{-\left(x^2+2y^2\right)/10^4}$ are the curves $e^{-\left(x^2+2y^2\right)/10^4} = k$ (k is a positive constant). This equation is equivalent to $x^2 + 2y^2 = K \;\Rightarrow$

$$\frac{x^2}{\left(\sqrt{K}\right)^2} + \frac{y^2}{\left(\sqrt{K/2}\right)^2} = 1,$$ where $K = -10^4 \ln k$, a family of ellipses. We sketch level curves for $K = 1, 2, 3,$ and 4. If the shark always swims in the direction of maximum increase of blood concentration, its direction at any point would coincide with the gradient vector. Then we know the shark's path is perpendicular to the level curves it intersects. We sketch one example of such a path.



(b) $\nabla C = -\frac{2}{10^4} e^{-\left(x^2+2y^2\right)/10^4}\,(x\,\mathbf{i} + 2y\,\mathbf{j})$. And ∇C points in the direction of most rapid increase in concentration; that is, ∇C is tangent to the most rapid increase curve. If $r(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ is a

parametrization of the most rapid increase curve, then $\dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j}$ is tangent to the curve, so $\dfrac{d\mathbf{r}}{dt} = \lambda \nabla C$ $\Rightarrow \dfrac{dx}{dt} = \lambda\left[-\dfrac{2}{10^4}e^{-(x^2+2y^2)/10^4}\right]x$ and $\dfrac{dy}{dt} = \lambda\left[-\dfrac{2}{10^4}e^{-(x^2+2y^2)/10^4}\right](2y)$. Therefore $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = 2\dfrac{y}{x} \Rightarrow \dfrac{dy}{y} = 2\dfrac{dx}{x} \Rightarrow \ln|y| = 2\ln|x|$ so that $y = kx^2$ for some constant k. But $y(x_0) = y_0 \Rightarrow y_0 = kx_0^2 \Rightarrow k = y_0/x_0^2$ ($x_0 = 0 \Rightarrow y_0 = 0 \Rightarrow$ the shark is already at the origin, so we can assume $x_0 \neq 0$.) Therefore the path the shark will follow is along the parabola $y = y_0(x/x_0)^2$.

3. (a) The area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height (the distance between the two parallel sides) and b_1, b_2 are the lengths of the bases (the parallel sides). From the figure in the text, we see that $h = x\sin\theta$, $b_1 = w - 2x$, and $b_2 = w - 2x + 2x\cos\theta$. Therefore the cross-sectional area of the rain gutter is

$$\begin{aligned} A(x,\theta) &= \tfrac{1}{2}x\sin\theta\,[(w-2x) + (w - 2x + 2x\cos\theta)] = (x\sin\theta)(w - 2x + x\cos\theta) \\ &= wx\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta,\ 0 < x \leq \tfrac{1}{2}w,\ 0 < \theta \leq \tfrac{\pi}{2} \end{aligned}$$

We look for the critical points of A: $\partial A/\partial x = w\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta$ and $\partial A/\partial\theta = wx\cos\theta - 2x^2\cos\theta + x^2(\cos^2\theta - \sin^2\theta)$, so $\partial A/\partial x = 0 \Leftrightarrow \sin\theta\,(w - 4x + 2x\cos\theta) = 0$ $\Leftrightarrow \cos\theta = \dfrac{4x - w}{2x} = 2 - \dfrac{w}{2x}$ ($0 < \theta \leq \frac{\pi}{2} \Rightarrow \sin\theta > 0$). If, in addition, $\partial A/\partial\theta = 0$, then

$$\begin{aligned} 0 &= wx\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) \\ &= wx\left(2 - \frac{w}{2x}\right) - 2x^2\left(2 - \frac{w}{2x}\right) + x^2\left[2\left(2 - \frac{w}{2x}\right)^2 - 1\right] \\ &= 2wx - \tfrac{1}{2}w^2 - 4x^2 + wx + x^2\left[8 - \frac{4w}{x} + \frac{w^2}{2x^2} - 1\right] = -wx + 3x^2 = x(3x - w) \end{aligned}$$

Since $x > 0$, we must have $x = \frac{1}{3}w$, in which case $\cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $k = \frac{\sqrt{3}}{6}w$, $b_1 = \frac{1}{3}w$, $b_2 = \frac{2}{3}w$, and $A = \frac{\sqrt{3}}{12}w^2$. As in Example 15.7.6 [ET 14.7.6], we can argue from the physical nature of this problem that we have found a relative maximum of A. Now checking the boundary of A, let $g(\theta) = A(w/2,\theta) = \frac{1}{2}w^2\sin\theta - \frac{1}{2}w^2\sin\theta + \frac{1}{4}w^2\sin\theta\cos\theta = \frac{1}{8}w^2\sin 2\theta$, $0 < \theta \leq \frac{\pi}{2}$. Clearly g is maximized when $\sin 2\theta = 1$ in which case $A = \frac{1}{8}w^2$. Also along the line $\theta = \frac{\pi}{2}$, let $h(x) = A\left(x, \frac{\pi}{2}\right) = wx - 2x^2$, $0 < x < \frac{1}{2}w \Rightarrow h'(x) = w - 4x = 0 \Leftrightarrow x = \frac{1}{4}w$, and $h\left(\frac{1}{4}w\right) = w\left(\frac{1}{4}w\right) - 2\left(\frac{1}{4}w\right)^2 = \frac{1}{8}w^2$. Since $\frac{1}{8}w^2 < \frac{\sqrt{3}}{12}w^2$, we conclude that the relative maximum found earlier was an absolute maximum.

(b) If the metal were bent into a semi-circular gutter of radius r, we would have $w = \pi r$ and $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{w}{\pi}\right)^2 = \dfrac{w^2}{2\pi}$. Since $\dfrac{w^2}{2\pi} > \dfrac{\sqrt{3}w^2}{12}$, it *would* be better to bend the metal into a gutter with a semicircular cross-section.

4. Since $(x+y+z)^r/\left(x^2+y^2+z^2\right)$ is a rational function with domain $\{(x,y,z) \mid (x,y,z) \neq (0,0,0)\}$, f is continuous on $\mathbb{R}^3$ if and only if $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(0,0,0) = 0$. Recall that $(a+b)^2 \leq 2a^2 + 2b^2$ and a double application of this inequality to $(x+y+z)^2$ gives $(x+y+z)^2 \leq 4x^2 + 4y^2 + 2z^2 \leq 4\left(x^2+y^2+z^2\right)$. Now for each r, $|(x+y+z)^r| = \left(|x+y+z|^2\right)^{r/2} = \left[(x+y+z)^2\right]^{r/2} \leq \left[4\left(x^2+y^2+z^2\right)\right]^{r/2} = 2^r\left(x^2+y^2+z^2\right)^{r/2}$ for $(x,y,z) \neq (0,0,0)$. Thus

$$|f(x,y,z) - 0| = \left|\frac{(x+y+z)^r}{x^2+y^2+z^2}\right| = \frac{|(x+y+z)^r|}{x^2+y^2+z^2} \leq 2^r\frac{\left(x^2+y^2+z^2\right)^{r/2}}{x^2+y^2+z^2} = 2^r\left(x^2+y^2+z^2\right)^{(r/2)-1} \text{ for}$$

$(x,y,z) \neq (0,0,0)$. Thus if $(r/2) - 1 > 0$, that is $r > 2$, then $2^r\left(x^2+y^2+z^2\right)^{(r/2)-1} \to 0$ as $(x,y,z) \to (0,0,0)$ and so $\lim\limits_{(x,y,z)\to(0,0,0)} (x+y+z)^r/\left(x^2+y^2+z^2\right) = 0$. Hence for $r > 2$, f is continuous on $\mathbb{R}^3$. Now if $r \leq 2$, then as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,0,0) = x^r/x^2 = x^{r-2}$ for $x \neq 0$. So when $r = 2$, $f(x,y,z) \to 1 \neq 0$ as $(x,y,z) \to (0,0,0)$ along the x-axis and when $r < 2$ the limit of $f(x,y,z)$ as $(x,y,z) \to (0,0,0)$ along the x-axis doesn't exist and thus can't be zero. Hence for $r \leq 2$ f isn't continuous at $(0,0,0)$ and thus is not continuous on $\mathbb{R}^3$.

5. Let $g(x,y) = xf\left(\dfrac{y}{x}\right)$. Then $g_x(x,y) = f\left(\dfrac{y}{x}\right) + xf'\left(\dfrac{y}{x}\right)\left(-\dfrac{y}{x^2}\right) = f\left(\dfrac{y}{x}\right) - \dfrac{y}{x}f'\left(\dfrac{y}{x}\right)$ and $g_y(x,y) = xf'\left(\dfrac{y}{x}\right)\left(\dfrac{1}{x}\right) = f'\left(\dfrac{y}{x}\right)$. Thus the tangent plane at (x_0, y_0, z_0) on the surface has equation

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left[f\left(\frac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) \quad \Rightarrow$$

$\left[f\left(\dfrac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\dfrac{y_0}{x_0}\right)\right]x + \left[f'\left(\dfrac{y_0}{x_0}\right)\right]y - z = 0$. But any plane whose equation is of the form $ax + by + cz = 0$ passes through the origin. Thus the origin is the common point of intersection.

6. (a) At $(x_1, y_1, 0)$ the equations of the tangent planes to $z = f(x,y)$ and $z = g(x,y)$ are P_1: $z - f(x_1,y_1) = f_x(x_1,y_1)(x - x_1) + f_y(x_1,y_1)(y - y_1)$ and P_2: $z - g(x_1,y_1) = g_x(x_1,y_1)(x - x_1) + g_y(x_1,y_1)(y - y_1)$, respectively. P_1 intersects the xy-plane in the line given by $f_x(x_1,y_1)(x - x_1) + f_y(x_1,y_1)(y - y_1) = -f(x_1,y_1)$, $z = 0$; and P_2 intersects the xy-plane in the line given by $g_x(x_1,y_1)(x - x_1) + g_y(x_1,y_1)(y - y_1) = -g(x_1,y_1)$, $z = 0$. The point $(x_2, y_2, 0)$ is the point of intersection of these two lines, since $(x_2, y_2, 0)$ is the point where the line of intersection of the two tangent planes intersects the xy-plane. Thus (x_2, y_2) is the solution of the simultaneous equations $f_x(x_1,y_1)(x_2 - x_1) + f_y(x_1,y_1)(y_2 - y_1) = -f(x_1,y_1)$ and $g_x(x_1,y_1)(x_2 - x_1) + g_y(x_1,y_1)(y_2 - y_1) = -g(x_1,y_1)$. For simplicity, rewrite $f_x(x_1,y_1)$ as f_x and similarly for f_y, g_x, g_y, f and g and solve the equations $(f_x)(x_2 - x_1) + (f_y)(y_2 - y_1) = -f$ and $(g_x)(x_2 - x_1) + (g_y)(y_2 - y_1) = -g$ simultaneously for $(x_2 - x_1)$ and $(y_2 - y_1)$. Then $y_2 - y_1 = \dfrac{gf_x - fg_x}{g_x f_y - f_x g_y}$ or $y_2 = y_1 - \dfrac{gf_x - fg_x}{f_x g_y - g_x f_y}$ and $(f_x)(x_2 - x_1) + \dfrac{(f_y)(gf_x - fg_x)}{g_x f_y - f_x g_y} = -f$ so

$$x_2 - x_1 = \frac{-f - [(f_y)(gf_x - fg_x)/(g_x f_y - f_x g_y)]}{f_x} = \frac{fg_y - f_y g}{g_x f_y - f_x g_y}.$$

Hence $x_2 = x_1 - \dfrac{fg_y - f_y g}{f_x g_y - g_x f_y}$.

(b) Let $f(x,y) = x^x + y^y - 1000$ and $g(x,y) = x^y + y^x - 100$. Then we wish to solve the system of equations $f(x,y) = 0$, $g(x,y) = 0$. Recall $\dfrac{d}{dx}[x^x] = x^x(1 + \ln x)$ (differentiate logarithmically), so $f_x(x,y) = x^x(1 + \ln x)$, $f_y(x,y) = y^y(1 + \ln y)$, $g_x(x,y) = yx^{y-1} + y^x \ln y$, and

$g_y(x,y) = x^y \ln x + xy^{x-1}$. Looking at the graph, we estimate the first point of intersection of the curves, and thus the solution to the system, to be approximately $(2.5, 4.5)$. Then following the method of part (a), $x_1 = 2.5$, $y_1 = 4.5$ and

$$x_2 = 2.5 - \frac{f(2.5,4.5)\,g_y(2.5,4.5) - f_y(2.5,4.5)\,g(2.5,4.5)}{f_x(2.5,4.5)\,g_y(2.5,4.5) - f_y(2.5,4.5)\,g_x(2.5,4.5)} \approx 2.447674117$$

$$y_2 = 4.5 - \frac{f_x(2.5,4.5)\,g(2.5,4.5) - f(2.5,4.5)\,g_x(2.5,4.5)}{f_x(2.5,4.5)\,g_y(2.5,4.5) - f_y(2.5,4.5)\,g_x(2.5,4.5)} \approx 4.555657467$$

Continuing this procedure, we arrive at the following values. (If you use a CAS, you may need to increase its computational precision.)

$x_1 = 2.5$	$y_1 = 4.5$
$x_2 = 2.447674117$	$y_2 = 4.555657467$
$x_3 = 2.449614877$	$y_3 = 4.551969333$
$x_4 = 2.449624628$	$y_4 = 4.551951420$
$x_5 = 2.449624628$	$y_5 = 4.551951420$

Thus, to six decimal places, the point of intersection is $(2.449625, 4.551951)$. The second point of intersection can be found similarly, or, by symmetry it is approximately $(4.551951, 2.449625)$.

7. (a) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z\partial x}\frac{\partial z}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z\partial y}\frac{\partial z}{\partial r}\right]\\ &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial y\partial x}\cos\theta\sin\theta\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}r\sin\theta + \dfrac{\partial u}{\partial y}r\cos\theta$ and

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\frac{\partial^2 u}{\partial y\partial x}r^2\sin\theta\cos\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta.$$ So

$$\begin{aligned}&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\\ &\quad= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial y\partial x}\cos\theta\sin\theta + \frac{\partial u}{\partial x}\frac{\cos\theta}{r} + \frac{\partial u}{\partial y}\frac{\sin\theta}{r}\\ &\qquad + \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial y\partial x}\sin\theta\cos\theta - \frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r} + \frac{\partial^2 u}{\partial z^2}\\ &\quad= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\end{aligned}$$

(b) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Then

$\dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \rho} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \rho} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial \rho} = \dfrac{\partial u}{\partial x}\sin\phi\cos\theta + \dfrac{\partial u}{\partial y}\sin\phi\sin\theta + \dfrac{\partial u}{\partial z}\cos\phi$, and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \rho^2} &= \sin\phi\cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial z\partial x}\frac{\partial z}{\partial \rho}\right] \\
&\qquad + \sin\phi\sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial z\partial y}\frac{\partial z}{\partial \rho}\right] \\
&\qquad + \cos\phi\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2 u}{\partial x\partial z}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\partial z}\frac{\partial y}{\partial \rho}\right] \\
&= 2\frac{\partial^2 u}{\partial y\partial x}\sin^2\phi\sin\theta\cos\theta + 2\frac{\partial^2 u}{\partial z\partial x}\sin\phi\cos\phi\cos\theta + 2\frac{\partial^2 u}{\partial y\partial z}\sin\phi\cos\phi\sin\theta \\
&\qquad + \frac{\partial^2 u}{\partial x^2}\sin^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\phi\sin^2\theta + \frac{\partial^2 u}{\partial z^2}\cos^2\phi
\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \phi} = \dfrac{\partial u}{\partial x}\rho\cos\phi\cos\theta + \dfrac{\partial u}{\partial y}\rho\cos\phi\sin\theta - \dfrac{\partial u}{\partial z}\rho\sin\phi$, and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \phi^2} &= 2\frac{\partial^2 u}{\partial y\partial x}\rho^2\cos^2\phi\sin\theta\cos\theta - 2\frac{\partial^2 u}{\partial x\partial z}\rho^2\sin\phi\cos\phi\cos\theta \\
&\qquad - 2\frac{\partial^2 u}{\partial y\partial z}\rho^2\sin\phi\cos\phi\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\cos^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\rho^2\cos^2\phi\sin^2\theta \\
&\qquad + \frac{\partial^2 u}{\partial z^2}\rho^2\sin^2\phi - \frac{\partial u}{\partial x}\rho\sin\phi\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\sin\theta - \frac{\partial u}{\partial z}\rho\cos\phi
\end{aligned}$$

And $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}\rho\sin\phi\sin\theta + \dfrac{\partial u}{\partial y}\rho\sin\phi\cos\theta$, while

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= -2\frac{\partial^2 u}{\partial y\partial x}\rho^2\sin^2\phi\cos\theta\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\sin^2\phi\sin^2\theta \\
&\qquad + \frac{\partial^2 u}{\partial y^2}\rho^2\sin^2\phi\cos^2\theta - \frac{\partial u}{\partial x}\rho\sin\phi\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\sin\theta
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} \\
&\quad = \frac{\partial^2 u}{\partial x^2}\left[\left(\sin^2\phi\cos^2\theta\right) + \left(\cos^2\phi\cos^2\theta\right) + \sin^2\theta\right] \\
&\qquad + \frac{\partial^2 u}{\partial y^2}\left[\left(\sin^2\phi\sin^2\theta\right) + \left(\cos^2\phi\sin^2\theta\right) + \cos^2\theta\right] + \frac{\partial^2 u}{\partial z^2}\left[\cos^2\phi + \sin^2\phi\right] \\
&\qquad + \frac{\partial u}{\partial x}\left[\frac{2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta}{\rho\sin\phi}\right] \\
&\qquad + \frac{\partial u}{\partial y}\left[\frac{2\sin^2\phi\sin\theta + \cos^2\phi\sin\theta - \sin^2\phi\sin\theta - \sin\theta}{\rho\sin\phi}\right]
\end{aligned}$$

But $2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta = \left(\sin^2\phi + \cos^2\phi - 1\right)\cos\theta = 0$ and similarly the coefficient of $\partial u/\partial y$ is 0. Also $\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\theta = \cos^2\theta\left(\sin^2\phi + \cos^2\phi\right) + \sin^2\theta = 1$, and similarly the coefficient of $\partial^2 u/\partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

8. The tangent plane to the surface $xy^2z^2 = 1$, at the point (x_0, y_0, z_0) is $y_0^2z_0^2\,(x - x_0) + 2x_0y_0z_0^2\,(y - y_0) + 2x_0y_0^2z_0\,(z - z_0) = 0 \;\Rightarrow$ $\left(y_0^2z_0^2\right)x + \left(2x_0y_0z_0^2\right)y + \left(2x_0y_0^2z_0\right)z = 5x_0y_0^2z_0^2 = 5$. Using the formula derived in Example 13.5.8 [ET 12.5.8], we find that the distance from $(0,0,0)$ to this tangent plane is

$D\,(x_0, y_0, z_0) = \dfrac{\left|5x_0y_0^2z_0^2\right|}{\sqrt{\left(y_0^2z_0^2\right)^2 + \left(2x_0y_0z_0^2\right)^2 + \left(2x_0y_0^2z_0\right)^2}}$. When D is a maximum, D^2 is a maximum and

$\nabla D^2 = \mathbf{0}$. Dropping the subscripts, let $f\,(x,y,z) = D^2 = \dfrac{25\,(xyz)^2}{y^2z^2 + 4x^2z^2 + 4x^2y^2}$. Now

use the fact that for points on the surface $xy^2z^2 = 1$ we have $z^2 = \dfrac{1}{xy^2}$, to get

$f\,(x,y) = D^2 = \dfrac{25x}{\dfrac{1}{x} + \dfrac{4x}{y^2} + 4x^2y^2} = \dfrac{25x^2y^2}{y^2 + 4x^2 + 4x^3y^4}$. Now $\nabla D^2 = \mathbf{0} \;\Rightarrow\; f_x = 0$ and $f_y = 0$.

$f_x = 0 \;\Rightarrow\; \dfrac{50xy^2\left(y^2 + 4x^2 + 4x^3y^4\right) - \left(8x + 12x^2y^4\right)\left(25x^2y^2\right)}{\left(y^2 + 4x^2 + 4x^3y^4\right)^2} = 0 \;\Rightarrow$

$xy^2\left(y^2 + 4x^2 + 4x^3y^4\right) - \left(4x + 6x^2y^4\right)x^2y^2 = 0 \;\Rightarrow\; xy^4 - 2x^4y^6 = 0 \;\Rightarrow$

$xy^4\left(1 - 2x^3y^2\right) = 0 \;\Rightarrow\; 1 = 2y^2x^3$ (since $x = 0$, $y = 0$ both give a minimum distance of 0). Also $f_y = 0$

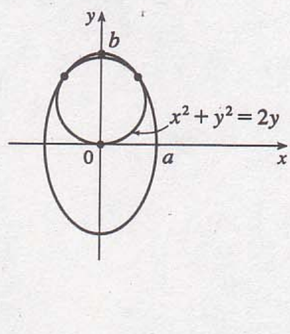
$\Rightarrow\; \dfrac{50x^2y\left(y^2 + 4x^2 + 4x^3y^4\right) - \left(2y + 16x^3y^3\right)25x^2y^2}{\left(y^2 + 4x^2 + 4x^3y^4\right)^2} = 0 \;\Rightarrow\; 4x^4y - 4x^5y^5 = 0 \;\Rightarrow$

$x^4y\left(1 - xy^4\right) = 0 \;\Rightarrow\; 1 = xy^4$. Now substituting $x = 1/y^4$ into $1 = 2y^2x^3$, we get $1 = 2y^{-10} \;\Rightarrow$

$y = \pm 2^{1/10} \;\Rightarrow\; x = 2^{-2/5} \;\Rightarrow\; z^2 = \dfrac{1}{xy^2} = \dfrac{1}{\left(2^{-2/5}\right)\left(2^{1/5}\right)} = 2^{1/5} \;\Rightarrow\; z = \pm 2^{1/10}$.

Therefore the tangent planes that are farthest from the origin are at the four points $\left(2^{-2/5}, \pm 2^{1/10}, \pm 2^{1/10}\right)$. These points all give a maximum since the minimum distance occurs when $x_0 = 0$ or $y_0 = 0$ in which case $D = 0$. The equations are $\left(2^{1/5}2^{1/5}\right)x \pm \left[(2)\left(2^{-2/5}\right)\left(2^{1/10}\right)\left(2^{1/5}\right)\right]y \pm \left[(2)\left(2^{-2/5}\right)\left(2^{1/5}\right)\left(2^{1/10}\right)\right]z = 5 \;\Rightarrow$ $\left(2^{2/5}\right)x \pm \left(2^{9/10}\right)y \pm \left(2^{9/10}\right)z = 5$.

9. Since we are minimizing the area of the ellipse, and the circle lies above the x-axis, the ellipse will intersect the circle for only one value of y. This y-value must satisfy both the equation of the circle and the equation of the ellipse. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; x^2 = \dfrac{a^2}{b^2}\left(b^2 - y^2\right)$. Substituting into the equation of the circle gives $\dfrac{a^2}{b^2}\left(b^2 - y^2\right) + y^2 - 2y = 0 \;\Rightarrow$



$\left(\dfrac{b^2 - a^2}{b^2}\right) y^2 - 2y + a^2 = 0$. In order for there to be only one solution to this quadratic equation, the discriminant must be 0, so $4 - 4a^2 \dfrac{b^2 - a^2}{b^2} = 0 \quad \Rightarrow \quad b^2 - a^2b^2 + a^4 = 0$. The area of the ellipse is $A\,(a, b) = \pi ab$, and we minimize this function subject to the constraint $g\,(a, b) = b^2 - a^2b^2 + a^4 = 0$.

Now $\nabla A = \lambda \nabla g \quad \Leftrightarrow \quad \pi b = \lambda\left(4a^3 - 2ab^2\right), \pi a = \lambda\left(2b - 2ba^2\right) \quad \Rightarrow \quad$ (1) $\lambda = \dfrac{\pi b}{2a\,(2a^2 - b^2)}$, (2) $\lambda = \dfrac{\pi a}{2b\,(1 - a^2)}$, (3) $b^2 - a^2b^2 + a^4 = 0$. Comparing (1) and (2) gives $\dfrac{\pi b}{2a\,(2a^2 - b^2)} = \dfrac{\pi a}{2b\,(1 - a^2)} \quad \Rightarrow$ $2\pi b^2 = 4\pi a^4 \quad \Leftrightarrow \quad a^2 = \frac{1}{\sqrt{2}} b$. Substitute this into (3) to get $b = \frac{3}{\sqrt{2}} \quad \Rightarrow \quad a = \sqrt{\frac{3}{2}}$.

16 Multiple Integrals ET 15

16.1 Double Integrals over Rectangles ET 15.1

1. (a) $\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = f\left(0,\frac{3}{2}\right)\Delta A + f(0,2)\,\Delta A + f\left(1,\frac{3}{2}\right)\Delta A + f(1,2)\,\Delta A$

$$= \left(-\tfrac{27}{4}\right)\tfrac{1}{2} + (-12)\,\tfrac{1}{2} + \left(1-\tfrac{27}{4}\right)\tfrac{1}{2} + (1-12)\,\tfrac{1}{2} = -17.75$$

(b) $\frac{1}{2}\left[f\left(1,\frac{3}{2}\right) + f(1,2) + f\left(2,\frac{3}{2}\right) + f(2,2)\right] = \frac{1}{2}\left[-\frac{23}{4} + (-11) + \left(-\frac{19}{4}\right) + (-10)\right]$

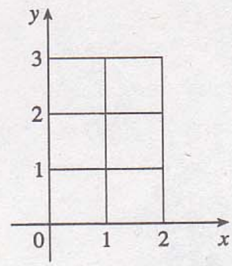
$$= \tfrac{1}{2}\left(-\tfrac{63}{2}\right) = -15.75$$

(c) $\frac{1}{2}\left[f(0,1) + f\left(0,\frac{3}{2}\right) + f(1,1) + f\left(1,\frac{3}{2}\right)\right] = \frac{1}{2}\left[-3 - \frac{27}{4} - 2 - \frac{23}{4}\right] = -8.75$

(d) $\frac{1}{2}\left[f(1,1) + f\left(1,\frac{3}{2}\right) + f(2,1) + f\left(2,\frac{3}{2}\right)\right] = \frac{1}{2}\left[-2 - \frac{23}{4} - 1 - \frac{19}{4}\right] = -6.75$

2. $V \approx (1)\left[f\left(\frac{1}{2},\frac{1}{2}\right) + f\left(\frac{1}{2},\frac{3}{2}\right) + f\left(\frac{3}{2},\frac{1}{2}\right) + f\left(\frac{3}{2},\frac{3}{2}\right)\right] = \left[\frac{61}{4} + \frac{45}{4} + \frac{53}{4} + \frac{37}{4}\right] = 49$

3. (a) The subrectangles are shown in the figure. The surface is the graph of $f(x,y) = x^2 + 4y$ and $\Delta A = 1$, so we estimate

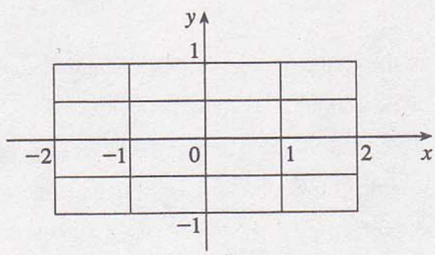


$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{3} f(x_i, y_j)\,\Delta A \\ &= f(1,1)\,\Delta A + f(1,2)\,\Delta A + f(1,3)\,\Delta A \\ &\qquad + f(2,1)\,\Delta A + f(2,2)\,\Delta A + f(2,3)\,\Delta A \\ &= 5\,(1) + 9\,(1) + 13\,(1) + 8\,(1) + 12\,(1) + 16\,(1) = 63 \end{aligned}$$

(b) $V \approx \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{3} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$

$$\begin{aligned} &= f\left(\tfrac{1}{2},\tfrac{1}{2}\right)\Delta A + f\left(\tfrac{1}{2},\tfrac{3}{2}\right)\Delta A + f\left(\tfrac{1}{2},\tfrac{5}{2}\right)\Delta A \\ &\qquad + f\left(\tfrac{3}{2},\tfrac{1}{2}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{3}{2}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{5}{2}\right)\Delta A \\ &= \tfrac{9}{4}\,(1) + \tfrac{25}{4}\,(1) + \tfrac{41}{4}\,(1) + \tfrac{17}{4}\,(1) + \tfrac{33}{4}\,(1) + \tfrac{49}{4}\,(1) = \tfrac{87}{2} = 43.5 \end{aligned}$$

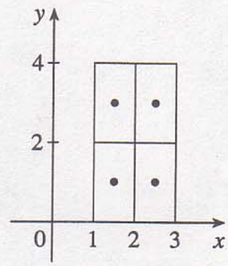
4. The subrectangles are shown in the figure.



Since $\Delta A = \frac{1}{2}$, we estimate

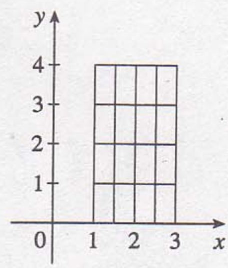
$$\begin{aligned}\iint_R \left(2x + x^2 y\right) dA &\approx \sum_{i=1}^{4} \sum_{j=1}^{4} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A \\ &= \tfrac{1}{2}\left[f(-2,-1) + f\left(-2,-\tfrac{1}{2}\right) + f(-2,0) + f\left(-2,\tfrac{1}{2}\right) + f(-1,-1)\right. \\ &\qquad + f\left(-1,-\tfrac{1}{2}\right) + f(-1,0) + f\left(-1,\tfrac{1}{2}\right) + f(0,-1) + f\left(0,-\tfrac{1}{2}\right) \\ &\qquad \left. + f(0,0) + f\left(0,\tfrac{1}{2}\right) + f(1,-1) + f\left(1,-\tfrac{1}{2}\right) + f(1,0) + f\left(1,\tfrac{1}{2}\right)\right] \\ &= \tfrac{1}{2}(-22) = -11\end{aligned}$$

5. (a) Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A = 2$, so we evaluate f at each midpoint and estimate



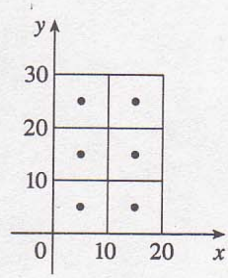
$$\begin{aligned}\textstyle\iint_R f(x,y)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A \\ &= f(1.5,1)\,\Delta A + f(1.5,3)\,\Delta A \\ &\qquad + f(2.5,1)\,\Delta A + f(2.5,3)\,\Delta A \\ &= 1(2) + (-8)(2) + 5(2) + (-1)(2) = -6\end{aligned}$$

(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus we estimate



$$\begin{aligned}\textstyle\iint_R f(x,y)\, dA &\approx \sum_{i=1}^{4} \sum_{j=1}^{4} f\left(x_i, y_j\right) \Delta A \\ &= f(1.5,1)\,\Delta A + f(1.5,2)\,\Delta A + f(1.5,3)\,\Delta A + f(1.5,4)\,\Delta A \\ &\quad + f(2,1)\,\Delta A + f(2,2)\,\Delta A + f(2,3)\,\Delta A + f(2,4)\,\Delta A \\ &\quad + f(2.5,1)\,\Delta A + f(2.5,2)\,\Delta A + f(2.5,3)\,\Delta A + f(2.5,4)\,\Delta A \\ &\quad + f(3,1)\,\Delta A + f(3,2)\,\Delta A + f(3,3)\,\Delta A + f(3,4)\,\Delta A \\ &= 1\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) + (-6)\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) + (-5)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) \\ &\quad + 5\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + 8\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) \\ &= -3.5\end{aligned}$$

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x,y)$ to be the depth of the water at (x,y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0,20] \times [0,30]$ and below the graph of $f(x,y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$.



Each subrectangle with its midpoint is shown in the figure. Then

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{3} f(\overline{x}_i,\overline{y}_j)\,\Delta A \\ &= \Delta A\,[f(5,5)+f(5,15)+f(5,25)+f(15,5)+f(15,15)+f(15,25)] \\ &= 100\,(3+7+10+3+5+8) = 3600 \end{aligned}$$

Thus, we estimate that the pool contains 3600 cubic feet of water.
Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$\begin{aligned} V &\approx \sum_{i=1}^{4}\sum_{j=1}^{6} f(x_i,y_j)\,\Delta A \\ &= 25\,[3+4+7+8+10+8+4+6+8+10+12+10+3+4 \\ &\qquad +5+6+8+7+2+2+2+3+4+4] \\ &= 25\,(140) = 3500 \end{aligned}$$

So we estimate that the pool contains 3500 ft^3 of water.

7. The values of $f(x,y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

8. From the level curves we see that $f\left(\frac{1}{2},\frac{1}{2}\right) \approx 11$. So, using the Midpoint Rule with only one subrectangle, we get $\iint_R f(x,y)\,dA \approx 1 \cdot f\left(\frac{1}{2},\frac{1}{2}\right) \approx 11$. Dividing R into four squares of equal size, we get $\iint_R f(x,y)\,dA \approx \frac{1}{4}\left[f\left(\frac{1}{4},\frac{1}{4}\right) + f\left(\frac{1}{4},\frac{3}{4}\right) + f\left(\frac{3}{4},\frac{1}{4}\right) + f\left(\frac{3}{4},\frac{3}{4}\right)\right] \approx \frac{1}{4}\,(11+13+9.5+11) \approx 11$. Using sixteen squares we get the same result. So $\iint_R f(x,y)\,dA \approx 11$.

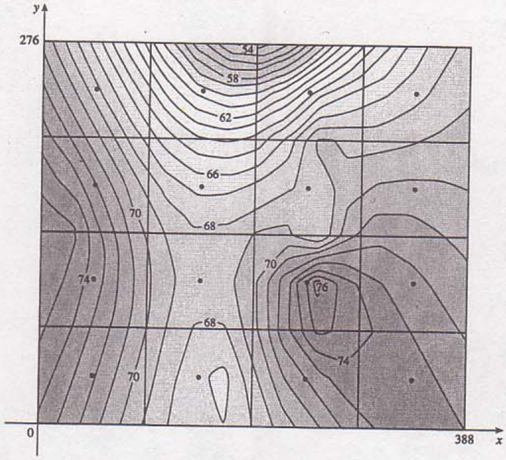
9. With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

$$\begin{aligned} \iint_R f(x,y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i,\overline{y}_j)\,\Delta A = \Delta A\,[f(1,1)+f(1,3)+f(3,1)+f(3,3)] \\ &\approx 4\,(27+4+14+17) = 248 \end{aligned}$$

10. As in Example 4, we place the origin at the southwest corner of the state. Then $R = [0,388] \times [0,276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x,y)$ to be the temperature at the location (x,y). The average temperature is given by

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{A(R)}\iint_R f(x,y)\,dA \\ &= \frac{1}{388 \cdot 276}\int_0^{276}\int_0^{388} f(x,y)\,dx\,dy \end{aligned}$$

We can use the Midpoint Rule with $m = n = 4$ to give a reasonable estimate of the value of the double integral.



Thus, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated. The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\begin{aligned}\int_0^{276}\int_0^{388} f(x,y)\,dx\,dy &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(\overline{x}_i,\overline{y}_j)\,\Delta A\\ &\approx \Delta A\,[72.2+73.6+72.1+68.2+67.4+68.5+66.7+60.3\\ &\qquad +72.0+74.9+68.4+63.7+73.2+72.3+70.3+67.7]\\ &= 6693\,(1111.5)\end{aligned}$$

Therefore, $f_{\text{ave}} \approx \dfrac{6693 \cdot 1111.5}{388 \cdot 276} \approx 69.5$, so the average temperature in Colorado on May 1, 1996, was approximately 69.5 °F.

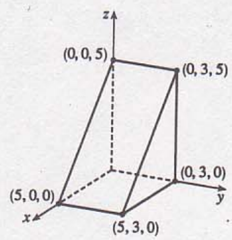
Alternatively, we can use the Midpoint Rule with $m = n = 2$ which is easier computationally but will most likely be less accurate since we have fewer subrectangles. In this case, $\Delta A = \frac{388}{2} \cdot \frac{276}{2} = 26{,}772$ and we can use the same grid to estimate the function values at the midpoints of the four subrectangles. Then

$$\begin{aligned}\int_0^{276}\int_0^{388} f(x,y)\,dx\,dy &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i,\overline{y}_j)\,\Delta A \approx 26{,}772\,[70.0+66.5+74.3+68.5]\\ &= 26{,}772 \cdot 279.3\end{aligned}$$

and $f_{\text{ave}} \approx \dfrac{26{,}772 \cdot 279.3}{388 \cdot 276} \approx 69.8$ °F.

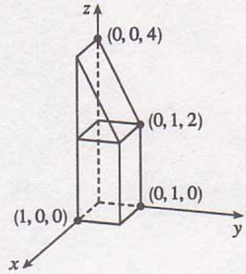
11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2,2] \times [1,6]$. S is a rectangular solid, thus $\iint_R 3\,dA = 4 \cdot 5 \cdot 3 = 60$.

12. $z = 5 - x \geq 0$ for $0 \leq x \leq 5$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 5 - x$ and above the rectangle $[0,5] \times [0,3]$. S is a triangular cylinder whose volume is 3 (area of triangle) $= 3\left(\frac{1}{2} \cdot 5 \cdot 5\right) = 37.5$. Thus, $\iint_R (5 - x)\,dA = 37.5$.

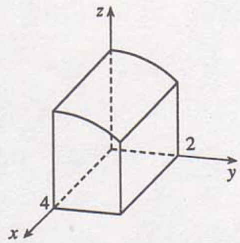


13. $z = f(x, y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0, 1] \times [0, 1] \times [0, 4]$ which lies below the plane $z = 4 - 2$. So

$$\iint_R (4 - y)\, dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



14. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x, y) = e^{-x^2 - y^2}$ (calling it `f`), load the `student` package, and then use the command

```
middlesum(middlesum(f,x=0..1,m),
                              y=0..1,m);
```

to get the estimate with $n = m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define `f` and then use nested `Sum` commands to calculate the estimates.

n	estimate
1	0.6065
4	0.5694
16	0.5606
64	0.5585
256	0.5579
1024	0.5578

16.

n	estimate
1	0.9922
4	0.9262
16	0.8797

n	estimate
64	0.8660
256	0.8625
1024	0.8616

17. If we divide R into mn subrectangles, $\iint_R k\, dA \approx \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$ for any choice of sample points $\left(x_{ij}^*, y_{ij}^*\right)$. But $f\left(x_{ij}^*, y_{ij}^*\right) = k$ always and $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \Delta A =$ area of $R = (b - a)(d - c)$. Thus, no matter how we choose the sample points, $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = k \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \Delta A = k(b - a)(d - c)$ and so

$$\begin{aligned} \iint_R k\, dA &= \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \lim_{m,n \to \infty} k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A \\ &= \lim_{m,n \to \infty} k(b - a)(d - c) = k(b - a)(d - c) \end{aligned}$$

18. On R, $0 \le x + y \le 2 < \pi$ and $\sin\theta \ge 0$ for $0 \le \theta \le \pi$. Thus $f(x, y) = \sin(x + y) \ge 0$ for all $(x, y) \in R$. Since $0 \le \sin(x + y) \le 1$, Property (9) gives $\iint_R 0\,dA \le \iint_R \sin(x + y)\,dA \le \iint_R 1\,dA$, so by Exercise 17 we have $0 \le \iint_R \sin(x + y)\,dA \le 1$.

16.2 Iterated Integrals ET 15.2

1. $\int_0^3 (2x + 3x^2y)\,dx = \left[x^2 + x^3y\right]_{x=0}^{x=3} = (9 + 27y) - (0 + 0) = 9 + 27y$,

$\int_0^4 (2x + 3x^2y)\,dy = \left[2xy + 3x^2\dfrac{y^2}{2}\right]_{y=0}^{y=4} = \left(8x + 3x^2 \cdot \dfrac{16}{2}\right) - (0 + 0) = 8x + 24x^2$

2. $\displaystyle\int_0^3 \frac{y}{x+2}\,dx = y\ln|x + 2|\big]_{x=0}^{x=3} = y\ln 5 - y\ln 2 = y\ln\frac{5}{2}$,

$\displaystyle\int_0^4 \frac{y}{x+2}\,dy = \frac{1}{x+2}\left[\frac{y^2}{2}\right]_{y=0}^{y=4} = \frac{1}{x+2}\left(\frac{16}{2} - 0\right) = \frac{8}{x+2}$

3. $\int_1^3 \int_0^1 (1 + 4xy)\,dx\,dy = \int_1^3 \left[x + 2x^2y\right]_{x=0}^{x=1} dy = \int_1^3 (1 + 2y)\,dy$

$= \left[y + y^2\right]_1^3 = (3 + 9) - (1 + 1) = 10$

4. $\int_2^4 \int_{-1}^1 (x^2 + y^2)\,dy\,dx = \int_2^4 \left[x^2y + \frac{1}{3}y^3\right]_{y=-1}^{y=1} dx = \int_2^4 \left[\left(x^2 + \frac{1}{3}\right) - \left(-x^2 - \frac{1}{3}\right)\right] dx$

$= \int_2^4 \left(2x^2 + \frac{2}{3}\right) dx = \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_2^4 = \left(\frac{128}{3} + \frac{8}{3}\right) - \left(\frac{16}{3} + \frac{4}{3}\right) = \frac{116}{3}$

5. $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y\,dy\,dx = \int_0^{\pi/2} \sin x\,dx \int_0^{\pi/2} \cos y\,dy$ (as in Example 5)

$= \left[-\cos x\right]_0^{\pi/2} \left[\sin y\right]_0^{\pi/2} = -(0 - 1)(1 - 0) = 1$

6. $\int_1^4 \int_0^2 (x + \sqrt{y})\,dx\,dy = \int_1^4 \left[\frac{1}{2}x^2 + x\sqrt{y}\right]_{x=0}^{x=2} dy = \int_1^4 (2 + 2\sqrt{y})\,dy$

$= \left[2y + 2 \cdot \frac{2}{3}y^{3/2}\right]_1^4 = \left(8 + \frac{4}{3} \cdot 8\right) - \left(2 + \frac{4}{3}\right) = \frac{46}{3}$

7. $\int_0^3 \int_0^1 \sqrt{x + y}\,dx\,dy = \int_0^3 \left[\frac{2}{3}(x + y)^{3/2}\right]_{x=0}^{x=1} dy = \frac{2}{3}\int_0^3 \left[(1 + y)^{3/2} - y^{3/2}\right] dy$

$= \frac{2}{3}\left[\frac{2}{5}(1 + y)^{5/2} - \frac{2}{5}y^{5/2}\right]_0^3 = \frac{4}{15}\left[32 - 3^{5/2} - 1\right] = \frac{4}{15}\left(31 - 9\sqrt{3}\right)$

8. $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y)\,dy\,dx = \int_0^{\pi/2} \left[-\cos(x + y)\right]_{y=0}^{y=\pi/2} dx = \int_0^{\pi/2} \left[\cos x - \cos\left(x + \frac{\pi}{2}\right)\right] dx$

$= \left[\sin x - \sin\left(x + \frac{\pi}{2}\right)\right]_0^{\pi/2} = (1 - 0) - (0 - 1) = 2$

9. $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx = \int_1^4 \left[x\ln|y| + \frac{1}{x} \cdot \frac{1}{2}y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x\ln 2 + \frac{3}{2x}\right) dx$

$= \left[\frac{1}{2}x^2\ln 2 + \frac{3}{2}\ln|x|\right]_1^4 = 8\ln 2 + \frac{3}{2}\ln 4 - \frac{1}{2}\ln 2$

$= \frac{15}{2}\ln 2 + 3\ln 4^{1/2} = \frac{21}{2}\ln 2$

10. $\int_1^2 \int_0^1 (x + y)^{-2}\,dx\,dy = \int_1^2 \left[-(x + y)^{-1}\right]_{x=0}^{x=1} dy = \int_1^2 \left[y^{-1} - (1 + y)^{-1}\right] dy$

$= \left[\ln y - \ln(1 + y)\right]_1^2 = \ln 2 - \ln 3 - 0 + \ln 2 = \ln\frac{4}{3}$

11. $\int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,dx\,dy = \left(\int_0^{\ln 5} e^{2x}\,dx\right)\left(\int_0^{\ln 2} e^{-y}\,dy\right) = \left[\frac{1}{2}e^{2x}\right]_0^{\ln 5}\left[-e^{-y}\right]_0^{\ln 2}$

$= \left(\frac{25}{2} - \frac{1}{2}\right)\left(-\frac{1}{2}+1\right) = 6$

12. $\displaystyle\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx = \int_0^1 \left[x\sqrt{x^2+y^2+1}\right]_{y=0}^{y=1} dx = \int_0^1 x\left(\sqrt{x^2+2}-\sqrt{x^2+1}\right)dx$

$= \frac{1}{3}\left[\left(x^2+2\right)^{3/2} - \left(x^2+1\right)^{3/2}\right]_0^1 = \frac{1}{3}\left[\left(3^{3/2}-2^{3/2}\right)-\left(2^{3/2}-1\right)\right]$

$= \frac{1}{3}\left(3\sqrt{3}-4\sqrt{2}+1\right)$

13. $\iint_R \left(6x^2y^3-5y^4\right)dA = \int_0^3\int_0^1 \left(6x^2y^3-5y^4\right)dy\,dx = \int_0^3 \left[\frac{3}{2}x^2y^4 - y^5\right]_{y=0}^{y=1} dx$

$= \int_0^3 \left(\frac{3}{2}x^2-1\right)dx = \left[\frac{1}{2}x^3 - x\right]_0^3 = \frac{27}{2} - 3 = \frac{21}{2}$

14. $\iint_R xye^y\,dA = \int_0^2\int_0^1 xye^y\,dy\,dx = \int_0^2 x\,dx\int_0^1 ye^y\,dy = \left[\frac{1}{2}x^2\right]_0^2\left[e^y\left(y-1\right)\right]_0^1$ (by integrating by parts)

$= \frac{1}{2}\left(4-0\right)\left(0+e^0\right) = 2$

15. $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1}\,dx\int_{-3}^3 y^2\,dy$

$= \left[\frac{1}{2}\ln\left(x^2+1\right)\right]_0^1\left[\frac{1}{3}y^3\right]_{-3}^3 = \frac{1}{2}\left(\ln 2 - \ln 1\right)\cdot\frac{1}{3}\left(27+27\right) = 9\ln 2$

16. $\displaystyle\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx = \int_0^1 \left(1+x^2\right)dx\int_0^1 \frac{1}{1+y^2}\,dy$

$= \left[x+\frac{1}{3}x^3\right]_0^1\left[\tan^{-1}y\right]_0^1 = \left(1+\frac{1}{3}-0\right)\left(\frac{\pi}{4}-0\right) = \frac{\pi}{3}$

17. $\int_0^{\pi/6}\int_0^{\pi/3} x\sin\left(x+y\right)dy\,dx$

$= \int_0^{\pi/6}\left[-x\cos\left(x+y\right)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6}\left[x\cos x - x\cos\left(x+\frac{\pi}{3}\right)\right]dx$

$= x\left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}\left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right]dx$

(by integrating by parts separately for each term)

$= \frac{\pi}{6}\left[\frac{1}{2}-1\right] - \left[-\cos x + \cos\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2}+0-\left(-1+\frac{1}{2}\right)\right]$

$= \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

18. $\int_0^1\int_0^1 xe^{xy}\,dy\,dx = \int_0^1\left[e^{xy}\right]_{y=0}^{y=1} dx = \int_0^1\left(e^x-1\right)dx = \left[e^x - x\right]_0^1 = e-2$

19. $\displaystyle\int_0^1\int_1^2 \frac{1}{x+y}\,dx\,dy = \int_0^1\left[\ln\left(x+y\right)\right]_{x=1}^{x=2} dy = \int_0^1\left[\ln\left(2+y\right)-\ln\left(1+y\right)\right]dy$

$= \Big[\left[\left(2+y\right)\ln\left(2+y\right)-\left(2+y\right)\right] - \left[\left(1+y\right)\ln\left(1+y\right)-\left(1+y\right)\right]\Big]_0^1$

(by integrating by parts separately for each term or by the Table of Integrals)

$= \left(3\ln 3\right) - 3 - \left(2\ln 2\right) + 2 - \left[\left(2\ln 2 - 2\right) - \left(0-1\right)\right] = 3\ln 3 - 4\ln 2 = \ln\frac{27}{16}$

20. $\int_0^1 \int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \int_0^1 \left[\frac{1}{2}\ln\left(x^2+y^2\right)\right]_{x=1}^{x=2} dy = \frac{1}{2}\int_0^1 \left[\ln\left(4+y^2\right) - \ln\left(1+y^2\right)\right] dy$

To evaluate the first term, we integrate by parts with $u = \ln\left(4+y^2\right) \Rightarrow du = \frac{2y}{4+y^2}\,dy$ and $dv = dy \Rightarrow v = y$. Then

$$\begin{aligned}\int \ln\left(4+y^2\right) dy &= y\ln\left(4+y^2\right) - \int \frac{2y^2}{4+y^2}\,dy = y\ln\left(4+y^2\right) - \int \left(2 - \frac{8}{4+y^2}\right) dy \\ &= y\ln\left(4+y^2\right) - 2y + 8\cdot\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = y\ln\left(4+y^2\right) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right)\end{aligned}$$

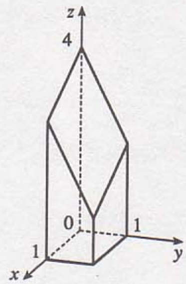
Similarly,

$$\int \ln\left(1+y^2\right) dy = y\ln\left(1+y^2\right) - 2y + 2\tan^{-1} y$$

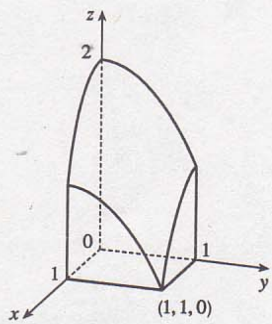
Thus,

$$\begin{aligned}\int_0^1 \int_1^2 \frac{x}{x^2+y^2}\,dx\,dy &= \frac{1}{2}\int_0^1 \left[\ln\left(4+y^2\right) - \ln\left(1+y^2\right)\right] dy \\ &= \frac{1}{2}\left[y\ln\left(4+y^2\right) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right) - y\ln\left(1+y^2\right) + 2y - 2\tan^{-1} y\right]_0^1 \\ &= \frac{1}{2}\left[\left(\ln 5 + 4\tan^{-1}\frac{1}{2} - \ln 2 - 2\tan^{-1} 1\right) - 0\right] \\ &= \frac{1}{2}\left[\ln 5 - \ln 2 + 4\tan^{-1}\frac{1}{2} - 2\left(\frac{\pi}{4}\right)\right] = \frac{1}{2}\ln\frac{5}{2} + 2\tan^{-1}\frac{1}{2} - \frac{\pi}{4}\end{aligned}$$

21. $z = f(x,y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1] \times [0,1]$.



22. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0,1] \times [0,1]$.



23. $V = \iint_R (2x + 5y + 1)\,dA = \int_1^4 \int_{-1}^0 (2x + 5y + 1)\,dx\,dy = \int_1^4 \left[x^2 + 5xy + x\right]_{x=-1}^{x=0} dy$
$= \int_1^4 5y\,dy = \frac{5}{2}y^2\Big]_1^4 = \frac{75}{2}$

24. $V = \iint_R (x^2+y^2)\,dA = \int_{-3}^{3}\int_{-2}^{2}(x^2+y^2)\,dx\,dy = \int_{-3}^{3}\left[\frac{1}{3}x^3 + y^2x\right]_{x=-2}^{x=2} dy$

$= \int_{-3}^{3}\left[\frac{16}{3} + 4y^2\right] dy = \left[\frac{16}{3}y + \frac{4}{3}y^3\right]_{-3}^{3} = 2\,(16+36) = 104$

25. $V = \int_{-2}^{2}\int_{-1}^{1}\left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\,dy = 4\int_0^2\int_0^1\left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\,dy$

$= 4\int_0^2\left[x - \frac{1}{12}x^3 - \frac{1}{9}y^2x\right]_{x=0}^{x=1} dy = 4\int_0^2\left(\frac{11}{12} - \frac{1}{9}y^2\right) dy = 4\left[\frac{11}{12}y - \frac{1}{27}y^3\right]_0^2 = 4\cdot\frac{83}{54} = \frac{166}{27}$

26. $V = \int_1^3\int_{-1}^{1}(y^2 - x^2)\,dx\,dy = 2\int_1^3\int_0^1 (y^2-x^2)\,dx\,dy = 2\int_1^3\left[y^2x - \frac{1}{3}x^3\right]_{x=0}^{x=1} dy$

$= 2\int_1^3\left(y^2 - \frac{1}{3}\right) dy = \frac{2}{3}\left[y^3 - y\right]_1^3 = 16$

27. Here we need the volume of the solid lying under the surface $z = x\sqrt{x^2+y}$ and above the square $R = [0,1]\times[0,1]$ in the xy-plane.

$$V = \int_0^1\int_0^1 x\sqrt{x^2+y}\,dx\,dy = \int_0^1 \frac{1}{3}\left[\left(x^2+y\right)^{3/2}\right]_{x=0}^{x=1} dy = \frac{1}{3}\int_0^1\left[(1+y)^{3/2} - y^{3/2}\right] dy$$

$$= \frac{1}{3}\cdot\frac{2}{5}\left[(1+y)^{5/2} - y^{5/2}\right]_0^1 = \frac{4}{15}\left(2\sqrt{2}-1\right)$$

28. Here we need the volume of the solid lying under the surface $z = 1 + (x-1)^2 + 4y^2$ and above the rectangle $R = [0,3]\times[0,2]$ in the xy-plane.

$$V = \int_0^3\int_0^2\left[1 + (x-1)^2 + 4y^2\right] dy\,dx = \int_0^3\left[y + (x-1)^2\,y + \frac{4}{3}y^3\right]_{y=0}^{y=2} dx$$

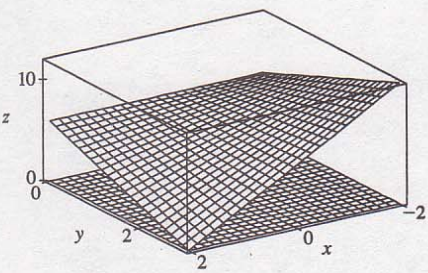
$$= \int_0^3\left[2 + 2\,(x-1)^2 + \frac{32}{3}\right] dx = \left[\frac{38}{3}x + \frac{2}{3}\,(x-1)^3\right]_0^3 = 44$$

29. In the first octant, $z \ge 0 \quad\Rightarrow\quad y \le 3$, so
$V = \int_0^3\int_0^2 (9-y^2)\,dx\,dy = \int_0^3\left[9x - y^2x\right]_{x=0}^{x=2} dy = \int_0^3 (18 - 2y^2)\,dy = \left[18y - \frac{2}{3}y^3\right]_0^3 = 36$

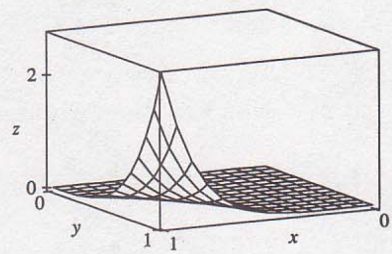
30. (a) Here we need the volume of the solid lying under the surface $z = 6 - xy$ and above the rectangle $R = [-2,2]\times[0,3]$ in the xy-plane.

$$V = \int_{-2}^{2}\int_0^3 (6-xy)\,dy\,dx$$
$$= \int_{-2}^{2}\left[6y - \frac{1}{2}xy^2\right]_{y=0}^{y=3} dx$$
$$= \int_{-2}^{2}\left(18 - \frac{9}{2}x\right) dx$$
$$= \left[18x - \frac{9}{4}x^2\right]_{-2}^{2} = 72$$

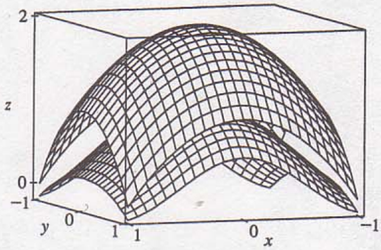
(b) The solid occupies the region between the two surfaces shown.



31. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command `Integrate[Integrate[f,{x,0,1}],{y,0,1}]`. We find that $\iint_R x^5y^3e^{xy}\,dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3d` (in Mathematica) to graph the function.



32. In Maple, we can calculate the integral by defining `f:=E^(-x^2)*cos(x^2+y^2);` and `g:=2-x^2-y^2;` and then [since $2 - x^2 - y^2 > e^{-x^2}\cos\left(x^2+y^2\right)$ for $-1 \le x \le 1$, $-1 \le y \le 1$] using the command `evalf(int(int(g-f,x=-1..1),y=-1..1),5);`. In Mathematica, we can use the command `N[Integrate[Integrate[f,{x,0,1}],{y,0,1}],5]`.



In each of these commands, the 5 indicates that we want only five significant digits; this speeds up the calculation considerably. We find that $\iint_R \left[\left(2-x^2-y^2\right) - \left(e^{-x^2}\cos\left(x^2+y^2\right)\right)\right] dA \approx 3.0271$. We can use the `plot3d` command (in Maple) or `Plot3d` (in Mathematica) to graph both functions on the same screen.

33. R is the rectangle $[-1,1]\times[0,5]$. Thus, $A(R) = 2\cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \tfrac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dx\,dy = \tfrac{1}{10}\int_0^5\left[\tfrac{1}{3}x^3y\right]_{x=-1}^{x=1} dy = \tfrac{1}{10}\int_0^5 \tfrac{2}{3}y\,dy$$
$$= \tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}$$

34. $A(R) = \frac{\pi}{2}\cdot 1 = \frac{\pi}{2}$, so

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \tfrac{1}{\pi/2}\int_0^{\pi/2}\int_0^1 x\sin xy\,dy\,dx = \tfrac{2}{\pi}\int_0^{\pi/2}\left[-\cos xy\right]_{y=0}^{y=1} dx$$
$$= \tfrac{2}{\pi}\int_0^{\pi/2}(1-\cos x)\,dx = \tfrac{2}{\pi}\left[x-\sin x\right]_0^{\pi/2} = 1 - \tfrac{2}{\pi}$$

35. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1\int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1\int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

36. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1: $g_x = \dfrac{d}{dx}g(x,y) = \dfrac{d}{dx}\displaystyle\int_a^x\left(\int_c^y f(s,t)\,dt\right)ds = \int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again: $g_{xy} = \dfrac{d}{dy}\displaystyle\int_c^y f(x,t)\,dt = f(x,y)$.

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x\int_c^y f(s,t)\,dt\,ds = \int_c^y\int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x,y)$. So $g_{xy} = g_{yx} = f(x,y)$.

16.3 Double Integrals over General Regions

ET 15.3

1. $\int_0^1 \int_0^{x^2} (x+2y)\,dy\,dx = \int_0^1 \left[xy+y^2\right]_{y=0}^{y=x^2} dx = \int_0^1 \left[x\left(x^2\right)+\left(x^2\right)^2-0-0\right] dx$
$= \int_0^1 \left(x^3+x^4\right) dx = \left[\frac{1}{4}x^4+\frac{1}{5}x^5\right]_0^1 = \frac{9}{20}$

2. $\int_1^2 \int_y^2 xy\,dx\,dy = \int_1^2 \left[\frac{1}{2}x^2 y\right]_{x=y}^{x=2} dy = \int_1^2 \frac{1}{2}y\left(4-y^2\right) dy = \frac{1}{2}\int_1^2 \left(4y-y^3\right) dy$
$= \frac{1}{2}\left[2y^2-\frac{1}{4}y^4\right]_1^2 = \frac{1}{2}\left(8-4-2+\frac{1}{4}\right) = \frac{9}{8}$

3. $\int_0^1 \int_y^{e^y} \sqrt{x}\,dx\,dy = \int_0^1 \left[\frac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y} dy = \frac{2}{3}\int_0^1 \left(e^{3y/2}-y^{3/2}\right) dy = \frac{2}{3}\left[\frac{2}{3}e^{3y/2}-\frac{2}{5}y^{5/2}\right]_0^1$
$= \frac{2}{3}\left(\frac{2}{3}e^{3/2}-\frac{2}{5}-\frac{2}{3}e^0+0\right) = \frac{4}{9}e^{3/2}-\frac{32}{45}$

4. $\int_0^1 \int_x^{2-x} \left(x^2-y\right) dy\,dx = \int_0^1 \left[x^2y-\frac{1}{2}y^2\right]_{y=x}^{y=2-x} dx = \int_0^1 \left[x^2(2-x)-\frac{1}{2}(2-x)^2-x^2(x)+\frac{1}{2}x^2\right] dx$
$= \int_0^1 \left(-2x^3+2x^2+2x-2\right) dx = \left[-\frac{1}{2}x^4+\frac{2}{3}x^3+x^2-2x\right]_0^1 = -\frac{5}{6}$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta = \int_0^{\pi/2} \left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta} d\theta = \int_0^{\pi/2} (\cos\theta)\,e^{\sin\theta}\,d\theta = e^{\sin\theta}\Big]_0^{\pi/2}$
$= e^{\sin(\pi/2)}-e^0 = e-1$

6. $\int_0^1 \int_0^v \sqrt{1-v^2}\,du\,dv = \int_0^1 \left[u\sqrt{1-v^2}\right]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2}\,dv = -\frac{1}{3}\left(1-v^2\right)^{3/2}\Big]_0^1$
$= -\frac{1}{3}(0-1) = \frac{1}{3}$

7. $\iint_D x^3y^2\,dA = \int_0^2 \int_{-x}^{x} x^3y^2\,dy\,dx = \int_0^2 \left[\frac{1}{3}x^3y^3\right]_{y=-x}^{y=x} dx = \frac{1}{3}\int_0^2 2x^6\,dx$
$= \frac{2}{3}\left[\frac{1}{7}x^7\right]_0^2 = \frac{2}{21}\left[2^7-0\right] = \frac{256}{21}$

8. $\displaystyle\iint_D \frac{4y}{x^3+2}\,dA = \int_1^2 \int_0^{2x} \frac{4y}{x^3+2}\,dy\,dx = \int_1^2 \left[\frac{2y^2}{x^3+2}\right]_{y=0}^{y=2x} dx = \int_1^2 \frac{8x^2}{x^3+2}\,dx$
$= \frac{8}{3}\ln\left|x^3+2\right|\Big]_1^2 = \frac{8}{3}(\ln 10-\ln 3) = \frac{8}{3}\ln\frac{10}{3}$

9. $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\,dx$
$= \frac{1}{2}\ln\left|x^2+1\right|\Big]_0^1 = \frac{1}{2}(\ln 2-\ln 1) = \frac{1}{2}\ln 2$

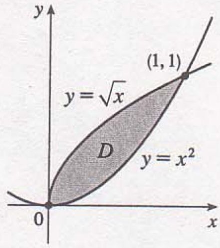
10. $\int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left[xe^{y^2}\right]_{x=0}^{x=y} dy = \int_0^1 ye^{y^2}\,dy = \frac{1}{2}e^{y^2}\Big]_0^1 = \frac{1}{2}(e-1)$

11. $\int_1^2 \int_y^{y^3} e^{x/y}\,dx\,dy = \int_1^2 \left[ye^{x/y}\right]_{x=y}^{x=y^3} dy = \int_1^2 \left(ye^{y^2}-ey\right) dy = \left[\frac{1}{2}e^{y^2}-\frac{1}{2}ey^2\right]_1^2 = \frac{1}{2}\left(e^4-4e\right)$

12. $\int_0^1 \int_0^y x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1 \left[-\frac{1}{3}\left(y^2-x^2\right)^{3/2}\right]_{x=0}^{x=y} dy = \frac{1}{3}\int_0^1 y^3\,dy = \frac{1}{3}\cdot\frac{1}{4}y^4\Big]_0^1 = \frac{1}{12}$

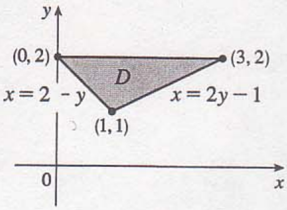
13. $\int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \left[x\sin y\right]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx = -\frac{1}{2}\cos x^2\Big]_0^1 = \frac{1}{2}(1-\cos 1)$

14.



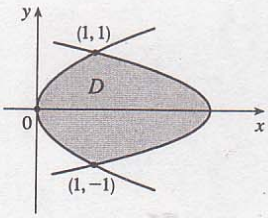
$$\begin{aligned}\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx &= \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}} dx \\ &= \int_0^1 \left(x^{3/2} + \tfrac{1}{2}x - x^3 - \tfrac{1}{2}x^4\right) dx \\ &= \left[\tfrac{2}{5}x^{5/2} + \tfrac{1}{4}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5\right]_0^1 = \tfrac{3}{10}\end{aligned}$$

15.



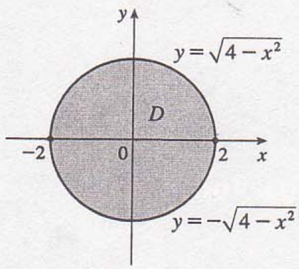
$$\begin{aligned}\int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy &= \int_1^2 \left[xy^3\right]_{x=2-y}^{x=2y-1} dy = \int_1^2 \left[(2y-1)-(2-y)\right] y^3\,dy \\ &= \int_1^2 \left(3y^4 - 3y^3\right) dy = \left[\tfrac{3}{5}y^5 - \tfrac{3}{4}y^4\right]_1^2 \\ &= \tfrac{96}{5} - 12 - \tfrac{3}{5} + \tfrac{3}{4} = \tfrac{147}{20}\end{aligned}$$

16.



$$\begin{aligned}\int_{-1}^1 \int_{y^2}^{3-2y^2} \left(y^2 - x\right) dx\,dy &= \int_{-1}^1 \left[xy^2 - \tfrac{1}{2}x^2\right]_{x=y^2}^{x=3-2y^2} dy \\ &= \int_{-1}^1 \left[\left(3-2y^2\right)y^2 - \tfrac{1}{2}\left(3-2y^2\right)^2 - y^2y^2 + \tfrac{1}{2}\left(y^2\right)^2\right] dy \\ &= \int_{-1}^1 \left(-\tfrac{9}{2}y^4 + 9y^2 - \tfrac{9}{2}\right) dy = \left[-\tfrac{9}{10}y^5 + 3y^3 - \tfrac{9}{2}y\right]_{-1}^1 = -\tfrac{24}{5}\end{aligned}$$

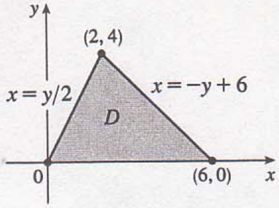
17.



$$\begin{aligned}\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x - y)\,dy\,dx &= \int_{-2}^2 \left[2xy - \tfrac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx \\ &= \int_{-2}^2 \left[2x\sqrt{4-x^2} - \tfrac{1}{2}\left(4-x^2\right) + 2x\sqrt{4-x^2} + \tfrac{1}{2}\left(4-x^2\right)\right] dx \\ &= \int_{-2}^2 4x\sqrt{4-x^2}\,dx = -\tfrac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^2 = 0\end{aligned}$$

(Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^2 4x\sqrt{4-x^2}\,dx = 0$.)

18.

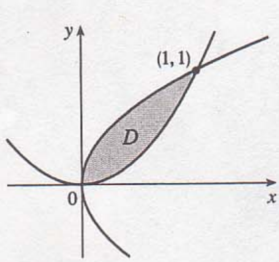


$$\int_0^4 \int_{y/2}^{6-y} ye^x \, dx \, dy = \int_0^4 \left[ye^x\right]_{x=y/2}^{x=6-y} dy$$
$$= \int_0^4 \left(ye^{6-y} - ye^{y/2}\right) dy$$
$$= \left[y\left(-e^{6-y} - 2e^{y/2}\right)\right]_0^4 + \left[-e^{6-y} + 4e^{y/2}\right]_0^4$$

(by integrating by parts separately for each term)

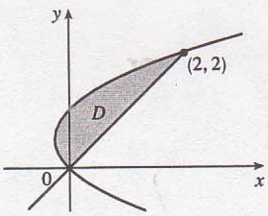
$$= -12e^2 + 3e^2 + e^6 - 4$$
$$= e^6 - 9e^2 - 4$$

19.



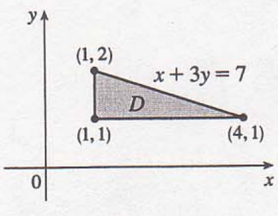
$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} \left(x^2 + y^2\right) dy \, dx = \int_0^1 \left[\left(x^2 y + \frac{y^3}{3}\right)\right]_{y=x^2}^{y=\sqrt{x}} dx$$
$$= \int_0^1 \left(x^{5/2} - x^4 + \tfrac{1}{3}x^{3/2} - \tfrac{1}{3}x^6\right) dx$$
$$= \left[\tfrac{2}{7}x^{7/2} - \tfrac{1}{5}x^5 + \tfrac{2}{15}x^{5/2} - \tfrac{1}{21}x^7\right]_0^1$$
$$= \tfrac{18}{105} = \tfrac{6}{35}$$

20.



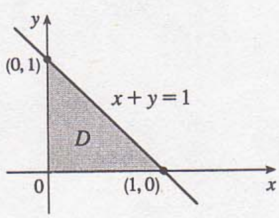
$$V = \int_0^2 \int_{y^2-y}^{y} \left(3x^2 + y^2\right) dx \, dy$$
$$= \int_0^2 \left[x^3 + y^2 x\right]_{x=y^2-y}^{x=y} dy$$
$$= \int_0^2 \left[2y^3 - \left(y^6 - 3y^5 + 4y^4 - 2y^3\right)\right] dy$$
$$= \left[-\frac{y^7}{7} + \frac{y^6}{2} - \frac{4y^5}{5} + y^4\right]_0^2 = \frac{144}{35}$$

21.



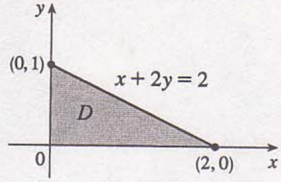
$$V = \int_1^2 \int_1^{7-3y} xy \, dx \, dy$$
$$= \int_1^2 \left[\tfrac{1}{2}x^2 y\right]_{x=1}^{x=7-3y} dy$$
$$= \tfrac{1}{2} \int_1^2 \left(48y - 42y^2 + 9y^3\right) dy$$
$$= \tfrac{1}{2}\left[24y^2 - 14y^3 + \tfrac{9}{4}y^4\right]_1^2 = \frac{31}{8}$$

22.



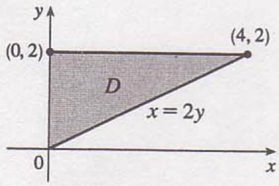
$$V = \int_0^1 \int_0^{1-x} \left(x^2 + y^2 + 4\right) dy \, dx = \int_0^1 \left[x^2 y + \tfrac{1}{3}y^3 + 4y\right]_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left[x^2 - x^3 + \tfrac{1}{3}(1-x)^3 + 4(1-x)\right] dx$$
$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(1-x)^4 - 2\left(1-x^2\right)\right]_0^1 = \tfrac{13}{6}$$

23.



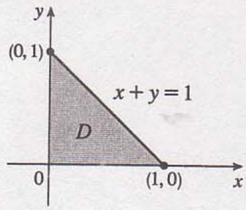
$$V = \int_0^2 \int_0^{1-x/2} \sqrt{9-x^2}\,dy\,dx = \int_0^2 \left[y\sqrt{9-x^2}\right]_{y=0}^{y=1-x/2} dx$$
$$= \int_0^2 \left(\sqrt{9-x^2} - \tfrac{1}{2}x\sqrt{9-x^2}\right) dx$$
$$= \int_0^2 \sqrt{9-x^2}\,dx + \tfrac{1}{4}\int_0^2 \left(-2x\sqrt{9-x^2}\right) dx$$
$$= \left[\tfrac{1}{2}x\sqrt{9-x^2} + \tfrac{9}{2}\sin^{-1}(x/3) + \tfrac{1}{6}\left(9-x^2\right)^{3/2}\right]_0^2$$
$$= \sqrt{5} + \tfrac{9}{2}\sin^{-1}\tfrac{2}{3} + \tfrac{5}{6}\sqrt{5} - \tfrac{1}{6}(27)$$
$$= \tfrac{1}{6}\left(11\sqrt{5} - 27\right) + \tfrac{9}{2}\sin^{-1}\tfrac{2}{3}$$

24.



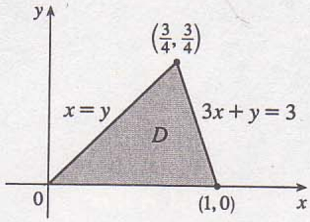
$$V = \int_0^2 \int_0^{2y} \sqrt{4-y^2}\,dx\,dy$$
$$= \int_0^2 \left[x\sqrt{4-y^2}\right]_{x=0}^{x=2y} dy = \int_0^2 2y\sqrt{4-y^2}\,dy$$
$$= \left[-\tfrac{2}{3}\left(4-y^2\right)^{3/2}\right]_0^2 = 0 + \tfrac{16}{3} = \tfrac{16}{3}$$

25.



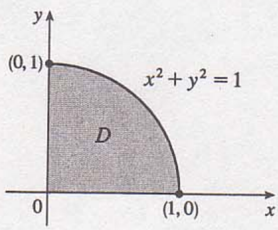
$$V = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$
$$= \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left[(1-x)^2 - \tfrac{1}{2}(1-x)^2\right] dx$$
$$= \int_0^1 \tfrac{1}{2}(1-x)^2\,dx = \left[-\tfrac{1}{6}(1-x)^3\right]_0^1 = \tfrac{1}{6}$$

26.



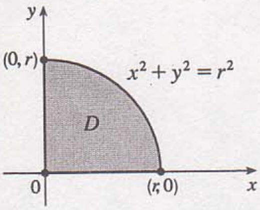
$$V = \int_0^{3/4} \int_y^{(3-y)/3} \tfrac{1}{3}(6-6x-2y)\,dx\,dy$$
$$= \int_0^{3/4} \left[\tfrac{2}{3}(3-y)\,x - x^2\right]_{x=y}^{x=(3-y)/3} dy$$
$$= \int_0^{3/4} \left[\tfrac{1}{9}(3-y)^2 - 2y + \tfrac{5}{3}y^2\right] dy$$
$$= \left[-\tfrac{1}{27}(3-y)^3 - y^2 + \tfrac{5}{9}y^3\right]_0^{3/4}$$
$$= -\tfrac{27}{64} - \tfrac{9}{16} + \tfrac{15}{64} + 1 = \tfrac{1}{4}$$

27.



$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$
$$= \int_0^1 \frac{1-x^2}{2}\,dx = \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

28.

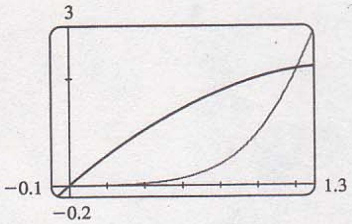


By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

$$\begin{aligned} V_1 &= \int_0^r \int_0^{\sqrt{r^2 - y^2}} \sqrt{r^2 - y^2}\, dx\, dy \\ &= \int_0^r \left[x\sqrt{r^2 - y^2} \right]_{x=0}^{x=\sqrt{r^2 - y^2}} dy \\ &= \int_0^r \left(r^2 - y^2\right) dy = \left[r^2 y - \tfrac{1}{3}y^3\right]_0^r = \tfrac{2}{3}r^3 \end{aligned}$$

Thus $V = \frac{16}{3}r^3$.

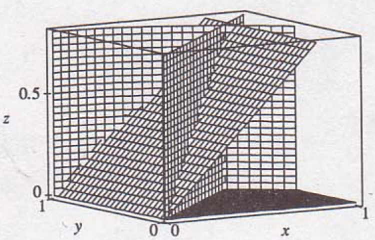
29.

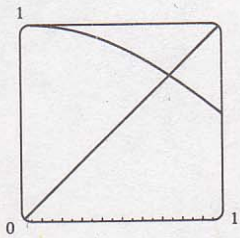


From the graph, it appears that the two curves intersect at $x = 0$ and at $x \approx 1.213$. Thus the desired integral is

$$\begin{aligned} \iint_D x\, dA &\approx \int_0^{1.213} \int_{x^4}^{3x - x^2} x\, dy\, dx \\ &= \int_0^{1.213} \left[xy\right]_{y=x^4}^{y=3x-x^2} dx \\ &= \int_0^{1.213} \left(3x^2 - x^3 - x^5\right) dx \\ &= \left[x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^{1.213} \approx 0.713 \end{aligned}$$

30.





The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$\begin{aligned} V &\approx \int_0^{0.7391} \int_x^{\cos x} z\, dy\, dx = \int_0^{0.7391} \int_x^{\cos x} x\, dy\, dx = \int_0^{0.7391} \left[xy\right]_{y=x}^{y=\cos x} dx \\ &= \int_0^{0.7391} \left(x\cos x - x^2\right) dx = \left[\cos x + x\sin x - \tfrac{1}{3}x^3\right]_0^{0.7391} \approx 0.1024 \end{aligned}$$

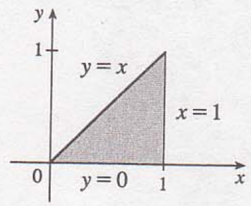
Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391} \int_0^x x\, dy\, dx + \int_{0.7391}^{\pi/2} \int_0^{\cos x} x\, dy\, dx \approx 0.4684$.

31. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} z\, dy\, dx = \int_0^2 \int_{x^3 - x}^{x^2 + x} \left(x^3 y^4 + xy^2\right) dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$.

32. For $|x| \le 1$ and $|y| \le 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. So the volume is given by $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[(8 - x^2 - 2y^2) - (2x^2 + y^2)\right] dy\,dx = \frac{13\pi}{2}$ (using a CAS).

33.



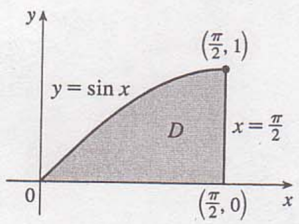
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le x, 0 \le x \le 1\} = \{(x,y) \mid y \le x \le 1, 0 \le y \le 1\}$$

we have

$$\int_0^1\int_0^x f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^1\int_y^1 f(x,y)\,dx\,dy$$

34.



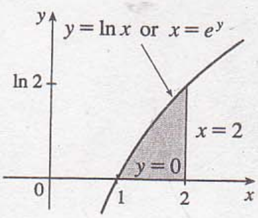
Because the region of integration is

$$D = \left\{(x,y) \mid 0 \le y \le \sin x, 0 \le x \le \tfrac{\pi}{2}\right\} = \left\{(x,y) \mid \sin^{-1} y \le x \le \tfrac{\pi}{2}, 0 \le y \le 1\right\}$$

we have

$$\int_0^{\pi/2}\int_0^{\sin x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^1\int_{\sin^{-1} y}^{\pi/2} f(x,y)\,dx\,dy$$

35.



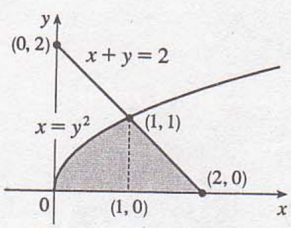
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\} = \{(x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^{\ln 2}\int_{e^y}^2 f(x,y)\,dx\,dy$$

36.



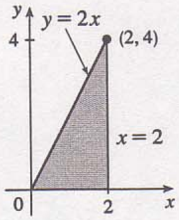
To reverse the order, we must break the region into two separate type I regions. Because the region of integration is

$$D = \left\{(x,y) \mid y^2 \le x \le 2 - y, 0 \le y \le 1\right\} = \left\{(x,y) \mid 0 \le y \le \sqrt{x}, 0 \le x \le 1\right\} \cup \{0 \le y \le 2 - x, 1 \le x \le 2\}$$

we have

$$\int_0^1\int_{y^2}^{2-y} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA = \int_0^1\int_0^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^2\int_0^{2-x} f(x,y)\,dy\,dx$$

37.



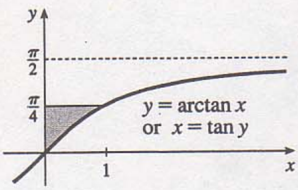
Because the region of integration is

$$D = \{(x,y) \mid y/2 \le x \le 2, 0 \le y \le 4\}$$
$$= \{(x,y) \mid 0 \le y \le 2x, 0 \le x \le 2\}$$

we have

$$\int_0^4 \int_{y/2}^2 f(x,y)\, dx\, dy = \iint_D f(x,y)\, dA$$
$$= \int_0^2 \int_0^{2x} f(x,y)\, dy\, dx$$

38.



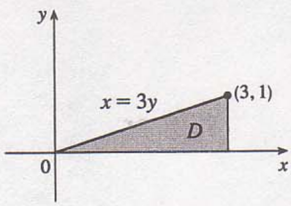
Because the region of integration is

$$D = \left\{(x,y) \mid \arctan x \le y \le \tfrac{\pi}{4}, 0 \le x \le 1\right\}$$
$$= \left\{(x,y) \mid 0 \le x \le \tan y, 0 \le y \le \tfrac{\pi}{4}\right\}$$

we have

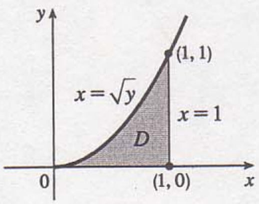
$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA$$
$$= \int_0^{\pi/4} \int_0^{\tan y} f(x,y)\, dx\, dy$$

39.



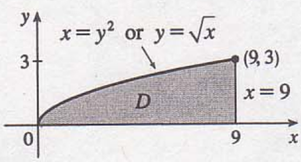
$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx$$
$$= \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx = \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\, dx$$
$$= \tfrac{1}{6}\, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6}$$

40.



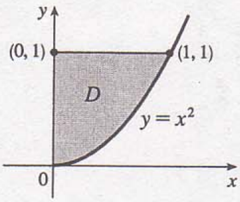
$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$
$$= \int_0^1 \left[\sqrt{x^3+1}\, y\right]_{y=0}^{y=x^2} dx$$
$$= \int_0^1 x^2 \sqrt{x^3+1}\, dx$$
$$= \tfrac{2}{9}\left(x^3+1\right)^{3/2}\Big]_0^1$$
$$= \tfrac{2}{9}\left(2^{3/2} - 1\right)$$

41.



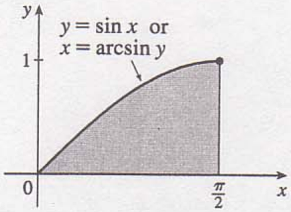
$$\int_0^3 \int_{y^2}^9 y \cos x^2\, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} y \cos x^2\, dy\, dx$$
$$= \int_0^9 \cos x^2 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx$$
$$= \int_0^9 \tfrac{1}{2} x \cos x^2\, dx = \tfrac{1}{4} \sin x^2\Big]_0^9$$
$$= \tfrac{1}{4} \sin 81$$

42.



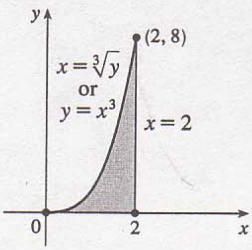
$$\int_0^1 \int_{x^2}^1 x^3 \sin\left(y^3\right) dy\,dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin\left(y^3\right) dx\,dy$$
$$= \int_0^1 \left[\frac{x^4}{4}\sin\left(y^3\right)\right]_{x=0}^{x=\sqrt{y}} dy$$
$$= \int_0^1 \tfrac{1}{4}y^2 \sin\left(y^3\right) dy$$
$$= -\tfrac{1}{12}\cos\left(y^3\right)\Big]_0^1 = \tfrac{1}{12}\left(1 - \cos 1\right)$$

43.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,[y]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,\sin x\,dx$$
$$\left[\text{Let } u = \cos x,\, du = -\sin x\,dx,\, dx = du/\left(-\sin x\right)\right]$$
$$= \int_1^0 -u\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

44.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$
$$= \int_0^2 e^{x^4}\,[y]_{y=0}^{y=x^3}\,dx$$
$$= \int_0^2 x^3 e^{x^4}\,dx$$
$$= \tfrac{1}{4}e^{x^4}\Big]_0^2 = \tfrac{1}{4}\left(e^{16}-1\right)$$

45. $D = \{(x,y) \mid 0 \le x \le 1, -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$

$\cup \{(x,y) \mid 0 \le x \le 1, -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0, -1 \le y \le -x-1\}$,

all type I.

$$\iint_D x^2\,dA = \int_0^1\int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0\int_{x+1}^1 x^2\,dy\,dx + \int_0^1\int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0\int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1\int_{1-x}^1 x^2\,dy\,dx \quad \text{[by symmetry of the regions and because } f(x,y) = x^2 \ge 0\text{]}$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

46. $D = \{(x,y) \mid -1 \le x \le 0, -1 \le y \le 1 + x^2\} \cup \{(x,y) \mid 0 \le x \le 1, \sqrt{x} \le y \le 1 + x^2\}$

$\cup \{(x,y) \mid 0 \le x \le 1, -1 \le y \le -\sqrt{x}\}$, all type I.

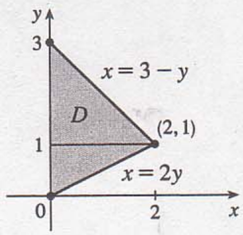
$$\begin{aligned}\iint_D xy\,dA &= \int_{-1}^0 \int_{-1}^{1+x^2} xy\,dy\,dx + \int_0^1 \int_{\sqrt{x}}^{1+x^2} xy\,dy\,dx + \int_0^1 \int_{-1}^{-\sqrt{x}} xy\,dy\,dx \\ &= \int_{-1}^0 \left[\tfrac{1}{2}xy^2\right]_{y=-1}^{y=1+x^2} dx + \int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=\sqrt{x}}^{y=1+x^2} dx + \int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=-1}^{y=-\sqrt{x}} dx \\ &= \int_{-1}^0 \left(x^3 + \tfrac{1}{2}x^5\right) dx + \int_0^1 \tfrac{1}{2}\left(x^5 + 2x^3 - x^2 + x\right) dx + \int_0^1 \tfrac{1}{2}\left(x^2 - x\right) dx \\ &= \left[\tfrac{1}{4}x^4 + \tfrac{1}{12}x^6\right]_{-1}^0 + \tfrac{1}{2}\left[\tfrac{1}{6}x^6 + \tfrac{1}{2}x^4 - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 + \tfrac{1}{2}\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^1 \\ &= -\tfrac{1}{3} + \tfrac{5}{12} - \tfrac{1}{12} = 0\end{aligned}$$

47. For $D = [0,1] \times [0,1]$, $0 \le \sqrt{x^3 + y^3} \le \sqrt{2}$ and $A(D) = 1$, so $0 \le \iint_D \sqrt{x^3 + y^3}\,dA \le \sqrt{2}$.

48. Since $D = \left\{(x,y) \mid x^2 + y^2 \le \tfrac{1}{4}\right\}$, $1 = e^0 \le e^{x^2+y^2} \le e^{1/4}$ and $A(D) = \tfrac{\pi}{4}$, so
$\tfrac{\pi}{4} \le \iint_D e^{x^2+y^2}\,dA \le \left(e^{1/4}\right)\tfrac{\pi}{4}$.

49. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$
$m \iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M \iint_D 1\,dA$ by (7) $\Rightarrow$ $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

50.



$$\begin{aligned}\iint_D f(x,y)\,dA &= \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy \\ &= \int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx\end{aligned}$$

51. $\iint_D (x^2 \tan x + y^3 + 4)\,dA = \iint_D x^2 \tan x\,dA + \iint_D y^3\,dA + \iint_D 4\,dA$. But $x^2 \tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2 \tan x\,dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\,dA = 0$. Thus

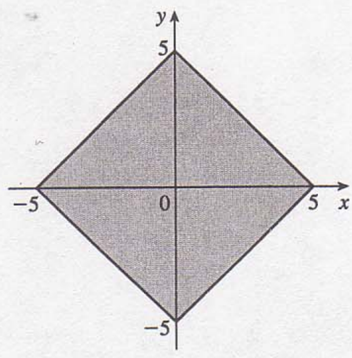
$$\iint_D (x^2 \tan x + y^3 + 4)\,dA = 4\iint_D dA = 4\,(\text{area of } D) = 4 \cdot \pi\left(\sqrt{2}\right)^2 = 8\pi$$

52. First,

$$\iint_D (2 - 3x + 4y)\,dA = \iint_D 2\,dA - \iint_D 3x\,dA + \iint_D 4y\,dA$$

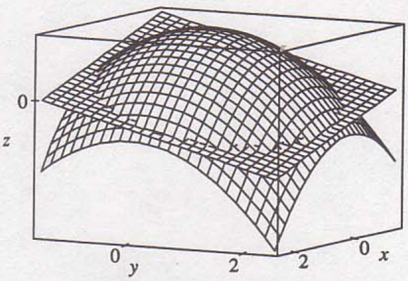
The region D, shown in the figure, is symmetric with respect to the y-axis and $3x$ is an odd function of x, so $\iint_D 3x\,dA = 0$. Similarly, $4y$ is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D 4y\,dA = 0$. Then

$$\begin{aligned}\iint_D (2 - 3x + 4y)\,dA &= \iint_D 2\,dA = 2\iint_D dA \\ &= 2\,(\text{area of } D) = 2\,(50) \\ &= 100\end{aligned}$$



53. Since $\sqrt{1-x^2-y^2} \geq 0$, we can interpret $\iint_D \sqrt{1-x^2-y^2}\,dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1-x^2-y^2}$ and above the region D in the xy-plane. $z = \sqrt{1-x^2-y^2}$ is equivalent to $x^2+y^2+z^2 = 1$, $z \geq 0$ which meets the xy-plane in the circle $x^2+y^2=1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi\,(1)^3\right] = \frac{2}{3}\pi$.

54. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13+4x-4x^2}}{2}$.



To find the two points of intersection of these curves, we use the CAS to solve $13+4x-4x^2=0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is

$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2} \left[(4-x^2-y^2)-(1-x-y)\right]\,dy\,dx = \frac{49\pi}{8}.$$

16.4 Double Integrals in Polar Coordinates ET 15.4

1. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2-x\}$. Thus $\iint_R f(x,y)\,dA = \int_0^2\int_0^{2-x} f(x,y)\,dy\,dx$.

3. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid -2 \leq x \leq 2, x \leq y \leq 2\}$. Thus $\iint_R f(x,y)\,dA = \int_{-2}^2\int_x^2 f(x,y)\,dy\,dx$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 1 \leq r \leq 3, 0 \leq \theta \leq \frac{\pi}{2}\right\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

5. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \leq r \leq 5, 0 \leq \theta \leq 2\pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

6. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \leq r \leq 2\sqrt{2}, \frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}\right\}$. Thus $\iint_R f(x,y)\,dA = \int_{\pi/4}^{5\pi/4}\int_0^{2\sqrt{2}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

7. The region R can be described in polar coordinates as $R = \{(r,\theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 5\}$. Thus,
$\iint_R x\,dA = \int_0^{2\pi}\int_0^5 (r\cos\theta)\,r\,dr\,d\theta = \left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^5 r^2\,dr\right) = [\sin\theta]_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^5 = 0$.

8. $\iint_R y\,dA = \int_0^{\pi/4}\int_0^3 (r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{\pi/4}\sin\theta\,d\theta\right)\left(\int_0^3 r^2\,dr\right) = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)(9) = 9\left(1-\frac{1}{\sqrt{2}}\right)$

9. $\iint_R xy\,dA = \int_0^{\pi/2}\int_2^5 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\,\int_2^5 r^3\,dr = \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\frac{1}{4}r^4\right]_2^5$
$= \frac{1}{2}\cdot\frac{5^4-2^4}{4} = \frac{609}{8}$

10. $\iint_R \sqrt{x^2+y^2}\,dA = \int_0^\pi \int_1^3 \sqrt{r^2}\,r\,dr\,d\theta = \left(\int_0^\pi d\theta\right)\left(\int_1^3 r^2\,dr\right) = [\theta]_0^\pi \left[\frac{1}{3}r^3\right]_1^3 = \pi\left(\frac{27-1}{3}\right) = \frac{26}{3}\pi$

11. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\,dr\right)$

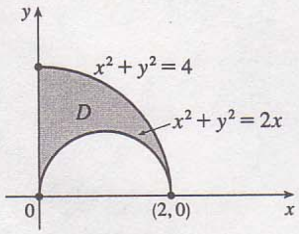
$$= [\theta]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)\left(e^{-4}-e^0\right) = \tfrac{\pi}{2}\left(1-e^{-4}\right)$$

12. $\displaystyle\iint_D \frac{1}{(1+x^2+y^2)^{3/2}}\,dA = \int_0^{\pi/2}\int_0^4 \frac{1}{(1+r^2)^{3/2}}\,r\,dr\,d\theta = \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^4 r\left(1+r^2\right)^{-3/2}dr\right)$

$$= [\theta]_0^{\pi/2}\left[-\left(1+r^2\right)^{-1/2}\right]_0^4 = \tfrac{\pi}{2}\left(1-\tfrac{1}{\sqrt{17}}\right)$$

13. $\int_0^{2\pi}\int_\theta^{2\theta} r^2\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{4}r^4\right]_{r=\theta}^{r=2\theta} d\theta = \frac{1}{4}\int_0^{2\pi} 15\theta^4\,d\theta = \frac{3}{4}\left[\theta^5\right]_0^{2\pi} = \frac{3}{4}\left(32\pi^5\right) = 24\pi^5$

14.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}\left(8\cos^4\theta\right)d\theta$$

$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta+\sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0+\tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

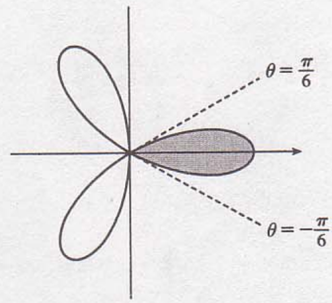
15. One loop is given by the region

$D = \{(r,\theta)\,|\,-\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta+\frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



16. By symmetry,

$$A = 2\int_{-\pi/2}^{\pi/2}\int_0^{1-\sin\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[r^2\right]_{r=0}^{r=1-\sin\theta} d\theta = \int_{-\pi/2}^{\pi/2}\left(1-2\sin\theta+\sin^2\theta\right)d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[1+\tfrac{1}{2}(1-\cos 2\theta)\right]d\theta = \int_{-\pi/2}^{\pi/2}\left(\tfrac{3}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta$$

since $2\sin\theta$ is an odd function. But $\frac{3}{2}-\frac{1}{2}\cos 2\theta$ is an even function, so

$A = \int_0^{\pi/2}(3-\cos 2\theta)\,d\theta = \left[3\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$.

17. By symmetry, the two loops of the lemniscate are equal in area, so

$A = 2\int_{-\pi/4}^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4}\left[r^2\right]_{r=0}^{r=2\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4} 4\cos 2\theta\,d\theta$

$= 8\int_0^{\pi/4}\cos 2\theta\,d\theta = 4\sin 2\theta\big]_0^{\pi/4} = 4.$

18. $2 = 4\sin\theta$ implies that $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so

$$A = \int_{\pi/6}^{5\pi/6} \int_2^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \left[\tfrac{1}{2}r^2\right]_{r=2}^{r=4\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6} \left(8\sin^2\theta - 2\right) d\theta$$
$$= \int_{\pi/6}^{5\pi/6} \left[4\left(1 - \cos 2\theta\right) - 2\right] d\theta = \left[2\theta - 2\sin 2\theta\right]_{\pi/6}^{5\pi/6} = \tfrac{4\pi}{3} + 2\sqrt{3}.$$

19. $V = \iint_{x^2+y^2\le 9} \left(x^2+y^2\right) dA = \int_0^{2\pi}\int_0^3 \left(r^2\right) r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^3 r^3\,dr = \left[\theta\right]_0^{2\pi} \left[\frac{1}{4}r^4\right]_0^3 = 2\pi\left(\frac{81}{4}\right) = \frac{81\pi}{2}$

20. The sphere $x^2+y^2+z^2=16$ intersects the xy-plane in the circle $x^2+y^2=16$, so

$$V = 2\iint_{4\le x^2+y^2\le 16} \sqrt{16-x^2-y^2}\,dA \quad \text{(by symmetry)}$$
$$= 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi} d\theta \int_2^4 r\left(16-r^2\right)^{1/2} dr$$
$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}\left(16-r^2\right)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)\left(0 - 12^{3/2}\right) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\pi$$

21. By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2} \sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi} d\theta \int_0^a r\sqrt{a^2-r^2}\,dr$$
$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}\left(a^2-r^2\right)^{3/2}\right]_0^a = 2(2\pi)\left(0+\tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

22. The paraboloid $z = 10 - 3x^2 - 3y^2$ intersects the plane $z=4$ when $4 = 10 - 3x^2 - 3y^2$ or $x^2+y^2=2$. So

$$V = \iint_{x^2+y^2\le 2} \left[\left(10-3x^2-3y^2\right) - 4\right] dA = \int_0^{2\pi}\int_0^{\sqrt{2}} \left(6-3r^2\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \left(6r - 3r^3\right) dr = \left[\theta\right]_0^{2\pi}\left[3r^2 - \tfrac{3}{4}r^4\right]_0^{\sqrt{2}} = 6\pi$$

23. The cone $z = \sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 1$ or $x^2+y^2=\frac{1}{2}$. So

$$V = \iint_{x^2+y^2\le 1/2}\left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2} - r\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}}\left(r\sqrt{1-r^2} - r^2\right) dr = \left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}\left(1-r^2\right)^{3/2} - \tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}}$$
$$= 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right) = \tfrac{\pi}{3}\left(2-\sqrt{2}\right)$$

24. The two paraboloids intersect when $3x^2+3y^2 = 4 - x^2 - y^2$ or $x^2+y^2=1$. So

$$V = \iint_{x^2+y^2\le 1}\left[\left(4-x^2-y^2\right) - 3\left(x^2+y^2\right)\right] dA = \int_0^{2\pi}\int_0^1 4\left(1-r^2\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^1 \left(4r - 4r^3\right) dr = \left[\theta\right]_0^{2\pi}\left[2r^2 - r^4\right]_0^1 = 2\pi$$

25. The given solid is the region inside the cylinder $x^2+y^2=4$ between the surfaces $z = \sqrt{64-4x^2-4y^2}$ and $z = -\sqrt{64-4x^2-4y^2}$. So

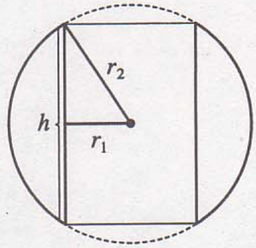
$$V = \iint_{x^2+y^2\le 4}\left[\sqrt{64-4x^2-4y^2} - \left(-\sqrt{64-4x^2-4y^2}\right)\right] dA$$
$$= \iint_{x^2+y^2\le 4} 2\sqrt{64-4x^2-4y^2}\,dA = 4\int_0^{2\pi}\int_0^2 \sqrt{16-r^2}\,r\,dr\,d\theta$$
$$= 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16-r^2}\,dr = 4\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}\left(16-r^2\right)^{3/2}\right]_0^2$$
$$= 8\pi\left(-\tfrac{1}{3}\right)\left(12^{3/2} - 16^{2/3}\right) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$

26. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2 + y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

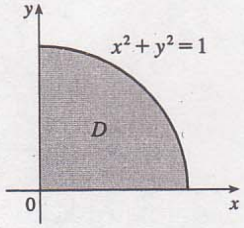
$$V = 2\iint_{r_1^2 \le x^2+y^2 \le r_2^2} \sqrt{r_2^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi}\int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr = \tfrac{4\pi}{3}\left[-\left(r_2^2 - r^2\right)^{3/2}\right]_{r_1}^{r_2} = \tfrac{4\pi}{3}\left(r_2^2 - r_1^2\right)^{3/2}$$

(b) A cross-sectional cut is shown in the figure. So $r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2$ or $\frac{1}{4}h^2 = r_2^2 - r_1^2$. Thus the volume in terms of h is $V = \frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3$.



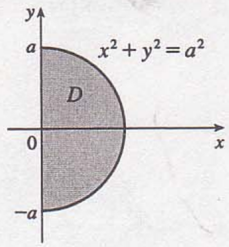
27.



$$\int_0^1\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx = \int_0^{\pi/2}\int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi/2} d\theta \int_0^1 re^{r^2}\, dr$$

$$= [\theta]_0^{\pi/2}\left[\tfrac{1}{2}e^{r^2}\right]_0^1 = \tfrac{1}{4}\pi\,(e-1)$$

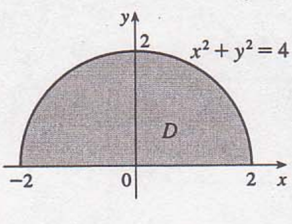
28.



$$\int_{-\pi/2}^{\pi/2}\int_0^a \left(r^2\right)^{3/2} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^a r^4\, dr = [\theta]_{-\pi/2}^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^a$$

$$= \tfrac{1}{5}\pi a^5$$

29.



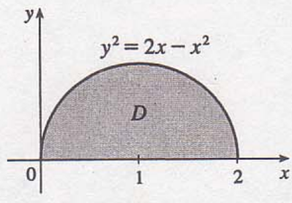
$$\int_0^{\pi}\int_0^2 (r\cos\theta)^2\,(r\sin\theta)^2\, r\, dr\, d\theta = \int_0^{\pi} (\sin\theta\cos\theta)^2\, d\theta \int_0^2 r^5\, dr$$

$$= \int_0^{\pi} \left(\tfrac{1}{2}\sin 2\theta\right)^2 d\theta \int_0^2 r^5\, dr$$

$$= \tfrac{1}{4}\left[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta\right]_0^{\pi}\left[\tfrac{1}{6}r^6\right]_0^2$$

$$= \tfrac{1}{4}\left(\tfrac{\pi}{2}\right)\left(\tfrac{64}{6}\right) = \tfrac{4\pi}{3}$$

30.



$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\, dr\, d\theta = \int_0^{\pi/2}\left[\tfrac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta = \int_0^{\pi/2}\left(\tfrac{8}{3}\cos^3\theta\right) d\theta = \tfrac{8}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \tfrac{16}{9}$$

31. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x,y) \mid x^2+y^2 \le 400\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20)=2$ to $f(0,20)=7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z-7=\frac{1}{8}(y-20) \;\Rightarrow\; z=\frac{1}{8}y+\frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y)=\frac{1}{8}y+\frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D=\{(r,\theta) \mid 0\le r\le 20, 0\le\theta\le 2\pi\}$ and substituting $x=r\cos\theta$, $y=r\sin\theta$ the integral becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta+\tfrac{9}{2}\right)r\,dr\,d\theta &= \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta+\tfrac{9}{4}r^2\right]_{r=0}^{r=20}d\theta\\ &= \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta+900\right)d\theta = \left[-\tfrac{1000}{3}\cos\theta+900\theta\right]_0^{2\pi}\\ &= 1800\pi\end{aligned}$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

32. (a) The total amount of water supplied each hour to the region within R feet of the sprinkler is

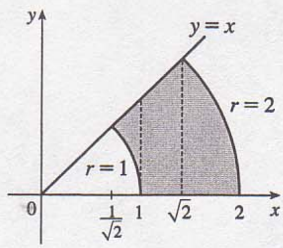
$$\begin{aligned}V &= \int_0^{2\pi}\int_0^R e^{-r}r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^R re^{-r}\,dr = [\theta]_0^{2\pi}\left[-re^{-r}-e^{-r}\right]_0^R\\ &= 2\pi\left[-Re^{-R}-e^{-R}+0+1\right] = 2\pi\left(1-Re^{-R}-e^{-R}\right)\text{ ft}^3\end{aligned}$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1-Re^{-R}-e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 1006 [ET 972].

33. $\int_{1/\sqrt{2}}^1\int_{\sqrt{1-x^2}}^x xy\,dy\,dx+\int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx+\int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} xy\,dy\,dx = \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta$

$$= \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2}d\theta = \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta$$

$$= \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



34. (a) $\iint_{D_a} e^{-(x^2+y^2)}dA = \int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1-e^{-a^2}\right)$ for each a. Then $\lim\limits_{a\to\infty}\pi\left(1-e^{-a^2}\right)=\pi$ since $e^{-a^2}\to 0$ as $a\to\infty$. Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)}dA=\pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}dA = \int_{-a}^a\int_{-a}^a e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-a}^a e^{-x^2}dx\right)\left(\int_{-a}^a e^{-y^2}dy\right)$ for each a. Then, from (a), $\pi = \iint_{\mathbb{R}^2}e^{-(x^2+y^2)}dA$, so

$\pi = \lim\limits_{a\to\infty}\iint_{S_a}e^{-(x^2+y^2)}\,dA = \lim\limits_{a\to\infty}\left(\int_{-a}^a e^{-x^2}dx\right)\left(\int_{-a}^a e^{-y^2}dy\right) = \left(\int_{-\infty}^{\infty}e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty}e^{-y^2}dy\right)$.

To evaluate $\lim\limits_{a\to\infty}\left(\int_{-a}^a e^{-x^2}dx\right)\left(\int_{-a}^a e^{-y^2}dy\right)$, we are using the fact that these integrals are bounded. This is

true since on $[-1,1]$, $0 < e^{-x^2} \le 1$ while on $(-\infty,-1)$, $0 < e^{-x^2} \le e^x$ and on $(1,\infty)$, $0 < e^{-x^2} < e^{-x}$. Hence $0 \le \int_{-\infty}^{\infty} e^{-x^2}\,dx \le \int_{-\infty}^{-1} e^x\,dx + \int_{-1}^{1} dx + \int_1^{\infty} e^{-x}\,dx = 2\left(e^{-1}+1\right)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pm\sqrt{\pi}$. But $e^{-x^2} \ge 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(d) Letting $t = \sqrt{2}x$, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right)dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$.

35. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\int_0^{\infty} x^2e^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(-\tfrac{1}{2}xe^{-x^2}\Big]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)$$
$$= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx \text{ (by l'Hospital's Rule)}$$
$$= \tfrac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}\,dx \text{ (since } e^{-x^2} \text{ is an even function)} = \tfrac{1}{4}\sqrt{\pi} \text{ [by Exercise 34(c)]}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \;\Rightarrow\; dx = 2u\,du \;\Rightarrow$

$$\int_0^{\infty} \sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^{\infty} u^2e^{-u^2}\,du$$
$$= 2\left(\tfrac{1}{4}\sqrt{\pi}\right) \text{ [by part(a)]} = \tfrac{1}{2}\sqrt{\pi}$$

16.5 Applications of Double Integrals — ET 15.5

1. $Q = \iint_D (x^2+3y^2)\,dA = \int_0^2\int_1^2 (x^2+3y^2)\,dy\,dx = \int_0^2 \left[x^2y+y^3\right]_{y=1}^{y=2} dx = \int_0^2 (x^2+7)\,dx$
$= \left[\frac{1}{3}x^3+7x\right]_0^2 = \frac{8}{3}+14 = \frac{50}{3}$ C

2. $Q = \iint_{0\le x^2+y^2\le 1} (1+x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 (1+r^2)\,r\,dr\,d\theta$
$= \int_0^{2\pi} d\theta\int_0^1 (r+r^3)\,dr = 2\pi\left[\frac{1}{2}r^2+\frac{1}{4}r^4\right]_0^1 = \frac{3\pi}{2}$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_{-1}^1\int_0^1 x^2\,dy\,dx = \int_{-1}^1 x^2\,dx\int_0^1 dy = \left[\frac{1}{3}x^3\right]_{-1}^1 [y]_0^1 = \frac{2}{3}$,
$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{3}{2}\int_{-1}^1\int_0^1 x^3\,dy\,dx = \frac{3}{2}\int_{-1}^1 x^3\,dx\int_0^1 dy = \frac{3}{2}\left[\frac{1}{4}x^4\right]_{-1}^1 [y]_0^1 = 0$,
$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{3}{2}\int_{-1}^1\int_0^1 x^2y\,dy\,dx = \frac{3}{2}\int_{-1}^1 x^2\,dx\int_0^1 y\,dy = \frac{3}{2}\left[\frac{1}{3}x^3\right]_{-1}^1\left[\frac{1}{2}y^2\right]_0^1 = \frac{1}{2}$. Hence $(\overline{x},\overline{y}) = \left(0,\frac{1}{2}\right)$.

4. $m = \int_0^3\int_0^2 y\,dx\,dy = \int_0^2 dx\int_0^3 y\,dy = 9$, $M_y = \int_0^3\int_0^2 xy\,dx\,dy = \int_0^2 x\,dx\int_0^3 y\,dy = 9$ and
$M_x = \int_0^3\int_0^2 y^2\,dx\,dy = \int_0^2 dx\int_0^3 y^2\,dy = 18$. Hence $m = 9$, $(\overline{x},\overline{y}) = (M_y/m, M_x/m) = (1,2)$.

5. $m = \int_0^2\int_{x/2}^{3-x} (x+y)\,dy\,dx = \int_0^2 \left[xy+\frac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left[x\left(3-\frac{3}{2}x\right)+\frac{1}{2}(3-x)^2-\frac{1}{8}x^2\right] dx$
$= \int_0^2 \left(-\frac{9}{8}x^2+\frac{9}{2}\right) dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right)+\frac{9}{2}x\right]_0^2 = 6$,
$M_y = \int_0^2\int_{x/2}^{3-x} (x^2+xy)\,dy\,dx = \int_0^2 \left[x^2y+\frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(\frac{9}{2}x-\frac{9}{8}x^3\right) dx = \frac{9}{2}$, and
$M_x = \int_0^2\int_{x/2}^{3-y} (xy+y^2)\,dy\,dx = \int_0^2 \left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(9-\frac{9}{2}x\right) dx = 9$.
Hence $m = 6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\dfrac{3}{4},\dfrac{3}{2}\right)$.

6. $m = \int_0^1 \int_y^{4-3y} x\,dx\,dy = \int_0^1 \left[\frac{1}{2}(4-3y)^2 - \frac{1}{2}y^2\right] dy = \left[-\frac{1}{18}(4-3y)^3 - \frac{1}{6}y^3\right]_0^1 = \frac{10}{3}$,
$M_y = \int_0^1 \int_y^{4-3y} x^2\,dx\,dy = \int_0^1 \left[\frac{1}{3}(4-3y)^3 - \frac{1}{3}y^3\right] dy = \left[-\frac{1}{36}(4-3y)^4 - \frac{1}{12}y^4\right]_0^1 = 7$,
$M_x = \int_0^1 \int_y^{4-3y} xy\,dx\,dy = \int_0^1 \left[\frac{1}{2}y(4-3y)^2 - \frac{1}{2}y^3\right] dy = \int_0^1 \left(8y - 12y^2 + 4y^3\right) dy = 1.$
Hence $m = \frac{10}{3}$, $(\overline{x}, \overline{y}) = (2.1, 0.3)$.

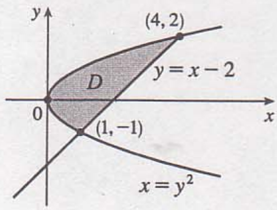
7. $m = \int_0^1 \int_{x^2}^1 xy\,dy\,dx = \int_0^1 \left(\frac{1}{2}x - \frac{1}{2}x^5\right) dx = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$,
$M_y = \int_0^1 \int_{x^2}^1 x^2 y\,dy\,dx = \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{2}x^6\right) dx = \frac{1}{6} - \frac{1}{14} = \frac{2}{21}$ and
$M_x = \int_0^1 \int_{x^2}^1 xy^2\,dy\,dx = \int_0^1 \left(\frac{1}{3}x - \frac{1}{3}x^7\right) dx = \frac{1}{6} - \frac{1}{24} = \frac{1}{8}$. Hence $m = \frac{1}{6}$, $(\overline{x}, \overline{y}) = \left(\frac{4}{7}, \frac{3}{4}\right)$.

8. $m = \int_{-3}^3 \int_0^{9-x^2} y\,dy\,dx = \int_{-3}^3 \frac{1}{2}\left(81 - 18x^2 + x^4\right) dx = 243 - 162 + \frac{243}{5} = \frac{648}{5} = 3^4 \cdot \frac{8}{5}$. $M_y = 0$ since ρ is independent of x and the region is symmetric about the y-axis, and

$$M_x = \int_{-3}^3 \int_0^{9-x^2} y^2\,dy\,dx = \int_{-3}^3 \frac{1}{3}\left(9-x^2\right)^3 dx = 2\int_0^3 \left(243 - 81x^2 + 9x^4 - \frac{1}{3}x^6\right) dx$$
$$= 2\left[3^6 - 3^6 + \frac{1}{5}3^7 - \frac{1}{21}3^7\right] = 2\left[3^6 \cdot \frac{21-5}{35}\right] = 3^6 \cdot \frac{32}{35}$$

Hence $m = \frac{648}{5}$, $(\overline{x}, \overline{y}) = \left(0, \frac{36}{7}\right)$.

9.



$$m = \int_{-1}^2 \int_{y^2}^{y+2} 3\,dx\,dy = \int_{-1}^2 \left(3y + 6 - 3y^2\right) dy = \frac{27}{2},$$
$$M_y = \int_{-1}^2 \int_{y^2}^{y+2} 3x\,dx\,dy = \int_{-1}^2 \frac{3}{2}\left[(y+2)^2 - y^4\right] dy$$
$$= \left[\frac{1}{2}(y+2)^3 - \frac{3}{10}y^5\right]_{-1}^2 = \frac{108}{5}$$

and

$$M_x = \int_{-1}^2 \int_{y^2}^{y+2} 3y\,dx\,dy = \int_{-1}^2 \left(3y^2 + 6y - 3y^3\right) dy$$
$$= \left[y^3 + 3y^2 - \frac{3}{4}y^4\right]_{-1}^2 = \frac{27}{4}$$

Hence $m = \frac{27}{2}$, $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$.

10. $m = \int_0^{\pi/2} \int_0^{\cos x} x\,dy\,dx = \int_0^{\pi/2} x\cos x\,dx = \left[x\sin x + \cos x\right]_0^{\pi/2} = \frac{\pi}{2} - 1$,
$M_y = \int_0^{\pi/2} \int_0^{\cos x} x^2\,dy\,dx = \int_0^{\pi/2} x^2 \cos x\,dx = \left[x^2 \sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2} = \frac{\pi^2}{4} - 2$, and
$M_x = \int_0^{\pi/2} \int_0^{\cos x} xy\,dy\,dx = \int_0^{\pi/2} \frac{1}{2}x\cos^2 x\,dx = \frac{1}{2}\left[x^2 - x + (x-1)\sin x\cos x\right]_0^{\pi/2} = \frac{\pi^2}{8} - \frac{\pi}{4}$. Hence
$m = \frac{\pi - 2}{2}$, $(\overline{x}, \overline{y}) = \left(\frac{\pi^2 - 8}{2(\pi - 2)}, \frac{\pi}{4}\right)$.

11. $\rho(x, y) = ky = kr\sin\theta$, $m = \int_0^{\pi/2} \int_0^1 kr^2 \sin\theta\,dr\,d\theta = \frac{1}{3}k\int_0^{\pi/2} \sin\theta\,d\theta = \frac{1}{3}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{3}k$,
$M_y = \int_0^{\pi/2} \int_0^1 kr^3 \sin\theta\cos\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{1}{8}k\left[-\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}k$,
$M_x = \int_0^{\pi/2} \int_0^1 kr^3 \sin^2\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{8}k\left[\theta + \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{16}k$. Hence $(\overline{x}, \overline{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

12. $\rho(x, y) = k\left(x^2 + y^2\right) = kr^2$, $m = \int_0^{\pi/2} \int_0^1 kr^3\,dr\,d\theta = \frac{\pi}{8}k$,
$M_y = \int_0^{\pi/2} \int_0^1 kr^4 \cos\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2} \cos\theta\,d\theta = \frac{1}{5}k\left[\sin\theta\right]_0^{\pi/2} = \frac{1}{5}k$,
$M_x = \int_0^{\pi/2} \int_0^1 kr^4 \sin\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2} \sin\theta\,d\theta = \frac{1}{5}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{5}k$. Hence $(\overline{x}, \overline{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$.

13. Placing the vertex opposite the hypotenuse at $(0, 0)$, $\rho(x, y) = k\left(x^2 + y^2\right)$. Then

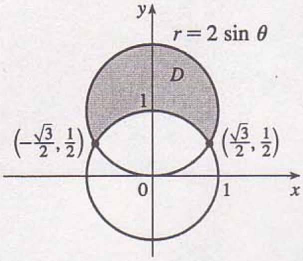
$$m = \int_0^a \int_0^{a-x} k\left(x^2 + y^2\right) dy\,dx = k\int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3\right] dx$$
$$= k\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4 - \frac{1}{12}(a-x)^4\right]_0^a = \frac{1}{6}ka^4$$

By symmetry,

$$M_y = M_x = \int_0^a \int_0^{a-x} ky\left(x^2+y^2\right)dy\,dx = k\int_0^a \left[\tfrac{1}{2}(a-x)^2 x^2 + \tfrac{1}{4}(a-x)^4\right]dx$$
$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence $(\overline{x},\overline{y}) = \left(\tfrac{2}{5}a, \tfrac{2}{5}a\right)$.

14.



$$\rho(x,y) = k/\sqrt{x^2+y^2} = k/r,$$
$$m = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} \frac{k}{r}\,r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6}\left[(2\sin\theta)-1\right]d\theta$$
$$= k\left[-2\cos\theta - \theta\right]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3} - \tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \tfrac{1}{2}k\int_{\pi/6}^{5\pi/6}\left(4\sin^3\theta - \sin\theta\right)d\theta$$
$$= \tfrac{1}{2}k\left[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6} = \sqrt{3}k$$

Hence $(\overline{x},\overline{y}) = \left(0, \frac{3\sqrt{3}}{2(3\sqrt{3}-\pi)}\right)$.

15. $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^1\int_{x^2}^1 y^2(xy)\,dy\,dx = \int_0^1\left[\tfrac{1}{4}xy^4\right]_{y=x^2}^{y=1}dx = \int_0^1 \tfrac{1}{4}\left(x-x^9\right)dx = \tfrac{1}{8} - \tfrac{1}{40} = \tfrac{1}{10}$,
$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^1\int_{x^2}^1 x^3y\,dy\,dx = \int_0^1\left[\tfrac{1}{2}x^3y^2\right]_{y=x^2}^{y=1}dx = \int_0^1 \tfrac{1}{2}\left(x^3-x^7\right)dx = \tfrac{1}{8} - \tfrac{1}{16} = \tfrac{1}{16}$,
$I_0 = I_x + I_y = \tfrac{13}{80}$.

16. $I_x = \int_0^{\pi/2}\int_0^1 \left(r^2\sin^2\theta\right)\left(kr^2\right)r\,dr\,d\theta = \tfrac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta = \tfrac{1}{6}k\left[\tfrac{1}{4}(2\theta - \sin 2\theta)\right]_0^{\pi/2} = \tfrac{\pi}{24}k$,
$I_y = \int_0^{\pi/2}\int_0^1 \left(r^2\cos^2\theta\right)\left(kr^2\right)r\,dr\,d\theta = \tfrac{1}{6}k\int_0^{\pi/6}\cos^2\theta\,d\theta = \tfrac{1}{6}k\left[\tfrac{1}{4}(2\theta + \sin 2\theta)\right]_0^{\pi/2} = \tfrac{\pi}{24}k$, and
$I_0 = I_x + I_y = \tfrac{\pi}{12}k$.

17. $I_x = \int_{-1}^2\int_{y^2}^{y+2} 3y^2\,dx\,dy = \int_{-1}^2\left(3y^3 + 6y^2 - 3y^4\right)dy = \left[\tfrac{3}{4}y^4 + 2y^3 - \tfrac{3}{5}y^5\right]_{-1}^2 = \tfrac{189}{20}$,
$I_y = \int_{-1}^2\int_{y^2}^{y+2} 3x^2\,dx\,dy = \int_{-1}^2\left[(y+2)^3 - y^6\right]dy = \left[\tfrac{1}{4}(y+2)^4 - \tfrac{1}{7}y^7\right]_{-1}^2 = \tfrac{1269}{28}$, and
$I_0 = I_x + I_y = \tfrac{1917}{35}$.

18. If we find the moments of inertia about the x- and y-axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x-axis is given by

$$I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^2\int_0^2 y^2(1+0.1x)\,dy\,dx = \int_0^2(1+0.1x)\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2}dx$$
$$= \tfrac{8}{3}\int_0^2(1+0.1x)\,dx = \tfrac{8}{3}\left[x + 0.1\cdot\tfrac{1}{2}x^2\right]_0^2 = \tfrac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^2\int_0^2 x^2(1+0.1x)\,dy\,dx = \int_0^2 x^2(1+0.1x)\left[y\right]_{y=0}^{y=2}dx$$
$$= 2\int_0^2\left(x^2+0.1x^3\right)dx = 2\left[\tfrac{1}{3}x^3 + 0.1\cdot\tfrac{1}{4}x^4\right]_0^2 = 2\left(\tfrac{8}{3} + 0.4\right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y-axis.

19. Using a CAS, we find $m = \iint_D \rho(x,y)\,dA = \int_0^\pi \int_0^{\sin x} xy\,dy\,dx = \dfrac{\pi^2}{8}$. Then

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{8}{\pi^2}\int_0^\pi \int_0^{\sin x} x^2 y\,dy\,dx = \frac{2\pi}{3} - \frac{1}{\pi} \text{ and}$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{8}{\pi^2}\int_0^\pi \int_0^{\sin x} xy^2\,dy\,dx = \frac{16}{9\pi}, \text{ so } (\overline{x},\overline{y}) = \left(\frac{2\pi}{3} - \frac{1}{\pi}, \frac{16}{9\pi}\right).$$

The moments of inertia are $I_x = \iint_D y^2 \rho(x,y)\,dA = \int_0^\pi \int_0^{\sin x} xy^3\,dy\,dx = \dfrac{3\pi^2}{64}$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^\pi \int_0^{\sin x} x^3 y\,dy\,dx = \dfrac{\pi^2}{16}\left(\pi^2 - 3\right)$, and $I_0 = I_x + I_y = \dfrac{\pi^2}{64}\left(4\pi^2 - 9\right)$.

20. Using a CAS, we find $m = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta = \frac{5}{3}\pi$,

$\overline{x} = \frac{1}{m}\iint_D x\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\cos\theta\,dr\,d\theta = \frac{21}{20}$ and

$\overline{y} = \frac{1}{m}\iint_D y\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\sin\theta\,dr\,d\theta = 0$, so $(\overline{x},\overline{y}) = \left(\frac{21}{20}, 0\right)$. The moments of inertia are $I_x = \iint_D y^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\sin^2\theta\,dr\,d\theta = \frac{33}{40}\pi$,

$I_y = \iint_D x^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\cos^2\theta\,dr\,d\theta = \frac{93}{40}\pi$, and $I_0 = I_x + I_y = \frac{63}{20}\pi$.

21. $I_x = \int_0^a\int_0^a \rho y^2\,dx\,dy = \rho\int_0^a dx \int_0^a y^2\,dy = \rho\,[x]_0^a\left[\frac{1}{3}y^3\right]_0^a = \rho a\left(\frac{1}{3}a^3\right) = \frac{1}{3}\rho a^4 = I_y$ by symmetry, and $m = \rho a^2$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} \Rightarrow \overline{\overline{x}} = \left[\left(\frac{1}{3}\rho a^4\right)/\left(\rho a^2\right)\right]^{1/2} = \frac{1}{\sqrt{3}}a$ and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} \Rightarrow \overline{\overline{y}} = \frac{1}{\sqrt{3}}a$.

22. $m = \int_0^\pi\int_0^{\sin x} \rho\,dy\,dx = \rho\int_0^\pi \sin x\,dx = \rho\left[-\cos x\right]_0^\pi = 2\rho$,

$$\begin{aligned} I_x &= \int_0^\pi\int_0^{\sin x} \rho y^2\,dy\,dx = \tfrac{1}{3}\rho\int_0^\pi \sin^3 x\,dx = \tfrac{1}{3}\rho\int_0^\pi\left(1-\cos^2 x\right)\sin x\,dx \\ &= \tfrac{1}{3}\rho\left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^\pi = \tfrac{4}{9}\rho, \end{aligned}$$

$$\begin{aligned} I_y &= \int_0^\pi\int_0^{\sin x} \rho x^2\,dy\,dx = \rho\int_0^\pi x^2\sin x\,dx = \rho\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^\pi \\ &\qquad \text{(by integrating by parts twice)} \qquad = \rho\left(\pi^2 - 4\right). \end{aligned}$$

Then $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{2}{9}$, so $\overline{\overline{y}} = \dfrac{\sqrt{2}}{3}$ and $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\pi^2-4}{2}$, so $\overline{\overline{x}} = \sqrt{\dfrac{\pi^2-4}{2}}$.

23. (a) $f(x,y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,1]\times[0,2]$, we can say

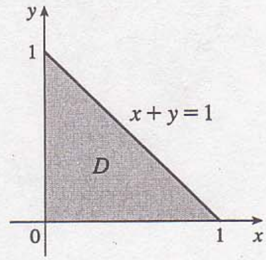
$$\begin{aligned} \iint_{\mathbb{R}^2} f(x,y)\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^2 Cx\,(1+y)\,dy\,dx \\ &= C\int_0^1 x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=2} dx = C\int_0^1 4x\,dx = C\left[2x^2\right]_0^1 = 2C \end{aligned}$$

Then $2C = 1 \Rightarrow C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{1}{2}x\,(1+y)\,dy\,dx$

$$= \int_0^1 \tfrac{1}{2}x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=1} dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{3}{2}\right) dx = \tfrac{3}{4}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{3}{8} \text{ or } 0.375$$

(c) $P(X+Y \leq 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y \leq 1) = \iint_D f(x,y)\,dA = \int_0^1 \int_0^{1-x} \tfrac{1}{2}x(1+y)\,dy\,dx$$
$$= \int_0^1 \tfrac{1}{2}x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}\right) dx$$
$$= \tfrac{1}{4}\int_0^1 \left(x^3 - 4x^2 + 3x\right) dx = \tfrac{1}{4}\left[\tfrac{x^4}{4} - 4\tfrac{x^3}{3} + 3\tfrac{x^2}{2}\right]_0^1$$
$$= \tfrac{5}{48} \approx 0.1042$$

24. (a) $f(x,y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\;dA = 1$. Here, $f(x,y) = 0$ outside the square $[0,1] \times [0,1]$, so $\iint_{\mathbb{R}^2} f(x,y)\;dA = \int_0^1 \int_0^1 4xy\,dy\,dx = \int_0^1 \left[2xy^2\right]_{y=0}^{y=1} dx = \int_0^1 2x\,dx = x^2\big]_0^1 = 1$. Thus, $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on Y, so

$$P\left(X \geq \tfrac{1}{2}\right) = \int_{1/2}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{1/2}^1 \int_0^1 4xy\,dy\,dx$$
$$= \int_{1/2}^1 \left[2xy^2\right]_{y=0}^{y=1} dx = \int_{1/2}^1 2x\,dx = x^2\big]_{1/2}^1 = \tfrac{3}{4}$$

(ii) $P\left(X \geq \tfrac{1}{2}, Y \leq \tfrac{1}{2}\right) = \int_{1/2}^{\infty} \int_{-\infty}^{1/2} f(x,y)\,dy\,dx = \int_{1/2}^1 \int_0^{1/2} 4xy\,dy\,dx$

$$= \int_{1/2}^1 \left[2xy^2\right]_{y=0}^{y=1/2} dx = \int_{1/2}^1 \tfrac{1}{2}x\,dx = \tfrac{1}{2} \cdot \tfrac{1}{2}x^2\big]_{1/2}^1 = \tfrac{3}{16}$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\,dA = \int_0^1 \int_0^1 x(4xy)\,dy\,dx = \int_0^1 2x^2\left[y^2\right]_{y=0}^{y=1} dx = 2\int_0^1 x^2\,dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^1 \int_0^1 y(4xy)\;dy\,dx = \int_0^1 4x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1} dx = \tfrac{4}{3}\int_0^1 x\,dx = \tfrac{4}{3}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{2}{3}$$

25. (a) $f(x,y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\;dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^{\infty} \int_0^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx = 0.1\int_0^{\infty} \int_0^{\infty} e^{-0.5x}e^{-0.2y}\,dy\,dx$$
$$= 0.1\int_0^{\infty} e^{-0.5x}\,dx \int_0^{\infty} e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy$$
$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$
$$= 0.1\lim_{t\to\infty}\left[-2\left(e^{-0.5t} - 1\right)\right] \lim_{t\to\infty}\left[-5\left(e^{-0.2t} - 1\right)\right]$$
$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$\begin{aligned}
P(Y \geq 1) &= \int_{-\infty}^{\infty} \int_{1}^{\infty} f(x,y)\, dy\, dx = \int_{0}^{\infty} \int_{1}^{\infty} 0.1e^{-(0.5x+0.2y)}\, dy\, dx \\
&= 0.1 \int_{0}^{\infty} e^{-0.5x}\, dx \int_{1}^{\infty} e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty} \int_{0}^{t} e^{-0.5x}\, dx \lim_{t\to\infty} \int_{1}^{t} e^{-0.2y}\, dy \\
&= 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_{0}^{t} \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_{1}^{t} \\
&= 0.1 \lim_{t\to\infty} \left[-2\left(e^{-0.5t}-1\right)\right] \lim_{t\to\infty} \left[-5\left(e^{-0.2t}-e^{-0.2}\right)\right] \\
&= (0.1)\cdot(-2)(0-1)\cdot(-5)\left(0-e^{-0.2}\right) = e^{-0.2} \approx 0.8187
\end{aligned}$$

(ii) $$\begin{aligned}
P(X \leq 2, Y \leq 4) &= \int_{-\infty}^{2} \int_{-\infty}^{4} f(x,y)\, dy\, dx = \int_{0}^{2} \int_{0}^{4} 0.1e^{-(0.5x+0.2y)}\, dy\, dx \\
&= 0.1 \int_{0}^{2} e^{-0.5x}\, dx \int_{0}^{4} e^{-0.2y}\, dy = 0.1 \left[-2e^{-0.5x}\right]_{0}^{2} \left[-5e^{-0.2y}\right]_{0}^{4} \\
&= (0.1)\cdot(-2)\left(e^{-1}-1\right)\cdot(-5)\left(e^{-0.8}-1\right) \\
&= \left(e^{-1}-1\right)\left(e^{-0.8}-1\right) = 1+e^{-1.8}-e^{-0.8}-e^{-1} \approx 0.3481
\end{aligned}$$

(c) The expected value of X is given by

$$\begin{aligned}
\mu_1 &= \iint_{\mathbb{R}^2} x f(x,y)\, dA = \int_{0}^{\infty} \int_{0}^{\infty} x \left[0.1e^{-(0.5x+0.2y)}\right] dy\, dx \\
&= 0.1 \int_{0}^{\infty} xe^{-0.5x}\, dx \int_{0}^{\infty} e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty} \int_{0}^{t} xe^{-0.5x}\, dx \lim_{t\to\infty} \int_{0}^{t} e^{-0.2y}\, dy
\end{aligned}$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\, dx$ (or we can use Formula 96 in the Table of Integrals):
$\int xe^{-0.5x}\, dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\, dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)\,e^{-0.5x}$. Thus

$$\begin{aligned}
\mu_1 &= 0.1 \lim_{t\to\infty} \left[-2(x+2)\,e^{-0.5x}\right]_{0}^{t} \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_{0}^{t} \\
&= 0.1 \lim_{t\to\infty} (-2)\left[(t+2)\,e^{-0.5t}-2\right] \lim_{t\to\infty} (-5)\left[e^{-0.2t}-1\right] \\
&= 0.1(-2)\left(\lim_{t\to\infty} \frac{t+2}{e^{0.5t}} - 2\right)(-5)(-1) = 2 \quad \text{(by l'Hospital's Rule)}
\end{aligned}$$

The expected value of Y is given by

$$\begin{aligned}
\mu_2 &= \iint_{\mathbb{R}^2} y f(x,y)\, dA = \int_{0}^{\infty} \int_{0}^{\infty} y \left[0.1e^{-(0.5+0.2y)}\right] dy\, dx \\
&= 0.1 \int_{0}^{\infty} e^{-0.5x}\, dx \int_{0}^{\infty} ye^{-0.2y}\, dy = 0.1 \lim_{t\to\infty} \int_{0}^{t} e^{-0.5x}\, dx \lim_{t\to\infty} \int_{0}^{t} ye^{-0.2y}\, dy
\end{aligned}$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\, dy$ (or again we can use Formula 96 in the Table of Integrals) which gives
$\int ye^{-0.2y}\, dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\, dy = -5(y+5)\,e^{-0.2y}$. Then

$$\begin{aligned}
\mu_2 &= 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_{0}^{t} \lim_{t\to\infty} \left[-5(y+5)\,e^{-0.2y}\right]_{0}^{t} \\
&= 0.1 \lim_{t\to\infty} \left[-2\left(e^{-0.5t}-1\right)\right] \lim_{t\to\infty} \left(-5\left[(t+5)\,e^{-0.2t}-5\right]\right) \\
&= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \quad \text{(by l'Hospital's Rule)}
\end{aligned}$$

26. (a) Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

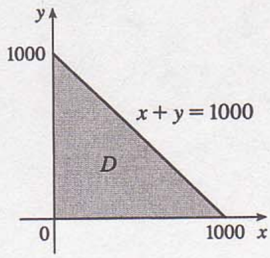
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6} e^{-(x+y)/1000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned} P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x,y)\,dy\,dx \\ &= \int_0^{1000} \int_0^{1000} 10^{-6} e^{-(x+y)/1000}\,dy\,dx \\ &= 10^{-6} \int_0^{1000} e^{-x/1000}\,dx \int_0^{1000} e^{-y/1000}\,dy \\ &= 10^{-6} \left[-1000e^{-x/1000}\right]_0^{1000} \left[-1000e^{-y/1000}\right]_0^{1000} \\ &= \left(e^{-1} - 1\right)^2 \approx 0.3996 \end{aligned}$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \leq 1000)$, or equivalently $P((X,Y) \in D)$ where D is the triangular region shown in the figure. Then



$$\begin{aligned} P(X + Y \leq 1000) &= \iint_D f(x,y)\,dA = \int_0^{1000} \int_0^{1000-x} 10^{-6} e^{-(x+y)/1000}\,dy\,dx \\ &= 10^{-6} \int_0^{1000} \left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx = -10^{-3} \int_0^{1000} \left(e^{-1} - e^{-x/1000}\right) dx \\ &= -10^{-3} \left[e^{-1}x + 1000e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642 \end{aligned}$$

27. The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$. The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5}$ and $f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$$\begin{aligned} f(x,y) = f_1(x)\,f_2(y) &= \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5} \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02} \\ &= \tfrac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} \end{aligned}$$

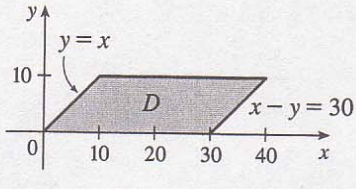
Then
$$\begin{aligned} P(40 \leq X \leq 50, 20 \leq Y \leq 25) &= \int_{40}^{50} \int_{20}^{25} f(x,y)\,dy\,dx \\ &= \tfrac{10}{\pi} \int_{40}^{50} \int_{20}^{25} e^{-2(x-45)^2 - 50(y-20)^2}\,dy\,dx \end{aligned}$$

Using a CAS or calculator to evaluate the integral, we get $P(40 \leq X \leq 50, 20 \leq Y \leq 25) \approx 0.500$.

28. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x)\, f_2(y) = \begin{cases} \frac{1}{50}e^{-x}y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \leq 30$. Thus the probability that they meet is $P((X,Y) \in D)$ where D is the parallelogram shown in the figure.



The integral is simpler to evaluate if we consider D as a type II region, so

$$P((X,Y) \in D) = \iint_D f(x,y)\, dx\, dy = \int_0^{10} \int_y^{y+30} \tfrac{1}{50} e^{-x} y\, dx\, dy = \tfrac{1}{50} \int_0^{10} y \left[-e^{-x}\right]_{x=y}^{x=y+30} dy$$

$$= \tfrac{1}{50} \int_0^{10} y \left(-e^{-(y+30)} + e^{-y}\right) dy = \tfrac{1}{50}\left(1 - e^{-30}\right) \int_0^{10} y e^{-y}\, dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is
$\frac{1}{50}\left(1 - e^{-30}\right)\left[-(y+1)e^{-y}\right]_0^{10} = \frac{1}{50}\left(1 - e^{-30}\right)\left(1 - 11e^{-10}\right) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

29. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\, dA$ is the number of infected individuals in an element of area dA, then $f(P, A)\, k\, dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P,A)\, dA = k \iint_D \frac{20 - d(P,A)}{20}\, dA$$

$$= k \iint_D \left[1 - \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{20}\right] dx\, dy$$

(b) If $A = (0,0)$, then

$$E = k \iint_D \left[1 - \frac{1}{20}\sqrt{x^2+y^2}\right] dx\, dy = k \int_0^{2\pi} \int_0^{10} \left(1 - \frac{r}{20}\right) r\, dr\, d\theta$$

$$= 2\pi k \left[\frac{r^2}{2} - \frac{r^3}{60}\right]_0^{10} = 2\pi k \left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k$$

For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$E = k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1-\frac{r}{20}\right) r\,dr\,d\theta$$

$$= k\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}-\frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta} d\theta$$

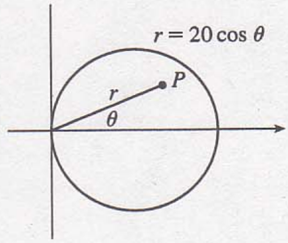
$$= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right)d\theta$$

$$= 200k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1-\sin^2\theta\right)\cos\theta\right]d\theta$$

$$= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2}$$

$$= 200k\left[\tfrac{\pi}{4}+0-\tfrac{2}{3}+\tfrac{2}{9}+\tfrac{\pi}{4}+0-\tfrac{2}{3}+\tfrac{2}{9}\right]$$

$$= 200k\left(\tfrac{\pi}{2}-\tfrac{8}{9}\right) \approx 136k$$



Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

16.6 Surface Area — ET 15.6

1. Here $z = f(x,y) = 2 + 3x + 4y$ and D is the rectangle $[0,5]\times[1,4]$, so by Formula 2 the area of the surface is

$$A(S) = \iint_D \sqrt{3^2+4^2+1}\,dA = \sqrt{26}\iint_D dA = \sqrt{26}\,A(D)$$
$$= \sqrt{26}\,(5)\,(3) = 15\sqrt{26}$$

2. $z = f(x,y) = 10 - 2x - 5y$ and D is the disk $x^2+y^2 \le 9$, so

$$A(S) = \iint_D \sqrt{(-2)^2+(-5)^2+1}\,dA = \sqrt{30}\iint_D dA = \sqrt{30}\,A(D)$$
$$= \sqrt{30}\left(\pi\cdot 3^2\right) = 9\sqrt{30}\,\pi$$

3. $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x+2y=6$, so D is the triangular region given by $\left\{(x,y)\mid 0\le x\le 2, 0\le y\le 3-\tfrac{3}{2}x\right\}$. Thus

$$A(S) = \iint_D \sqrt{(-3)^2+(-2)^2+1}\,dA = \sqrt{14}\iint_D dA = \sqrt{14}\,A(D)$$
$$= \sqrt{14}\left(\tfrac{1}{2}\cdot 2\cdot 3\right) = 3\sqrt{14}$$

4. $z = f(x,y) = x + y^2$ with $0\le x\le y$, $0\le y\le 1$. Thus, by Formula 3,

$$A(S) = \iint_D \sqrt{1+1+4y^2}\,dA = \int_0^1\int_0^y \sqrt{2+4y^2}\,dx\,dy = \int_0^1\left[x\sqrt{2+4y^2}\right]_{x=0}^{x=y} dy$$
$$= \int_0^1 y\sqrt{2+4y^2}\,dy = 2\left(\tfrac{1}{24}\right)\left(2+4y^2\right)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(6\sqrt{6}-2\sqrt{2}\right)$$
$$= \tfrac{3}{\sqrt{6}} - \tfrac{1}{3\sqrt{2}}$$

5. $y^2+z^2=9 \quad\Rightarrow\quad z=\sqrt{9-y^2}$. $f_x=0$, $f_y=-y\left(9-y^2\right)^{-1/2} \quad\Rightarrow$

$$A(S)=\int_0^4\int_0^2\sqrt{0^2+\left[-y\left(9-y^2\right)^{-1/2}\right]^2+1}\,dy\,dx=\int_0^4\int_0^2\sqrt{\frac{y^2}{9-y^2}+1}\,dy\,dx$$
$$=\int_0^4\int_0^2\frac{3}{\sqrt{9-y^2}}\,dy\,dx=3\int_0^4\left[\sin^{-1}\frac{y}{3}\right]_{y=0}^{y=2}dx=3\left[\left(\sin^{-1}\frac{2}{3}\right)x\right]_0^4=12\sin^{-1}\frac{2}{3}$$

6. $z=f(x,y)=4-x^2-y^2$ and D is the projection of the paraboloid $z=4-x^2-y^2$ onto the xy-plane, that is, $D=\left\{(x,y)\mid x^2+y^2\le 4\right\}$. So $f_x=-2x$, $f_y=-2y \quad\Rightarrow$

$$A(S)=\iint_D\sqrt{(-2x)^2+(-2y)^2+1}\,dA=\iint_D\sqrt{4\left(x^2+y^2\right)+1}\,dA=\int_0^{2\pi}\int_0^2\sqrt{4r^2+1}\,r\,dr\,d\theta$$
$$=\int_0^{2\pi}\left[\tfrac{1}{12}\left(4r^2+1\right)^{3/2}\right]_{r=0}^{r=2}d\theta=\int_0^{2\pi}\tfrac{1}{12}\left(17\sqrt{17}-1\right)d\theta=\tfrac{\pi}{6}\left(17\sqrt{17}-1\right).$$

7. $z=f(x,y)=y^2-x^2$ with $1\le x^2+y^2\le 4$. Then

$$A(S)=\iint_D\sqrt{1+4x^2+4y^2}\,dA=\int_0^{2\pi}\int_1^2\sqrt{1+4r^2}\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_1^2\sqrt{1+4r^2}\,r\,dr$$
$$=[\theta]_0^{2\pi}\left[\tfrac{1}{12}\left(1+4r^2\right)^{3/2}\right]_1^2=\tfrac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

8. $z=f(x,y)=\frac{2}{3}\left(x^{3/2}+y^{3/2}\right)$ and $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$. Then $f_x=x^{1/2}$, $f_y=y^{1/2}$ and

$$A(S)=\iint_D\sqrt{\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+1}\,dA=\int_0^1\int_0^1\sqrt{x+y+1}\,dy\,dx$$
$$=\int_0^1\left[\frac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1}dx=\frac{2}{3}\int_0^1\left[(x+2)^{3/2}-(x+1)^{3/2}\right]dx$$
$$=\frac{2}{3}\left[\frac{2}{5}(x+2)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_0^1=\frac{4}{15}\left(3^{5/2}-2^{5/2}-2^{5/2}+1\right)$$
$$=\frac{4}{15}\left(3^{5/2}-2^{7/2}+1\right)$$

9. $z=f(x,y)=xy$ with $0\le x^2+y^2\le 1$, so $f_x=y$, $f_y=x \quad\Rightarrow$

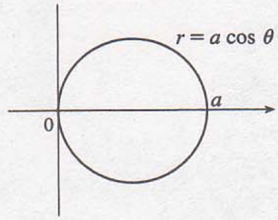
$$A(S)=\iint_D\sqrt{y^2+x^2+1}\,dA=\int_0^{2\pi}\int_0^1\sqrt{r^2+1}\,r\,dr\,d\theta=\int_0^{2\pi}\left[\tfrac{1}{3}\left(r^2+1\right)^{3/2}\right]_{r=0}^{r=1}d\theta$$
$$=\int_0^{2\pi}\tfrac{1}{3}\left(2\sqrt{2}-1\right)d\theta=\tfrac{2\pi}{3}\left(2\sqrt{2}-1\right).$$

10. Given the sphere $x^2+y^2+z^2=4$, when $z=1$, we get $x^2+y^2=3$ so $D=\left\{(x,y)\mid x^2+y^2\le 3\right\}$ and $z=f(x,y)=\sqrt{4-x^2-y^2}$. Thus

$$A(S)=\iint_D\sqrt{\left[(-x)\left(4-x^2-y^2\right)^{-1/2}\right]^2+\left[(-y)\left(4-x^2-y^2\right)^{-1/2}\right]^2+1}\,dA=\int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{\frac{r^2}{4-r^2}+1}\,r\,dr\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2r}{\sqrt{4-r^2}}\,dr\,d\theta=\int_0^{2\pi}\left[-2\left(4-r^2\right)^{1/2}\right]_{r=0}^{r=\sqrt{3}}d\theta=\int_0^{2\pi}(-2+4)\,d\theta=2\theta\Big]_0^{2\pi}=4\pi.$$

11. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x\left(a^2 - x^2 - y^2\right)^{-1/2}$, $z_y = -y\left(a^2 - x^2 - y^2\right)^{-1/2}$,

$$\begin{aligned}
A(S) &= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left[-a\sqrt{a^2 - r^2}\right]_{r=0}^{r=a\cos\theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\theta} - a\right) d\theta = 2a^2 \int_0^{\pi/2}\left(1 - \sqrt{1 - \cos^2\theta}\right) d\theta \\
&= 2a^2 \int_0^{\pi/2} d\theta - 2a^2 \int_0^{\pi/2} \sqrt{\sin^2\theta}\, d\theta = a^2\pi - 2a^2 \int_0^{\pi/2} \sin\theta\, d\theta = a^2(\pi - 2)
\end{aligned}$$



12. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z - 2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z - 2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3 - 2)^2 = 4$, that is, $D = \left\{(x, y) \mid x^2 + y^2 \leq 3\right\}$.

$$\begin{aligned}
A(S) &= \iint_D \sqrt{\left[(-x)\left(4 - x^2 - y^2\right)^{-1/2}\right]^2 + \left[(-y)\left(4 - x^2 - y^2\right)^{-1/2}\right]^2 + 1}\, dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\, dr}{\sqrt{4 - r^2}}\, d\theta = \int_0^{2\pi} \left[-2\left(4 - r^2\right)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta \\
&= \textstyle\int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\big]_0^{2\pi} = 4\pi
\end{aligned}$$

13. (a) The midpoints of the four squares are $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{1}{4}, \frac{3}{4}\right)$, $\left(\frac{3}{4}, \frac{1}{4}\right)$, and $\left(\frac{3}{4}, \frac{3}{4}\right)$; the derivatives of the function $f(x, y) = x^2 + y^2$ are $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, so the Midpoint Rule gives

$$\begin{aligned}
A(S) &= \textstyle\int_0^1 \int_0^1 \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dy\, dx \\
&\approx \tfrac{1}{4}\left(\sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2 + \left[2\left(\tfrac{1}{4}\right)\right]^2 + 1} + \sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2 + \left[2\left(\tfrac{3}{4}\right)\right]^2 + 1}\right. \\
&\qquad \left. + \sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2 + \left[2\left(\tfrac{1}{4}\right)\right]^2 + 1} + \sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2 + \left[2\left(\tfrac{3}{4}\right)\right]^2 + 1}\right) \\
&= \tfrac{1}{4}\left(\sqrt{\tfrac{3}{2}} + 2\sqrt{\tfrac{7}{2}} + \sqrt{\tfrac{11}{2}}\right) \approx 1.8279
\end{aligned}$$

(b) A CAS estimates the integral to be $A(S) = \int_0^1 \int_0^1 \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_0^1 \int_0^1 \sqrt{4x^2 + 4y^2 + 1}\, dy\, dx \approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

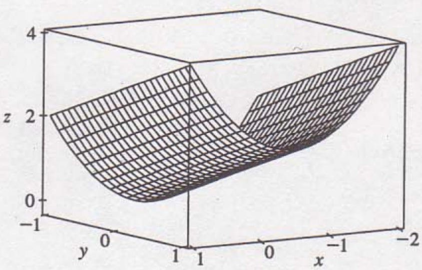
14. (a) With $m = n = 2$ we have four squares with midpoints $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}\right)$, and $\left(\frac{3}{2}, \frac{3}{2}\right)$. Since $z = f(x, y) = xy + x^2 + y^2$, the Midpoint Rule gives

$$\begin{aligned} A(S) &= \int_0^2 \int_0^2 \sqrt{(y+2x)^2 + (x+2y)^2 + 1}\, dy\, dx \\ &\approx 1\left(\sqrt{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{5}{2}\right)^2 + \left(\tfrac{7}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{7}{2}\right)^2 + \left(\tfrac{5}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{9}{2}\right)^2 + \left(\tfrac{9}{2}\right)^2 + 1}\right) \\ &= \frac{\sqrt{22}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{166}}{2} \approx 17.619 \end{aligned}$$

(b) Using a CAS, we have $A(S) = \int_0^2 \int_0^2 \sqrt{(y+2x)^2 + (x+2y)^2 + 1}\, dy\, dx \approx 17.7165$. This is within about 0.1 of the Midpoint Rule estimate.

15. Since $f(x, y) = x^2 + 2y$, we have $f_x = 2x$, $f_y = 2$. We use a CAS to calculate the integral $A(S) = \int_0^1 \int_0^1 \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_0^1 \int_0^1 \sqrt{4x^2 + 5}\, dy\, dx = \int_0^1 \sqrt{4x^2 + 5}\, dx$, and find that $A(S) = \frac{3}{2} + \frac{5}{8} \ln 5$.

16. $f(x, y) = 1 + x + y + x^2 \quad \Rightarrow \quad f_x = 1 + 2x$, $f_y = 1$. We use a CAS to calculate the integral $A(S) = \int_{-2}^1 \int_{-1}^1 \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-2}^1 \int_{-1}^1 \sqrt{(1+2x)^2 + 2}\, dy\, dx = 2 \int_{-2}^1 \sqrt{4x^2 + 4x + 3}\, dx$ and find that $A(S) = 3\sqrt{11} + 2 \sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or $A(S) = 3\sqrt{11} + \ln\left(10 + 3\sqrt{11}\right)$.



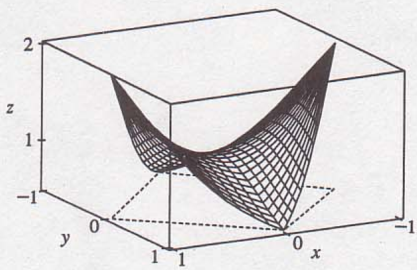
17. $f(x, y) = 1 + x^2 y^2 \quad \Rightarrow \quad f_x = 2xy^2$, $f_y = 2x^2 y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral $A(S) = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2 y^4 + 4x^4 y^2 + 1}\, dy\, dx$, and find that $A(S) \approx 3.3213$.

18. Let $f(x, y) = \dfrac{1 + x^2}{1 + y^2}$. Then $f_x = \dfrac{2x}{1 + y^2}$,

$$f_y = \left(1 + x^2\right)\left[-\frac{2y}{\left(1 + y^2\right)^2}\right] = -\frac{2y\left(1 + x^2\right)}{\left(1 + y^2\right)^2}.$$

We use a CAS to estimate

$\int_{-1}^1 \int_{-(1-|x|)}^{1-|x|} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx \approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1 - |x|) \le y \le 1 - |x|$ as the y-range in our plot command.



19. Here $z = f(x, y) = ax + by + c$, $f_x(x, y) = a$, $f_y(x, y) = b$, so
$A(S) = \iint_D \sqrt{a^2 + b^2 + 1}\, dA = \sqrt{a^2 + b^2 + 1} \iint_D dA = \sqrt{a^2 + b^2 + 1}\, A(D)$.

20. Let S be the upper hemisphere. Then $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$, so

$$A(S) = \iint_D \sqrt{\left[-x\left(a^2 - x^2 - y^2\right)^{-1/2}\right]^2 + \left[-y\left(a^2 - x^2 - y^2\right)^{-1/2}\right]^2 + 1}\, dA$$

$$= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA = \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta$$

$$= \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta = 2\pi \lim_{t \to a^-} \left[-a\sqrt{a^2 - r^2}\right]_0^t = 2\pi \lim_{t \to a^-} -a\left[\sqrt{a^2 - t^2} - a\right]$$

$= 2\pi(-a)(-a) = 2\pi a^2$. Thus the surface area of the entire sphere is $4\pi a^2$.

21. If we project the surface onto the xz-plane, then the surface lies "above" the disk $x^2 + z^2 \leq 25$ in the xz-plane. We have $y = f(x, z) = x^2 + z^2$ and, adapting Formula 2, the area of the surface is

$$A(S) = \iint_{x^2 + z^2 \leq 25} \sqrt{[f_x(x, z)]^2 + [f_z(x, z)]^2 + 1}\, dA = \iint_{x^2 + z^2 \leq 25} \sqrt{4x^2 + 4z^2 + 1}\, dA$$

Converting to polar coordinates $x = r\cos\theta$, $z = r\sin\theta$ we have

$$A(S) = \int_0^{2\pi} \int_0^5 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^5 r\left(4r^2 + 1\right)^{1/2} dr$$

$$= [\theta]_0^{2\pi} \left[\tfrac{1}{12}\left(4r^2 + 1\right)^{3/2}\right]_0^5 = \tfrac{\pi}{6}\left(101\sqrt{101} - 1\right)$$

22. First we find the area of the face of the surface that intersects the positive y-axis. As in Exercise 21, we can project the face onto the xz-plane, so the surface lies "above" the disk $x^2 + z^2 \leq 1$. Then $z = f(x, z) = \sqrt{1 - z^2}$ and the area is

$$A(S) = \iint_{x^2 + z^2 \leq 1} \sqrt{[f_x(x, z)]^2 + [f_z(x, z)]^2 + 1}\, dA = \iint_{x^2 + z^2 \leq 1} \sqrt{0 + \left(\frac{-z}{\sqrt{1 - z^2}}\right)^2 + 1}\, dA$$

$$= \iint_{x^2 + z^2 \leq 1} \sqrt{\frac{z^2}{1 - z^2} + 1}\, dA = \int_{-1}^1 \int_{-\sqrt{1 - z^2}}^{\sqrt{1 - z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz$$

$$= 4\int_0^1 \int_0^{\sqrt{1 - z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz \quad \text{(by the symmetry of the surface)}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t \to 1^-} 4\int_0^t \int_0^{\sqrt{1 - z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz = \lim_{t \to 1^-} 4\int_0^t \frac{\sqrt{1 - z^2}}{\sqrt{1 - z^2}}\, dz$$

$$= \lim_{t \to 1^-} 4\int_0^t dz = \lim_{t \to 1^-} 4t = 4$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

16.7 Triple Integrals ET 15.7

1. $\int_0^1 \int_{-1}^2 \int_0^3 xyz^2\,dz\,dx\,dy = \int_0^1 \int_{-1}^2 xy\left[\frac{1}{3}z^3\right]_{z=0}^{z=3} dx\,dy = \int_0^1 \int_{-1}^2 9xy\,dx\,dy$

$= \int_0^1 \left[\frac{9}{2}x^2 y\right]_{x=-1}^{x=2} dy = \int_0^1 \frac{27}{2}y\,dy = \frac{27}{4}y^2\Big]_0^1 = \frac{27}{4}$

2. $\int_0^2 \int_{-3}^0 \int_{-1}^1 (x^2+yz)\,dz\,dy\,dx = \int_0^2 \int_{-3}^0 \left[x^2 z + \frac{1}{2}yz^2\right]_{z=-1}^{z=1} dy\,dx = \int_0^2 \int_{-3}^0 2x^2\,dy\,dx$

$= \int_0^2 \left[2x^2 y\right]_{y=-3}^{y=0} dx = \int_0^2 6x^2\,dx = 2x^3\Big]_0^2 = 16$

$\int_{-1}^1 \int_{-3}^0 \int_0^2 (x^2+yz)\,dx\,dy\,dz = \int_{-1}^1 \int_{-3}^0 \left[\frac{1}{3}x^3 + xyz\right]_{x=0}^{x=2} dy\,dz = \int_{-1}^1 \int_{-3}^0 \left(\frac{8}{3}+2yz\right) dy\,dz$

$= \int_{-1}^1 \left[\frac{8}{3}y + y^2 z\right]_{y=-3}^{y=0} dz = \int_{-1}^1 (8-9z)\,dz = \left[8z - \frac{9}{2}z^2\right]_{-1}^1 = 16$

$\int_{-1}^1 \int_0^2 \int_{-3}^0 (x^2+yz)\,dy\,dx\,dz = \int_{-1}^1 \int_0^2 \left[x^2 y + \frac{1}{2}y^2 z\right]_{y=-3}^{y=0} dx\,dz = \int_{-1}^1 \int_0^2 \left(3x^2 - \frac{9}{2}z\right) dx\,dz$

$= \int_{-1}^1 \left[x^3 - \frac{9}{2}xz\right]_{x=0}^{x=2} dz = \int_{-1}^1 (8-9z)\,dz = \left[8z - \frac{9}{2}z^2\right]_{-1}^1 = 16$

3. $\int_0^1 \int_0^z \int_0^{x+z} 6xz\,dy\,dx\,dz = \int_0^1 \int_0^z \left[6xyz\right]_{y=0}^{y=x+z} dx\,dz = \int_0^1 \int_0^z 6xz\,(x+z)\,dx\,dz$

$= \int_0^1 \left[2x^3 z + 3x^2 z^2\right]_{x=0}^{x=z} dz = \int_0^1 \left(2z^4 + 3z^4\right) dz = \int_0^1 5z^4\,dz = z^5\Big]_0^1 = 1$

4. $\int_1^2 \int_0^x \int_0^{1-y} x^3 y^2 z\,dz\,dy\,dx = \int_1^2 \int_0^x \left[\frac{1}{2}x^3 y^2 z^2\right]_{z=0}^{z=1-y} dy\,dx = \int_1^2 \int_0^x \frac{1}{2}x^3 y^2 (1-y)^2\,dy\,dx$

$= \int_1^2 \int_0^x \left(\frac{1}{2}x^3 y^2 - x^3 y^3 + \frac{1}{2}x^3 y^4\right) dy\,dx$

$= \int_1^2 \left[\frac{1}{6}x^3 y^3 - \frac{1}{4}x^3 y^4 + \frac{1}{10}x^3 y^5\right]_{y=0}^{y=x} dx$

$= \int_1^2 \left(\frac{1}{6}x^6 - \frac{1}{4}x^7 + \frac{1}{10}x^8\right) dx = \left[\frac{1}{42}x^7 - \frac{1}{32}x^8 + \frac{1}{90}x^9\right]_1^2$

$= \frac{128}{42} - \frac{256}{32} + \frac{512}{90} - \frac{1}{42} + \frac{1}{32} - \frac{1}{90} = \frac{7387}{10080}$

5. $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy = \int_0^3 \int_0^1 \left[xze^y\right]_{x=0}^{x=\sqrt{1-z^2}} dz\,dy = \int_0^3 \int_0^1 ze^y\sqrt{1-z^2}\,dz\,dy$

$= \int_0^3 \left[-\frac{1}{3}\left(1-z^2\right)^{3/2} e^y\right]_{z=0}^{z=1} dy = \int_0^3 \frac{1}{3}e^y\,dy = \frac{1}{3}e^y\Big]_0^3 = \frac{1}{3}\left(e^3-1\right)$

6. $\int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dx\,dy\,dz = \int_0^1 \int_0^z \left[xze^{-y^2}\right]_{x=0}^{x=y} dy\,dz = \int_0^1 \int_0^z yze^{-y^2}\,dy\,dz = \int_0^1 \left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z} dz$

$= \int_0^1 -\frac{1}{2}z\left(e^{-z^2}-1\right) dz = \frac{1}{2}\int_0^1 \left(z - ze^{-z^2}\right) dz$

$= \frac{1}{2}\left[\frac{1}{2}z^2 + \frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}\left(1 + e^{-1} - 0 - 1\right) = \frac{1}{4e}$

7. $\iiint_E 2x\,dV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^y 2x\,dz\,dx\,dy = \int_0^2 \int_0^{\sqrt{4-y^2}} \left[2xz\right]_{z=0}^{z=y} dx\,dy = \int_0^2 \int_0^{\sqrt{4-y^2}} 2xy\,dx\,dy$

$= \int_0^2 \left[x^2 y\right]_{x=0}^{x=\sqrt{4-y^2}} dy = \int_0^2 \left(4-y^2\right) y\,dy = \left[2y^2 - \frac{1}{4}y^4\right]_0^2 = 4$

8. $\iiint_E yz\cos\left(x^5\right)\,dV = \int_0^1 \int_0^x \int_x^{2x} yz\cos\left(x^5\right)\,dz\,dy\,dx = \int_0^1 \int_0^x \left[\frac{1}{2}yz^2\cos\left(x^5\right)\right]_{z=x}^{z=2x} dy\,dx$

$= \frac{1}{2}\int_0^1 \int_0^x 3x^2 y\cos\left(x^5\right) dy\,dx = \frac{1}{2}\int_0^1 \left[\frac{3}{2}x^2 y^2\cos\left(x^5\right)\right]_{y=0}^{y=x} dx$

$= \frac{3}{4}\int_0^1 x^4\cos\left(x^5\right) dx = \frac{3}{4}\left[\frac{1}{5}\sin\left(x^5\right)\right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20}\sin 1$

9. Here $E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1 + x + y\}$, so

$$\begin{aligned}\iiint_E 6xy\,dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx \\ &= \int_0^1 \int_0^{\sqrt{x}} 6xy\,(1 + x + y)\,dy\,dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^1 \left(3x^2 + 3x^3 + 2x^{5/2}\right) dx = \left[x^3 + \tfrac{3}{4}x^4 + \tfrac{4}{7}x^{7/2}\right]_0^1 = \tfrac{65}{28}\end{aligned}$$

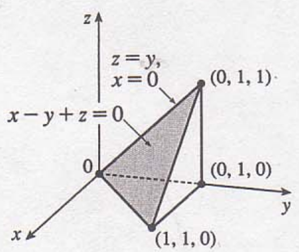
10. Here E is the region that lies under the plane $3x + 2y + z = 6$ and above the region in the xy-plane bounded by the lines $x = 0$, $y = 0$ and $3x + 2y = 6$, so

$$\begin{aligned}\iiint_E x\,dV &= \int_0^2 \int_0^{3-3x/2} \int_0^{6-3x-2y} x\,dz\,dy\,dx = \int_0^2 \int_0^{3-3x/2} \left(6x - 3x^2 - 2xy\right) dy\,dx \\ &= \int_0^2 \left[\left(6x - 3x^2\right)\left(3 - \frac{3}{2}x\right) - x\left(3 - \frac{3}{2}x\right)^2\right] dx = 9\int_0^2 \left(x - x^2 + \frac{1}{4}x^3\right) dx \\ &= 9\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{16}x^4\right]_0^2 = 3.\end{aligned}$$

11. Here E is the region that lies below the plane with x-, y-, and z-intercepts 1, 2, and 3 respectively, that is, below the plane $2z + 6x + 3y = 6$ and above the region in the xy-plane bounded by the lines $x = 0$, $y = 0$ and $6x + 3y = 6$. So

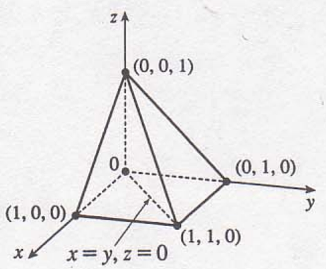
$$\begin{aligned}\iiint_E xy\,dV &= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} xy\,dz\,dy\,dx = \int_0^1 \int_0^{2-2x} \left(3xy - 3x^2y - \tfrac{3}{2}xy^2\right) dy\,dx \\ &= \int_0^1 \left[\tfrac{3}{2}xy^2 - \tfrac{3}{2}x^2y^2 - \tfrac{1}{2}xy^3\right]_{y=0}^{y=2-2x} dx = \int_0^1 \left(2x - 6x^2 + 6x^3 - 2x^4\right) dx \\ &= \left[x^2 - 2x^3 + \tfrac{3}{2}x^4 - \tfrac{2}{5}x^5\right]_0^1 = \tfrac{1}{10}.\end{aligned}$$

12.



$$\begin{aligned}\int_0^1 \int_0^y \int_0^{y-z} xz\,dx\,dz\,dy &= \int_0^1 \int_0^y \tfrac{1}{2}\,(y - z)^2\,z\,dz\,dy \\ &= \tfrac{1}{2}\int_0^1 \left[\tfrac{1}{2}y^2z^2 - \tfrac{2}{3}yz^3 + \tfrac{1}{4}z^4\right]_{z=0}^{z=y} dy \\ &= \tfrac{1}{24}\int_0^1 y^4\,dy = \tfrac{1}{24}\left[\tfrac{1}{5}y^5\right]_0^1 = \tfrac{1}{120}\end{aligned}$$

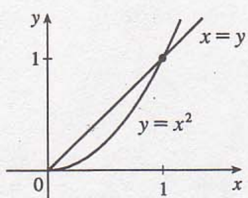
13.



By symmetry $\iiint_E z\,dV = 2\iiint_{E'} z\,dV$ where E' is the part of E to the left [as viewed from $(10, 10, 0)$] of the plane $x = y$. So

$$\begin{aligned}\iiint_E z\,dV &= \int_0^1 \int_y^1 \int_0^{1-x} 2z\,dz\,dx\,dy = \int_0^1 \int_y^1 (1 - x)^2\,dx\,dy \\ &= \int_0^1 \left[-\tfrac{1}{3}\,(1 - x)^3\right]_{x=y}^{x=1} dy = \int_0^1 \tfrac{1}{3}\,(1 - y)^3\,dy \\ &= \tfrac{1}{12}\,(1 - y)^4\Big]_0^1 = \tfrac{1}{12}\end{aligned}$$

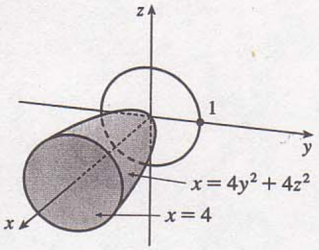
14.



E is the solid above the region shown in the xy-plane and below the plane $z = x$. Thus,

$$\begin{aligned}\iiint_E (x + 2y)\,dV &= \int_0^1 \int_{x^2}^x \int_0^x (x + 2y)\,dz\,dy\,dx \\ &= \int_0^1 \int_{x^2}^x \left(x^2 + 2yx\right) dy\,dx = \int_0^1 \left[x^2y + xy^2\right]_{y=x^2}^{y=x} dx \\ &= \int_0^1 \left(2x^3 - x^4 - x^5\right) dx = \left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\right]_0^1 = \tfrac{2}{15}\end{aligned}$$

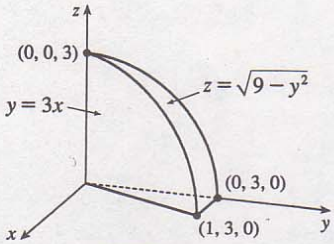
15.



The projection E on the yz-plane is the disk $y^2 + z^2 \leq 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^{4} x\,dx\right] dA$$
$$= \tfrac{1}{2}\iint_D \left[4^2 - \left(4y^2 + 4z^2\right)^2\right] dA = 8\int_0^{2\pi}\int_0^1 \left(1 - r^4\right) r\,dr\,d\theta$$
$$= 8\int_0^{2\pi} d\theta \int_0^1 \left(r - r^5\right) dr = 8\,(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}$$

16.

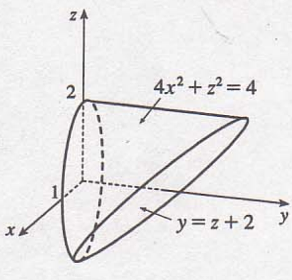


$$\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1 \int_{3x}^3 \tfrac{1}{2}\left(9 - y^2\right) dy\,dx = \int_0^1 \left[\tfrac{9}{2}y - \tfrac{1}{6}y^3\right]_{y=3x}^{y=3} dx$$
$$= \int_0^1 \left[9 - \tfrac{27}{2}x + \tfrac{9}{2}x^3\right] dx = \left[9x - \tfrac{27}{4}x^2 + \tfrac{9}{8}x^4\right]_0^1 = \tfrac{27}{8}$$

17. The plane $2x + 3y + 6z = 12$ intersects the xy-plane when $2x + 3y + 6\,(0) = 12 \;\Rightarrow\; y = 4 - \tfrac{2}{3}x$. So $E = \left\{(x, y, z) \mid 0 \leq x \leq 6, 0 \leq y \leq 4 - \tfrac{2}{3}x, 0 \leq z \leq \tfrac{1}{6}\,(12 - 2x - 3y)\right\}$ and

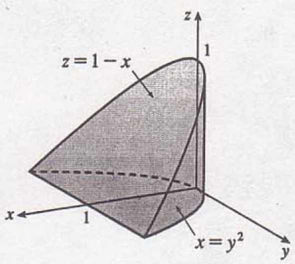
$$V = \int_0^6 \int_0^{4-2x/3} \int_0^{(12-2x-3y)/6} dz\,dy\,dx = \tfrac{1}{6}\int_0^6 \int_0^{4-2x/3} (12 - 2x - 3y)\,dy\,dx$$
$$= \frac{1}{6}\int_0^6 \left[12y - 2xy - \frac{3}{2}y^2\right]_{y=0}^{y=4-2x/3} dx = \frac{1}{6}\int_0^6 \left[\frac{(12-2x)^2}{3} - \frac{3}{2}\,\frac{12-2x}{9}\right] dx$$
$$= \tfrac{1}{36}\int_0^6 (12 - 2x)^2\,dx = \left[\tfrac{1}{36}\left(-\tfrac{1}{6}\right)(12 - 2x)^3\right]_0^6 = 8$$

18.



$$V = \int_{-1}^{1} \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} \int_0^{z+2} dy\,dz\,dx = 2\int_0^1 \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} \int_0^{z+2} dy\,dz\,dx$$
$$= 2\int_0^1 \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} (z + 2)\,dz\,dx = 2\int_0^1 \left[\frac{1}{2}z^2 + 2z\right]_{z=-2\sqrt{1-x^2}}^{z=2\sqrt{1-x^2}} dx$$
$$= 2\int_0^1 8\sqrt{1 - x^2}\,dx = 16\left[\tfrac{1}{2}x\sqrt{1 - x^2} + \tfrac{1}{2}\sin^{-1}x\right]_0^1 = 8\tfrac{\pi}{2} = 4\pi$$

19.



$$V = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} dz\,dy\,dx = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (1 - x)\,dy\,dx$$
$$= \int_0^1 2\sqrt{x}\,(1 - x)\,dx = \int_0^1 2\left(\sqrt{x} - x^{3/2}\right) dx$$
$$= 2\left[\tfrac{2}{3}x^{3/2} - \tfrac{2}{5}x^{5/2}\right]_0^1 = 2\left(\tfrac{2}{3} - \tfrac{2}{5}\right) = \tfrac{8}{15}$$

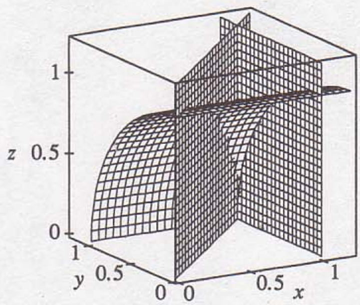
20. The paraboloids $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$ intersect when $x^2 + y^2 = 18 - x^2 - y^2 \Rightarrow$ $2x^2 + 2y^2 = 18 \Rightarrow x^2 + y^2 = 9$. Thus, $E = \{(x, y, z) \mid x^2 + y^2 \le 9, x^2 + y^2 \le z \le 18 - x^2 - y^2\}$. Let $D = \{(x, y) \mid x^2 + y^2 \le 9\}$. Then

$$\begin{aligned} V &= \iiint_E dV = \iint_D \left(\int_{x^2+y^2}^{18-x^2-y^2} dz\right) dA = \iint_D (18 - 2x^2 - 2y^2)\, dA \\ &= \int_0^{2\pi}\int_0^3 (18 - 2r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \left[9r^2 - \tfrac{1}{2}r^4\right]_{r=0}^{r=3} d\theta = \int_0^{2\pi} \tfrac{81}{2}\, d\theta = 81\pi \end{aligned}$$

21. (a) The wedge can be described as the region

$$\begin{aligned} D &= \{(x, y, z) \mid y^2 + z^2 \le 1, 0 \le x \le 1, 0 \le y \le x\} \\ &= \left\{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{1 - y^2}\right\} \end{aligned}$$

So the integral expressing the volume of the wedge is $\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$.



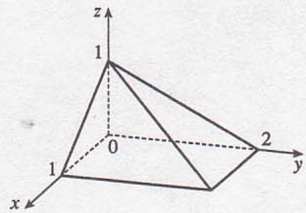
(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx = \frac{\pi}{4} - \frac{1}{3}$. (Or use Formulas 30 and 87 from the Table of Integrals.)

22. (a) Note that $\Delta V_{ijk} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, so the Midpoint Rule gives

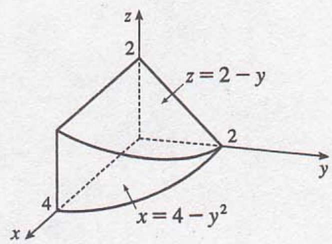
$$\begin{aligned} \iiint_B f(x, y, z)\, dV &\approx \tfrac{1}{8}\left[f\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) + f\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{3}{4}\right) + f\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)\right. \\ &\qquad \left. + f\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right)\right] \\ &= \tfrac{1}{8}\left[e^{-3(1/4)^2} + 3e^{-2(1/4)^2-(3/4)^2} + 3e^{-(1/4)^2-2(3/4)^2} + e^{-3(3/4)^2}\right] \approx 0.42968 \end{aligned}$$

(b) A CAS estimates the integral to be $\iiint_B e^{-x^2-y^2-z^2}\, dV \approx 0.42$. The estimate in part (a) is correct to one decimal place, and is larger than the actual value of the integral.

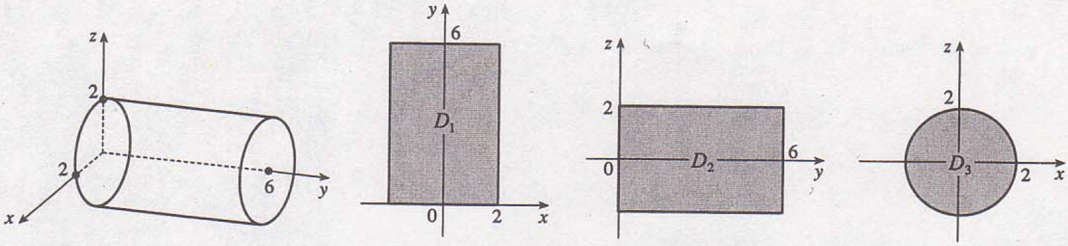
23. $E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le z \le 1 - x, 0 \le y \le 2 - 2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1 - x$, $y = 2 - 2z$.



24. $E = \left\{(x, y, z) \mid 0 \le y \le 2, 0 \le z \le 2 - y, 0 \le x \le 4 - y^2\right\}$, the solid bounded by the three coordinate planes, the plane $z = 2 - y$, and the cylindrical surface $x = 4 - y^2$.



25.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x, y) \mid -2 \leq x \leq 2, 0 \leq y \leq 6\}$$

$$D_2 = \{(y, z) \mid -2 \leq z \leq 2, 0 \leq y \leq 6\}$$

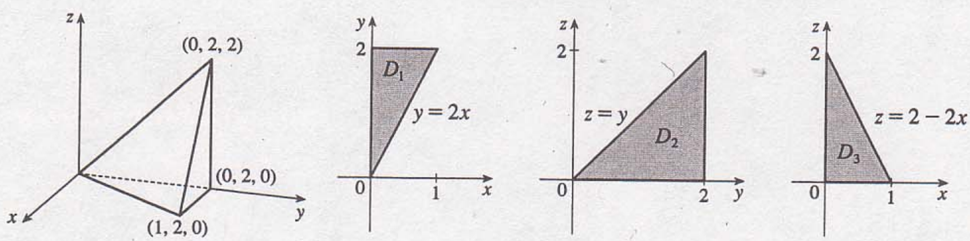
$$D_3 = \{(x, z) \mid x^2 + z^2 \leq 4\}$$

Therefore

$$\begin{aligned} E &= \left\{(x, y, z) \mid -\sqrt{4 - x^2} \leq z \leq \sqrt{4 - x^2},\ -2 \leq x \leq 2, 0 \leq y \leq 6\right\} \\ &= \left\{(x, y, z) \mid -\sqrt{4 - z^2} \leq x \leq \sqrt{4 - z^2},\ -2 \leq z \leq 2, 0 \leq y \leq 6\right\} \end{aligned}$$

$$\begin{aligned} \iiint_E f(x, y, z)\, dV &= \int_{-2}^{2} \int_0^6 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z)\, dz\, dy\, dx = \int_0^6 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z)\, dz\, dx\, dy \\ &= \int_0^6 \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z)\, dx\, dz\, dy = \int_{-2}^{2} \int_0^6 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z)\, dx\, dy\, dz \\ &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^6 f(x, y, z)\, dy\, dz\, dx = \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_0^6 f(x, y, z)\, dy\, dx\, dz \end{aligned}$$

26.



If D_1 and D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$D_1 = \{(x, y) \mid 0 \leq x \leq 1, 2x \leq y \leq 2\} = \{(x, y) \mid 0 \leq y \leq 2, 0 \leq x \leq y/2\}$,
$D_2 = \{(y, z) \mid 0 \leq y \leq 2, 0 \leq z \leq y\} = \{(y, z) \mid 0 \leq z \leq 2, z \leq y \leq 2\}$, and
$D_3 = \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq 2 - 2x\} = \{(x, z) \mid 0 \leq z \leq 2, 0 \leq x \leq (2 - z)/2\}$

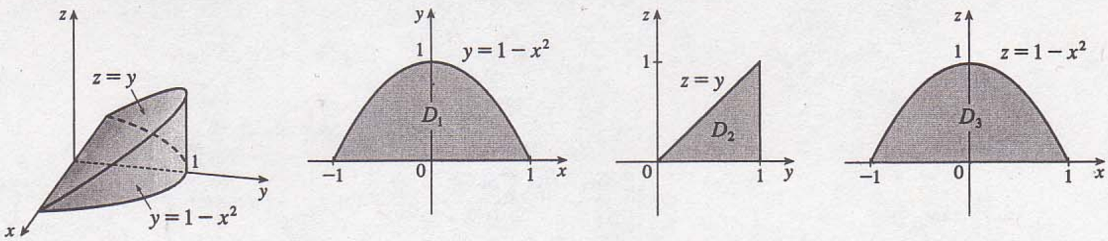
Therefore

$$\begin{aligned}
E &= \{(x,y,z) \mid 0 \le x \le 1, 2x \le y \le 2, 0 \le z \le y - 2x\} \\
&= \{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y/2, 0 \le z \le y - 2x\} \\
&= \{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le y, 0 \le x \le (y-z)/2\} \\
&= \{(x,y,z) \mid 0 \le z \le 2, z \le y \le 2, 0 \le x \le (y-z)/2\} \\
&= \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 2 - 2x, z + 2x \le y \le 2\} \\
&= \{(x,y,z) \mid 0 \le z \le 2, 0 \le x \le (2-z)/2, z + 2x \le y \le 2\}
\end{aligned}$$

Then

$$\begin{aligned}
\iiint_E f(x,y,z)\, dV &= \int_0^1 \int_{2x}^2 \int_0^{y-2x} f(x,y,z)\, dz\, dy\, dx \\
&= \int_0^2 \int_0^{y/2} \int_0^{y-2x} f(x,y,z)\, dz\, dx\, dy \\
&= \int_0^2 \int_0^y \int_0^{(y-z)/2} f(x,y,z)\, dx\, dz\, dy \\
&= \int_0^2 \int_z^2 \int_0^{(y-z)/2} f(x,y,z)\, dx\, dy\, dz \\
&= \int_0^1 \int_0^{2-2x} \int_{z+2x}^2 f(x,y,z)\, dy\, dz\, dx \\
&= \int_0^2 \int_0^{(2-z)/2} \int_{z+2x}^2 f(x,y,z)\, dy\, dx\, dz
\end{aligned}$$

27.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1 - x^2\} = \left\{(x,y) \mid 0 \le y \le 1, -\sqrt{1-y} \le x \le \sqrt{1-y}\right\},$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}, \text{ and}$$

$$D_3 = \{(x,z) \mid -1 \le x \le 1, 0 \le z \le 1 - x^2\} = \{(x,z) \mid 0 \le z \le 1, -\sqrt{1-z} \le x \le \sqrt{1-z}\}.$$

Therefore

$$\begin{aligned}
E &= \{(x,y,z) \mid -1 \le x \le 1, 0 \le y \le 1 - x^2, 0 \le z \le y\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 1, -\sqrt{1-y} \le x \le \sqrt{1-y}, 0 \le z \le y\right\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, -\sqrt{1-y} \le x \le \sqrt{1-y}\right\} \\
&= \left\{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, -\sqrt{1-y} \le x \le \sqrt{1-y}\right\} \\
&= \{(x,y,z) \mid -1 \le x \le 1, 0 \le z \le 1 - x^2, z \le y \le 1 - x^2\} \\
&= \{(x,y,z) \mid 0 \le z \le 1, -\sqrt{1-z} \le x \le \sqrt{1-z}, z \le y \le 1 - x^2\}
\end{aligned}$$

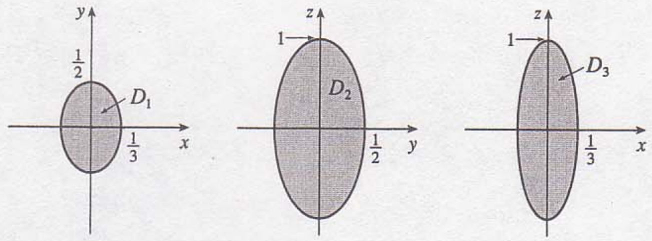
Then

$$\iiint_E f(x,y,z)\,dV = \int_{-1}^{1}\int_0^{1-x^2}\int_0^{y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\int_0^{y} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{y}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_z^1\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_0^{1-x^2}\int_z^{1-x^2} f(x,y,z)\,dy\,dz\,dx = \int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_z^{1-x^2} f(x,y,z)\,dy\,dx\,dz$$

28.



If D_1, D_2 and D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1 = \{(x,y) \mid 9x^2 + 4y^2 \le 1\}$, $D_2 = \{(y,z) \mid 4y^2 + z^2 \le 1\}$, $D_3 = \{(x,z) \mid 9x^2 + z^2 \le 1\}$. Therefore

$$\iiint_E f(x,y,z)\,dV = \int_{-1/3}^{1/3}\int_{-\sqrt{1-9x^2}/2}^{\sqrt{1-9x^2}/2}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}/3}^{\sqrt{1-4y^2}/3}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dx\,dy$$

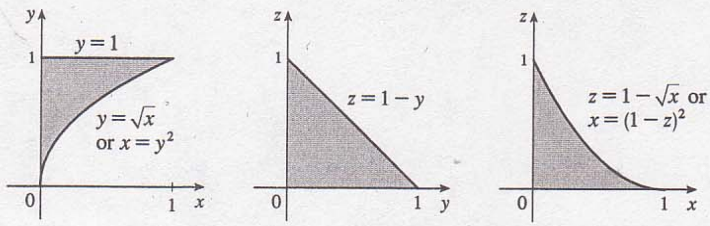
$$= \int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}}\int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-z^2}/2}^{\sqrt{1-z^2}/2}\int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{-1/3}^{1/3}\int_{-\sqrt{1-9x^2}}^{\sqrt{1-9x^2}}\int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-z^2}/3}^{\sqrt{1-z^2}/3}\int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x,y,z)\,dy\,dx\,dz$$

29.



The diagrams show the projections of E on the xy-, yz-, and xz-planes. Therefore

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy$$

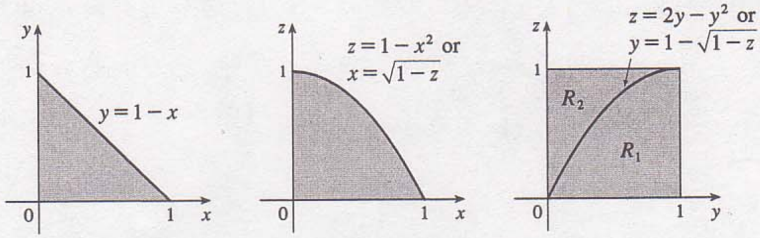
$$= \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

30.



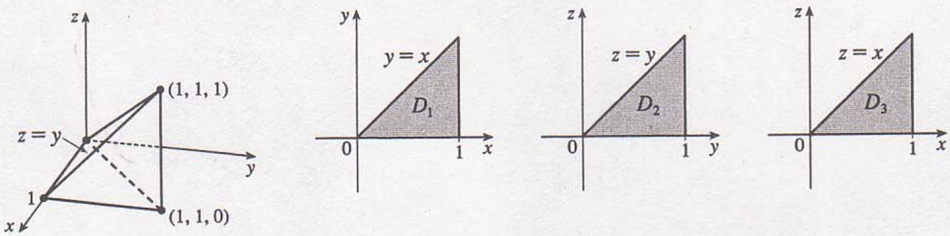
The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\, dy\, dz\, dx = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\, dy\, dx\, dz$$
$$= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\, dz\, dx\, dy = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\, dz\, dy\, dx$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1-y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y,z) in R_1, $0 \le x \le 1 - y$ and for (y,z) in R_2, $0 \le x \le \sqrt{1-z}$, and so the given integral is also equal to

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\, dx\, dy\, dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x,y,z)\, dx\, dy\, dz$$
$$= \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\, dx\, dz\, dy + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\, dx\, dz\, dy.$$

31.



$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\, dz\, dx\, dy = \iiint_E f(x,y,z)\, dV$ where $E = \{(x,y,z) \mid 0 \le z \le y, y \le x \le 1, 0 \le y \le 1\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$D_1 = \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\}$,

$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}$, and

$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1, z \le x \le 1\}$.

Thus we also have

$$E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, y \le x \le 1\}$$
$$= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x, z \le y \le x\}$$
$$= \{(x,y,z) \mid 0 \le z \le 1, z \le x \le 1, z \le y \le x\}.$$

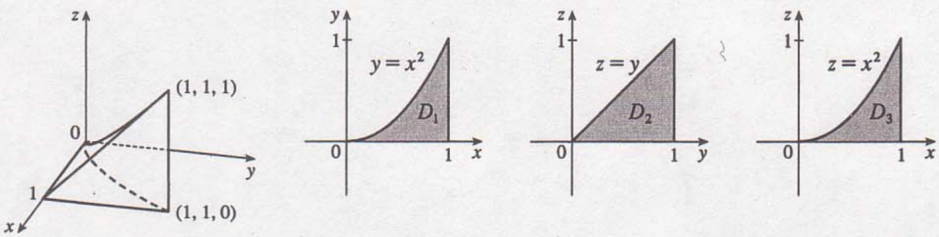
Then

$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz$$

32.



$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y\}$. If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\} = \{(x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1\}$,

$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}$,

$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x^2\} = \{(x,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1\}$. Thus we also have

$$E = \{(x,y,z) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1, 0 \le z \le y\}$$

$$= \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, \sqrt{y} \le x \le 1\}$$

$$= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, \sqrt{y} \le x \le 1\}$$

$$= \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x^2, z \le y \le x^2\}$$

$$= \{(x,y,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1, z \le y \le x^2\}$$

Then

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz$$

33. $m = \iiint_E \rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 2(1+x+y)\,dy\,dx = \int_0^1 \left[2y + 2xy + y^2\right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 \left(2\sqrt{x} + 2x^{3/2} + x\right) dx = \left[\frac{4}{3}x^{3/2} + \frac{4}{5}x^{5/2} + \frac{1}{2}x^2\right]_0^1 = \frac{79}{30}$

$M_{yz} = \iiint_E x\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 2x(1+x+y)\,dy\,dx = \int_0^1 \left[2xy + 2x^2y + xy^2\right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 \left(2x^{3/2} + 2x^{5/2} + x^2\right) dx = \left[\frac{4}{5}x^{5/2} + \frac{4}{7}x^{7/2} + \frac{1}{3}x^3\right]_0^1 = \frac{179}{105}$

$M_{xz} = \iiint_E y\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2y\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 2y(1+x+y)\,dy\,dx = \int_0^1 \left[y^2 + xy^2 + \frac{2}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 \left(x + x^2 + \frac{2}{3}x^{3/2}\right) dx = \left[\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{4}{15}x^{5/2}\right]_0^1 = \frac{11}{10}$

$M_{xy} = \iiint_E z\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2z\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} \left[z^2\right]_{z=0}^{z=1+x+y} dy\,dx = \int_0^1 \int_0^{\sqrt{x}} (1+x+y)^2\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} \left(1 + 2x + 2y + 2xy + x^2 + y^2\right) dy\,dx$

$= \int_0^1 \left[y + 2xy + y^2 + xy^2 + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \left(\sqrt{x} + \frac{7}{3}x^{3/2} + x + x^2 + x^{5/2}\right) dx$

$= \left[\frac{2}{3}x^{3/2} + \frac{14}{15}x^{5/2} + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{2}{7}x^{7/2}\right]_0^1 = \frac{571}{210}$

Thus the mass is $\frac{79}{30}$ and the center of mass is $(\overline{x},\overline{y},\overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\frac{358}{553}, \frac{33}{79}, \frac{571}{553}\right)$.

34. $m = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^1 \int_0^{1-y^2} (1-z)\,dz\,dy = 4\int_{-1}^1 \left[z - \frac{1}{2}z^2\right]_{z=0}^{z=1-y^2} dy$

$= 2\int_{-1}^1 (1-y^4)\,dy = \frac{16}{5}$,

$M_{yz} = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^1 \int_0^{1-y^2} (1-z)^2\,dz\,dy = 2\int_{-1}^1 \left[-\frac{1}{3}(1-z)^3\right]_{z=0}^{z=1-y^2} dy$

$= \frac{2}{3}\int_{-1}^1 (1-y^6)\,dy = \left(\frac{4}{3}\right)\left(\frac{6}{7}\right) = \frac{24}{21}$

$M_{xz} = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^1 \int_0^{1-y^2} 4y(1-z)\,dz\,dy$

$= \int_{-1}^1 \left[4y(1-y^2) - 2y(1-y^2)^2\right] dy = \int_{-1}^1 (2y - 2y^5)\,dy = 0$ (the integrand is odd)

$M_{xy} = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^1 \int_0^{1-y^2} (4z - 4z^2)\,dz\,dy$

$= 2\int_{-1}^1 \left[(1-y^2)^2 - \frac{2}{3}(1-y^2)^3\right] dy = 2\int_{-1}^1 \left[\frac{1}{3} - y^4 + \frac{2}{3}y^6\right] dy$

$= \left[\frac{4}{3}y - \frac{4}{5}y^5 + \frac{8}{21}y^7\right]_0^1 = \frac{96}{105} = \frac{32}{35}$

Thus, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$

35. $m = \int_0^a \int_0^a \int_0^a (x^2 + y^2 + z^2)\, dx\, dy\, dz = \int_0^a \int_0^a \left[\frac{1}{3}x^3 + xy^2 + xz^2\right]_{x=0}^{x=a} dy\, dz$

$= \int_0^a \int_0^a \left(\frac{1}{3}a^3 + ay^2 + az^2\right) dy\, dz = \int_0^a \left[\frac{1}{3}a^3 y + \frac{1}{3}ay^3 + ayz^2\right]_{y=0}^{y=a} dz$

$= \int_0^a \left(\frac{2}{3}a^4 + a^2 z^2\right) dz = \left[\frac{2}{3}a^4 z + \frac{1}{3}a^2 z^3\right]_0^a = \frac{2}{3}a^5 + \frac{1}{3}a^5 = a^5$

$M_{yz} = \int_0^a \int_0^a \int_0^a \left[x^3 + x\left(y^2 + z^2\right)\right] dx\, dy\, dz = \int_0^a \int_0^a \left[\frac{1}{4}a^4 + \frac{1}{2}a^2\left(y^2 + z^2\right)\right] dy\, dz$

$= \int_0^a \left(\frac{1}{4}a^5 + \frac{1}{6}a^5 + \frac{1}{2}a^3 z^2\right) dz = \frac{1}{4}a^6 + \frac{1}{3}a^6 = \frac{7}{12}a^6$

$= M_{xz} = M_{xy}$ by symmetry of E and $\rho(x, y, z)$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

36. $m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[(1-x)\, y - y^2\right] dy\, dx$

$= \int_0^1 \left[\frac{1}{2}(1-x)^3 - \frac{1}{3}(1-x)^3\right] dx = \frac{1}{6}\int_0^1 (1-x)^3\, dx = \frac{1}{24}$

$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\left(x - x^2\right) y - xy^2\right] dy\, dx$

$= \int_0^1 \left[\frac{1}{2}x(1-x)^3 - \frac{1}{3}x(1-x)^3\right] dx = \frac{1}{6}\int_0^1 \left(x - 3x^2 + 3x^3 - x^4\right) dx$

$= \frac{1}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = \frac{1}{120}$

$M_{xz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[(1-x)\, y^2 - y^3\right] dy\, dx$

$= \int_0^1 \left[\frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4\right] dx = \frac{1}{12}\left[-\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{60}$

$M_{xy} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{1}{2}y(1-x-y)^2\right] dy\, dx$

$= \frac{1}{2}\int_0^1 \int_0^{1-x} \left[(1-x)^2 y - 2(1-x)\, y^2 + y^3\right] dy\, dx$

$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}(1-x)^4 - \frac{2}{3}(1-x)^4 + \frac{1}{4}(1-x)^4\right] dx$

$= \frac{1}{24}\int_0^1 (1-x)^4\, dx = -\frac{1}{24}\left[\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{120}$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

37. (a) $m = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^4 \left(x^2 + y^2 + z^2\right) dx\, dz\, dy$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1}\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^4 x\left(x^2 + y^2 + z^2\right) dx\, dz\, dy$,

$\overline{y} = m^{-1}\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^4 y\left(x^2 + y^2 + z^2\right) dx\, dz\, dy$, and

$\overline{z} = m^{-1}\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^4 z\left(x^2 + y^2 + z^2\right) dx\, dz\, dy$

(c) $I_z = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^4 \left(x^2 + y^2\right)\left(x^2 + y^2 + z^2\right) dx\, dz\, dy$

38. (a) $m = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$,

$\overline{y} = m^{-1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$,

$\overline{z} = m^{-1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

(c) $I_z = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2)\,(1+x+y+z)\, dz\, dx\, dy$

39. (a) $m = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (1+x+y+z)\, dz\, dy\, dx = \frac{3\pi}{32} + \frac{11}{24}$

(b)
$$(\overline{x}, \overline{y}, \overline{z}) = \Big(m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y x\,(1+x+y+z)\, dz\, dy\, dx,\; m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y y\,(1+x+y+z)\, dz\, dy\, dx,\; m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y z\,(1+x+y+z)\, dz\, dy\, dx \Big)$$

$$= \left(\frac{28}{9\pi+44}, \frac{30\pi+128}{45\pi+220}, \frac{45\pi+208}{135\pi+660} \right)$$

(c) $$I_z = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (x^2+y^2)\,(1+x+y+z)\, dz\, dy\, dx = \frac{68+15\pi}{240}$$

40. (a) $m = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)\, dz\, dy\, dx = \frac{56}{5} = 11.2$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} x\,(x^2+y^2)\, dz\, dy\, dx \approx 0.375$,

$\overline{y} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} y\,(x^2+y^2)\, dz\, dy\, dx = \frac{45\pi}{64} \approx 2.209$,

$\overline{z} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,(x^2+y^2)\, dz\, dy\, dx = \frac{15}{16} = 0.9375$.

(c) $I_z = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)^2\, dz\, dy\, dx = \frac{10{,}464}{175} \approx 59.79$

41. $I_x = \int_0^L \int_0^L \int_0^L k\,(y^2+z^2)\, dz\, dy\, dx = k \int_0^L \int_0^L \left(Ly^2 + \frac{1}{3}L^3\right) dy\, dx = k \int_0^L \frac{2}{3}L^4\, dx = \frac{2}{3}kL^5$.

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

42. Let k be the density. Then

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} k\,(y^2+z^2)\, dx\, dy\, dz = ka \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} (y^2+z^2)\, dy\, dz$$

$$= ak \int_{-c/2}^{c/2} \left[\tfrac{1}{3}y^3 + z^2 y\right]_{y=-b/2}^{y=b/2} dz = ak \int_{-c/2}^{c/2} \left(\tfrac{1}{12}b^3 + bz^2\right) dz = ak \left[\tfrac{1}{12}b^3 z + \tfrac{1}{3}bz^3\right]_{-c/2}^{c/2}$$

$$= ak \left(\tfrac{1}{12}b^3 c + \tfrac{1}{12}bc^3\right) = \tfrac{1}{12}kabc\,(b^2+c^2)$$

By symmetry, $I_y = \frac{1}{12}kabc\,(a^2+c^2)$ and $I_z = \frac{1}{12}kabc\,(a^2+b^2)$.

43. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x,y,z)\,dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^2\int_0^2\int_0^2 Cxyz\,dz\,dy\,dx\\ &= C\int_0^2 x\,dx\int_0^2 y\,dy\int_0^2 z\,dz = C\left[\frac{x^2}{2}\right]_0^2\left[\frac{y^2}{2}\right]_0^2\left[\frac{z^2}{2}\right]_0^2\\ &= 8C\end{aligned}$$

Then we must have $8C = 1 \;\Rightarrow\; C = \frac{1}{8}$.

(b) $$\begin{aligned}P(X\le 1, Y\le 1, Z\le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx\\ &= \int_0^1\int_0^1\int_0^1 \tfrac{1}{8}xyz\,dz\,dy\,dx = \tfrac{1}{8}\int_0^1 x\,dx\int_0^1 y\,dy\int_0^1 z\,dz\\ &= \frac{1}{8}\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1\left[\frac{z^2}{2}\right]_0^1 = \tfrac{1}{8}\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{64}\end{aligned}$$

(c) $P(X+Y+Z\le 1) = P((X,Y,Z)\in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1$. The plane $x+y+z=1$ meets the xy-plane in the line $x+y=1$, so we have

$$\begin{aligned}P(X+Y+Z\le 1) &= \iiint_E f(x,y,z)\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y}\tfrac{1}{8}xyz\,dz\,dy\,dx\\ &= \tfrac{1}{8}\int_0^1\int_0^{1-x} xy\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x-y}\,dy\,dx\\ &= \tfrac{1}{16}\int_0^1\int_0^{1-x} xy\,(1-x-y)^2\,dy\,dx\\ &= \tfrac{1}{16}\int_0^1\int_0^{1-x}\left[\left(x^3-2x^2+x\right)y + \left(2x^2-2x\right)y^2 + xy^3\right]dy\,dx\\ &= \tfrac{1}{16}\int_0^1\left[\left(x^3-2x^2+x\right)\tfrac{1}{2}y^2 + \left(2x^2-2x\right)\tfrac{1}{3}y^3 + x\left(\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1-x}dx\\ &= \tfrac{1}{192}\int_0^1\left(x-4x^2+6x^3-4x^4+x^5\right)dx = \tfrac{1}{192}\left(\tfrac{1}{30}\right) = \tfrac{1}{5760}\end{aligned}$$

44. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x,y,z)\,dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx\\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx\\ &= C\int_0^{\infty} e^{-0.5x}\,dx\int_0^{\infty} e^{-0.2y}\,dy\int_0^{\infty} e^{-0.1z}\,dz\\ &= C\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx\;\lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy\;\lim_{t\to\infty}\int_0^t e^{-0.1z}\,dz\\ &= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t\lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t\\ &= C\lim_{t\to\infty}\left[-2\left(e^{-0.5t}-1\right)\right]\lim_{t\to\infty}\left[-5\left(e^{-0.2t}-1\right)\right]\lim_{t\to\infty}\left[-10\left(e^{-0.1t}-1\right)\right]\\ &= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1) = 100C\end{aligned}$$

So we must have $100C = 1 \;\Rightarrow\; C = \frac{1}{100}$.

(b) We have no restriction on Z, so

$$P(X \le 1, Y \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^{\infty} \tfrac{1}{100} e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx$$

$$= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz$$

$$= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \quad \text{[by part (a)]}$$

$$= \tfrac{1}{100}\left(2-2e^{-0.5}\right)\left(5-5e^{-0.2}\right)(10) = \left(1-e^{-0.5}\right)\left(1-e^{-0.2}\right) \approx 0.07132$$

(c) $P(X \le 1, Y \le 1, Z \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx$

$$= \int_0^1\int_0^1\int_0^1 \tfrac{1}{100} e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx$$

$$= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^1 e^{-0.1z}\,dz$$

$$= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \left[-10e^{-0.1z}\right]_0^1$$

$$= \left(1-e^{-0.5}\right)\left(1-e^{-0.2}\right)\left(1-e^{-0.1}\right) \approx 0.006787$$

45. $V(E) = L^3$,

$$f_{\text{ave}} = \frac{1}{L^3}\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz = \frac{1}{L^3}\int_0^L x\,dx \int_0^L y\,dy \int_0^L z\,dz$$

$$= \frac{1}{L^3}\left[\frac{x^2}{2}\right]_0^L \left[\frac{y^2}{2}\right]_0^L \left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3}\frac{L^2}{2}\frac{L^2}{2}\frac{L^2}{2} = \frac{L^3}{8}$$

46. $V(E) = \frac{(1)(1)(1)}{6} = \frac{1}{6}$. The equation of the plane through the last three vertices is $x + y + z = 1$, so

$$f_{\text{ave}} = \tfrac{1}{1/6}\int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= 6\int_0^1\int_0^{1-x}\left[(x+y)(1-x-y) + \tfrac{1}{2}(1-x-y)^2\right] dy\,dx$$

$$= 3\int_0^1\int_0^{1-x}\left(1-2xy-x^2-y^2\right) dy\,dx = 3\int_0^1\int_0^{1-x}\left[1-(x+y)^2\right] dy\,dx$$

$$= 3\int_0^1\left[y - \tfrac{1}{3}(x+y)^3\right]_{y=0}^{y=1-x} dx = 3\int_0^1\left(1-x-\tfrac{1}{3}+\tfrac{1}{3}x^3\right) dx = \int_0^1\left(x^3-3x+2\right) dx$$

$$= \tfrac{1}{4} - \tfrac{3}{2} + 2 = \tfrac{3}{4}$$

47. The triple integral will attain its maximum when the integrand $1 - x^2 - 2y^2 - 3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E, and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E. So we require that $x^2 + 2y^2 + 3z^2 \le 1$. This describes the region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

Discovery Project □ Volumes of Hyperspheres

In this project we use V_n to denote the n-dimensional volume of an n-dimensional hypersphere.

1. The interior of the circle is the set of points $\left\{(x,y) \mid -r \le y \le r,\ -\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2}\right\}$. So, substituting $y = r\sin\theta$ and then using Formula 64 to evaluate the integral, we get

$$V_2 = \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\,dy = \int_{-r}^{r} 2\sqrt{r^2-y^2}\,dy = \int_{-\pi/2}^{\pi/2} 2r\sqrt{1-\sin^2\theta}\,(r\cos\theta\,d\theta)$$

$$= 2r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = 2r^2\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2r^2\left(\tfrac{\pi}{2}\right) = \pi r^2$$

2. The region of integration is

$\left\{(x,y,z) \mid -r \le z \le r, -\sqrt{r^2-z^2} \le y \le \sqrt{r^2-z^2}, -\sqrt{r^2-z^2-y^2} \le x \le \sqrt{r^2-z^2-y^2}\right\}$. Substituting $y = \sqrt{r^2-z^2}\sin\theta$ and using Formula 64 to integrate $\cos^2\theta$, we get

$$V_3 = \int_{-r}^{r}\int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}}\int_{-\sqrt{r^2-z^2-y^2}}^{\sqrt{r^2-z^2-y^2}} dx\,dy\,dz = \int_{-r}^{r}\int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} 2\sqrt{r^2-z^2-y^2}\,dy\,dz$$

$$= \int_{-r}^{r}\int_{-\pi/2}^{\pi/2} 2\sqrt{r^2-z^2}\sqrt{1-\sin^2\theta}\left(\sqrt{r^2-z^2}\cos\theta\,d\theta\right)dz$$

$$= 2\left[\int_{-r}^{r}(r^2-z^2)\,dz\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right] = 2\left(\frac{4r^3}{3}\right)\left(\frac{\pi}{2}\right) = \frac{4\pi r^3}{3}$$

3. Here we substitute $y = \sqrt{r^2-w^2-z^2}\sin\theta$ and, later, $w = r\sin\phi$. Because $\int_{-\pi/2}^{\pi/2}\cos^p\theta\,d\theta$ seems to occur frequently in these calculations, it is useful to find a general formula for that integral. From Exercises 39 and 40 in Section 8.1 [ET 7.1], we have

$$\int_0^{\pi/2}\sin^{2k}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}\,\frac{\pi}{2} \text{ and } \int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)}$$

and from the symmetry of the sine and cosine functions, we can conclude that

$$\int_{-\pi/2}^{\pi/2}\cos^{2k}x\,dx = 2\int_0^{\pi/2}\sin^{2k}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)\,\pi}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k} \tag{1}$$

$$\int_{-\pi/2}^{\pi/2}\cos^{2k+1}x\,dx = 2\int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2\cdot 2\cdot 4\cdot 6\cdot\cdots\cdot 2k}{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)} \tag{2}$$

Thus
$$\begin{aligned}
V_4 &= \int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\int_{-\sqrt{r^2-w^2-z^2-y^2}}^{\sqrt{r^2-w^2-z^2-y^2}} dx\,dy\,dz\,dw \\
&= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}} \sqrt{r^2-w^2-z^2-y^2}\,dy\,dz\,dw \\
&= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\pi/2}^{\pi/2} \left(r^2-w^2-z^2\right)\cos^2\theta\,d\theta\,dz\,dw \\
&= 2\left[\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}} \left(r^2-w^2-z^2\right)dz\,dw\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right] \\
&= 2\left(\tfrac{\pi}{2}\right)\left[\int_{-r}^{r}\tfrac{4}{3}\left(r^2-w^2\right)^{3/2}dw\right] = \pi\left(\tfrac{4}{3}\right)\int_{-\pi/2}^{\pi/2} r^4\cos^4\phi\,d\phi = \frac{4\pi}{3}r^4\cdot\frac{1\cdot 3\cdot\pi}{2\cdot 4} = \frac{\pi^2 r^4}{2}
\end{aligned}$$

4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2}\cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

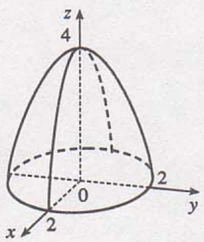
$$\begin{aligned}
V_4 &= \int_{-r}^{r}\int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}}\cdots\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1\,dx_2\cdots dx_{n-1}\,dx_n \\
&= 2\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta_2\,d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2}\cos^3\theta_3\,d\theta_3\right]\cdots\left[\int_{-\pi/2}^{\pi/2}\cos^{n-1}\theta_{n-1}\,d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2}\cos^n\theta_n\,d\theta_n\right]r^n \\
&= \begin{cases}
\left[2\cdot\dfrac{\pi}{2}\right]\left[\dfrac{2\cdot 2}{1\cdot 3}\cdot\dfrac{1\cdot 3\pi}{2\cdot 4}\right]\left[\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\cdot\dfrac{1\cdot 3\cdot 5\pi}{2\cdot 4\cdot 6}\right]\cdots\left[\dfrac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\dfrac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right]r^n & n \text{ even} \\[2ex]
2\left[\dfrac{\pi}{2}\cdot\dfrac{2\cdot 2}{1\cdot 3}\right]\left[\dfrac{1\cdot 3\pi}{2\cdot 4}\cdot\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\right]\cdots\left[\dfrac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\dfrac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right]r^n & n \text{ odd}
\end{cases}
\end{aligned}$$

By canceling within each set of brackets, we find that

$$V_4 = \begin{cases}
\dfrac{2\pi}{2}\cdot\dfrac{2\pi}{4}\cdot\dfrac{2\pi}{6}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot n}r^n = \dfrac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n & n \text{ even} \\[2ex]
2\cdot\dfrac{2\pi}{3}\cdot\dfrac{2\pi}{5}\cdot\dfrac{2\pi}{7}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{2\,(2\pi)^{(n-1)/2}}{3\cdot 5\cdot 7\cdot\cdots\cdot n}r^n = \dfrac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}r^n & n \text{ odd}
\end{cases}$$

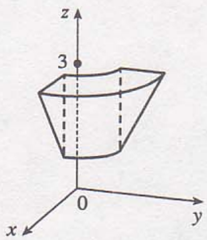
16.8 Triple Integrals in Cylindrical and Spherical Coordinates ET 15.8

1. The region of integration is given in cylindrical coordinates by $E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}$. This represents the solid region bounded above by $z = 4 - r^2 = 4 - x^2 - y^2$, a paraboloid, and below by the xy-plane.



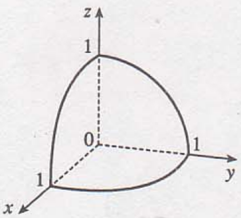
$$\begin{aligned}
\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^2\left(4r - r^3\right)dr\,d\theta \\
&= \int_0^{2\pi}\left[2r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=2}d\theta \\
&= \int_0^{2\pi}(8-4)\,d\theta = 4\theta\Big]_0^{2\pi} = 8\pi
\end{aligned}$$

2. The region of integration is given in cylindrical coordinates by
$E = \left\{(r,\theta,z) \mid 0 \le \theta \le \frac{\pi}{2}, 1 \le r \le 3, r \le z \le 3\right\}$. This represents the solid in the first octant between the cylinders $r = 1$ and $r = 3$ and bounded below by $z = r = \sqrt{x^2+y^2}$, a cone, and above by the plane $z = 3$.



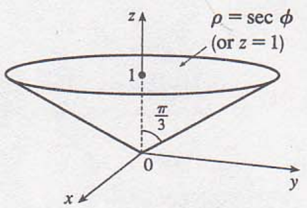
$$\begin{aligned}\int_1^3 \int_0^{\pi/2} \int_r^3 r\,dz\,d\theta\,dr &= \int_1^3 \int_0^{\pi/2} \left(3r - r^2\right) d\theta\,dr \\ &= \int_1^3 \tfrac{\pi}{2}\left(3r - r^2\right) dr = \tfrac{\pi}{2}\left[\tfrac{3}{2}r^2 - \tfrac{1}{3}r^3\right]_1^3 \\ &= \tfrac{\pi}{2}\left(\tfrac{27}{2} - \tfrac{27}{3} - \tfrac{3}{2} + \tfrac{1}{3}\right) = \tfrac{5\pi}{3}\end{aligned}$$

3. The region of integration is given in spherical coordinates by
$E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\right\}$. Thus E is the solid in the first octant bounded by the sphere $\rho = x^2 + y^2 + z^2 = 1$ and the three coordinate planes.



$$\begin{aligned}\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/2} \int_0^{\pi/2} \left[\tfrac{1}{3}\rho^3 \sin\phi\right]_{\rho=0}^{\rho=1} d\theta\,d\phi \\ &= \int_0^{\pi/2} \int_0^{\pi/2} \tfrac{1}{3}\sin\phi\,d\theta\,d\phi = \int_0^{\pi/2} \tfrac{1}{3}\sin\phi\,[\theta]_{\theta=0}^{\theta=\pi/2}\,d\phi \\ &= \tfrac{1}{3}\int_0^{\pi/2} \tfrac{\pi}{2}\sin\phi\,d\phi = \tfrac{\pi}{6}\left[-\cos\phi\right]_0^{\pi/2} = \tfrac{\pi}{6}\end{aligned}$$

4. The region of integration is given in spherical coordinates by
$E = \left\{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{3}, 0 \le \rho \le \sec\phi\right\}$. Since $\rho = \sec\phi$ is equivalent to $\rho\cos\phi = z = 1$, E is the solid bounded by the cone $\phi = \frac{\pi}{3}$ and the plane $z = 1$.



$$\begin{aligned}\int_0^{\pi/3} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/3} \int_0^{2\pi} \left[\tfrac{1}{3}\rho^3 \sin\phi\right]_{\rho=0}^{\rho=\sec\phi} d\theta\,d\phi \\ &= \frac{1}{3}\int_0^{\pi/3}\int_0^{2\pi} \frac{\sin\phi}{\cos^3\phi}\,d\theta\,d\phi = \frac{2\pi}{3}\int_0^{\pi/3} \left(\tan\phi\sec^2\phi\right) d\phi \\ &= \frac{2\pi}{3}\left[\frac{\tan^2\phi}{2}\right]_0^{\pi/3} = \pi\end{aligned}$$

5. The solid E is most conveniently described if we use cylindrical coordinates:
$E = \left\{(r,\theta,z) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2\right\}$. Then
$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$.

6. The solid E is most conveniently described if we use spherical coordinates:
$E = \left\{(\rho,\theta,\phi) \mid 1 \le \rho \le 2, \frac{\pi}{2} \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\right\}$. Then
$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2} \int_{\pi/2}^{2\pi} \int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

7. In cylindrical coordinates, E is given by $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz$$
$$= [\theta]_0^{2\pi}\left[\tfrac{1}{3}r^3\right]_0^4 [z]_{-5}^4 = (2\pi)\left(\tfrac{64}{3}\right)(9) = 384\pi$$

8. The paraboloid $z = 1 - x^2 - y^2$ intersects the xy-plane in the circle $x^2 + y^2 = r^2 = 1$ or $r = 1$, so in cylindrical coordinates, E is given by $\left\{(r,\theta,z) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1, 0 \le z \le 1 - r^2\right\}$. Thus

$$\iiint_E \left(x^3 + xy^2\right) dV = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} \left(r^3\cos^3\theta + r^3\cos\theta\sin^2\theta\right) r\,dz\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^4\cos\theta\,[z]_{z=0}^{z=1-r^2}\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^1 r^4\left(1-r^2\right)\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\left[\frac{1}{5}r^5 - \frac{1}{7}r^7\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{\pi/2} \frac{2}{35}\cos\theta\,d\theta = \frac{2}{35}\,[\sin\theta]_0^{\pi/2} = \frac{2}{35}$$

9. In cylindrical coordinates E is bounded by the cylinders $r = 1$ and $r = 2$, the plane $z = x + 2 = r\cos\theta + 2$, and the xy-plane, so E is given by $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 1 \le r \le 2, 0 \le z \le r\cos\theta + 2\}$. Thus

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{2+r\cos\theta} (r\sin\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^2 r^2\sin\theta\,[z]_{z=0}^{z=2+r\cos\theta}\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_1^2 \left(2r^2 + r^3\cos\theta\right)\sin\theta\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{2}{3}r^3 + \tfrac{1}{4}r^4\cos\theta\right]_{r=1}^{r=2}\sin\theta\,d\theta$$
$$= \int_0^{2\pi}\left(\tfrac{14}{3} + \tfrac{15}{4}\cos\theta\right)\sin\theta\,d\theta = \left[-\tfrac{14}{3}\cos\theta - \tfrac{15}{8}\cos^2\theta\right]_0^{2\pi} = 0$$

10. In cylindrical coordinates, E is bounded by the cylinder $r = 1$ and the planes $z = 0$, $z = y = r\sin\theta$ with $y \ge 0 \quad\Rightarrow\quad 0 \le \theta \le \pi$, so E is given by $\{(r,\theta,z) \mid 0 \le \theta \le \pi, 0 \le r \le 1, 0 \le z \le r\sin\theta\}$. Thus

$$\iiint_E xz\,dV = \int_0^{\pi}\int_0^1\int_0^{r\sin\theta} r^2 z\cos\theta\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^1 \left[\tfrac{1}{2}z^2\right]_{z=0}^{z=r\sin\theta} r^2\cos\theta\,dr\,d\theta$$
$$= \tfrac{1}{2}\int_0^{\pi}\int_0^1 r^4\sin^2\theta\cos\theta\,dr\,d\theta = \tfrac{1}{2}\int_0^{\pi}\left[\tfrac{1}{5}r^5\right]_{r=0}^{r=1}\sin^2\theta\cos\theta\,d\theta$$
$$= \tfrac{1}{10}\int_0^{\pi}\left(\sin^2\theta\cos\theta\right)d\theta = \tfrac{1}{30}\sin^3\theta\Big]_0^{\pi} = 0$$

11. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So $E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2r\}$ and

$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\,z\right]_{z=0}^{z=2r} dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1} d\theta = \tfrac{2}{5}\int_0^{2\pi}\cos^2\theta\,d\theta$$
$$= \frac{2}{5}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{5}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{2\pi}{5}$$

12. (a) $V = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$= 4\int_0^{\pi/2} \int_0^{a\cos\theta} \int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= -\tfrac{4}{3}\int_0^{\pi/2} \left[\left(a^2-r^2\right)^{3/2}\right]_{r=0}^{r=a\cos\theta} d\theta$$

$$= -\tfrac{4}{3}\int_0^{\pi/2} \left[\left(a^2-a^2\cos^2\theta\right)^{3/2} - a^3\right] d\theta$$

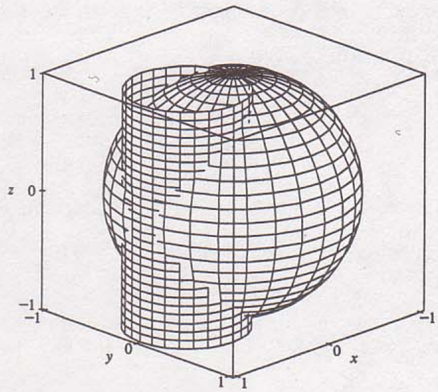
$$= -\tfrac{4}{3}\int_0^{\pi/2} \left[\left(a^2\sin^2\theta\right)^{3/2} - a^3\right] d\theta$$

$$= -\tfrac{4}{3}\int_0^{\pi/2} \left(a^3\sin^3\theta - a^3\right) d\theta$$

$$= -\frac{4a^3}{3}\int_0^{\pi/2} \left[\sin\theta\left(1-\cos^2\theta\right) - 1\right] d\theta$$

$$= -\frac{4a^3}{3}\left[-\cos\theta + \tfrac{1}{3}\cos^3\theta - \theta\right]_0^{\pi/2} = -\frac{4a^3}{3}\left(-\tfrac{\pi}{2} + \tfrac{2}{3}\right) = \tfrac{2}{9}a^3(3\pi - 4)$$

(b)



To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

```
sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d([cos(theta),theta,z],
theta=0..2*Pi,z=-1..1,coords=cylindrical):
with(plots): display3d({sphere,cylinder});
```

In Mathematica, we can use

```
sphere=SphericalPlot3d[1,{theta,0,2Pi},{phi,0,Pi}]
cylinder=ParametricPlot3d[{Sin[theta],Cos[theta],z},
{theta,0,2Pi},{z,-1,1}]
Show[{sphere,cylinder}]
```

13. The paraboloids intersect when $x^2+y^2 = 36-3x^2-3y^2 \;\Rightarrow\; D = \{(x,y) \mid x^2+y^2 \le 9\}$. So, in cylindrical coordinates, $E = \{(r,\theta,z) \mid r^2 \le z \le 36-r^2, 0 \le r \le 3, 0 \le \theta \le 2\pi\}$ and

$$V = \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \left(36r - 4r^3\right) dr\,d\theta$$
$$= \int_0^{2\pi}\left[18r^2 - r^4\right]_{r=0}^{r=3} d\theta = \int_0^{2\pi} 81\,d\theta = 162\pi$$

14. $M_{yz} = \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r^2\cos\theta\,dz\,dr\,d\theta = 0 = M_{xz}$ by the symmetry of the region, and

$$M_{xy} = \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} rz\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \left[\tfrac{1}{2}r\left(36-3r^2\right)^2 - \tfrac{1}{2}r^5\right] dr\,d\theta$$
$$= \int_0^{2\pi}\left[-\tfrac{1}{36}\left(36-3r^2\right)^3 - \tfrac{1}{12}r^6\right]_{r=0}^{r=3} d\theta = \int_0^{2\pi}\left(-3^4 + 36^2\right) d\theta = 2430\pi$$

Hence $(\overline{x},\overline{y},\overline{z}) = (0,0,15)$.

15. The paraboloid $z = 4x^2+4y^2$ intersects the plane $z = a$ when $a = 4x^2+4y^2$ or $x^2+y^2 = \frac{1}{4}a$. So, in cylindrical coordinates, $E = \{(r,\theta,z) \mid 0 \le r \le \frac{1}{2}\sqrt{a}, 0 \le \theta \le 2\pi, 4r^2 \le z \le a\}$. Thus

$$m = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Kr\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(ar - 4r^3\right) dr\,d\theta$$
$$= K\int_0^{2\pi}\left[\tfrac{1}{2}ar^2 - r^4\right]_{r=0}^{r=\sqrt{a}/2} d\theta = K\int_0^{2\pi}\tfrac{1}{16}a^2\,d\theta = \tfrac{1}{8}a^2\pi K$$

Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(\tfrac{1}{2}a^2r - 8r^5\right)dr\,d\theta$$
$$= K\int_0^{2\pi}\left[\tfrac{1}{4}a^2r^2 - \tfrac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2} d\theta = K\int_0^{2\pi}\tfrac{1}{24}a^3\,d\theta = \tfrac{1}{12}a^3\pi K$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0, 0, \frac{2}{3}a\right)$.

16. Since density is proportional to the distance from the z-axis, we can say $\rho(x,y,z) = K\sqrt{x^2+y^2}$. Then

$$m = 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} Kr^2\,dz\,dr\,d\theta = 2K\int_0^{2\pi}\int_0^{a} r^2\sqrt{a^2-r^2}\,dr\,d\theta$$
$$= 2K\int_0^{2\pi}\left[\tfrac{1}{8}r\left(2r^2-a^2\right)\sqrt{a^2-r^2} + \tfrac{1}{8}a^4\sin^{-1}(r/a)\right]_{r=0}^{r=a} d\theta$$
$$= 2K\int_0^{2\pi}\left[\left(\tfrac{1}{8}a^4\right)\left(\tfrac{\pi}{2}\right)\right]d\theta = \tfrac{1}{4}a^4\pi^2K.$$

17. In spherical coordinates, B is represented by $\{(\rho,\theta,\phi)\mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$. Thus

$$\iiint_B \left(x^2+y^2+z^2\right)dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \left(\rho^2\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^1\rho^4\,d\rho$$
$$= \left[-\cos\phi\right]_0^{\pi}\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{5}\rho^5\right]_0^1 = (2)(2\pi)\left(\tfrac{1}{5}\right) = \tfrac{4\pi}{5}$$

18. In spherical coordinates, H is represented by $\left\{(\rho,\theta,\phi)\mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\iiint_H \left(x^2+y^2\right)dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \left(\rho^2\sin^2\phi\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^1\rho^4\,d\rho$$
$$= \left[\theta\right]_0^{2\pi}\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^1 = \tfrac{4\pi}{15}$$

19. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi)\mid 1\le\rho\le 2, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_1^2\rho^3\,d\rho = \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\theta\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_1^2$$
$$= \left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right)\left(\tfrac{15}{4}\right) = \tfrac{15\pi}{16}$$

20. $$\iiint_E xe^{(x^2+y^2+z^2)^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\sin\phi\cos\theta)\,e^{\rho^4}\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta$$
$$= \int_0^{\pi/2}\cos\theta\,d\theta\int_0^{\pi/2}\sin^2\phi\,d\phi\int_1^2\rho^3e^{\rho^4}\,d\rho$$
$$= \left[\sin\theta\right]_0^{\pi/2}\left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^{\pi/2}\left[\tfrac{1}{4}e^{\rho^4}\right]_1^2$$
$$= (1)\left(\tfrac{\pi}{4}\right)\left[\tfrac{1}{4}\left(e^{16}-e\right)\right] = \tfrac{1}{16}\pi\left(e^{16}-e\right)$$

21. $$\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^2 (\rho)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{2\pi}d\theta\int_0^{\pi/6}\sin\phi\,d\phi\int_0^2\rho^3\,d\rho = \left[\theta\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/6}\left[\tfrac{1}{4}\rho^4\right]_0^2$$
$$= (2\pi)\left(1-\tfrac{\sqrt{3}}{2}\right)(4) = 8\pi\left(1-\tfrac{\sqrt{3}}{2}\right) = 4\pi\left(2-\sqrt{3}\right)$$

22. $$\iiint_E xyz\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_2^4 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\int_2^4\rho^5\,d\rho = \left[\tfrac{1}{4}\sin^4\phi\right]_0^{\pi/3}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}\rho^6\right]_2^4 = 0$$

23. Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\tfrac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4\cos\phi}\sin\phi\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/3}\left(\tfrac{64}{3}\cos^3\phi\right)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left[-\tfrac{16}{3}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/3}d\theta$$
$$= \int_0^{2\pi} -\tfrac{16}{3}\left(\tfrac{1}{16}-1\right)d\theta = 5\theta\big]_0^{2\pi} = 10\pi$$

24. In spherical coordinates, the sphere $x^2+y^2+z^2=4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2+y^2}$ is represented by $\phi = \frac{\pi}{4}$. Thus, the solid is given by $\left\{(\rho,\theta,\phi)\,\middle|\,0\le\rho\le 2, 0\le\theta\le 2\pi, \frac{\pi}{4}\le\phi\le\frac{\pi}{2}\right\}$ and
$$V = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^2\,d\rho$$
$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2}\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{3}\rho^3\right]_0^2 = \left(\tfrac{\sqrt{2}}{2}\right)(2\pi)\left(\tfrac{8}{3}\right) = \tfrac{8\sqrt{2}\pi}{3}$$

25. By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi}\rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\cos\phi\sin\phi\left(64\cos^4\phi\right)d\phi\,d\theta$$
$$= \int_0^{2\pi}64\left[-\tfrac{1}{6}\cos^6\phi\right]_{\phi=0}^{\phi=\pi/3}d\theta = \int_0^{2\pi}\tfrac{21}{2}\,d\theta = 21\pi$$

Hence $(\overline{x},\overline{y},\overline{z}) = (0,0,2.1)$.

26. (a) Placing the center of the base at $(0,0,0)$, $\rho(x,y,z) = K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^a\rho^3\,d\rho$$
$$= K\left[\theta\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_0^a = K(2\pi)(1)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{2}\pi K a^4$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\int_0^a\rho^4\,d\rho$$
$$= K\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^a = K(2\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{5}a^5\right) = \tfrac{1}{5}\pi K a^5$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{2}{5}a\right)$.

(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a\left(K\rho^3\sin\phi\right)\left(\rho^2\sin^2\phi\right)d\rho\,d\phi\,d\theta = K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^a\rho^5\,d\rho$
$$= K\left[\theta\right]_0^{2\pi}\left[-\cos\phi+\tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\tfrac{1}{6}\rho^6\right]_0^a = K(2\pi)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{6}a^6\right) = \tfrac{2}{9}\pi K a^6$$

27. (a) The density function is $\rho(x,y,z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$. Then $M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \frac{1}{2}\pi K a^4\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi = \frac{1}{8}\pi K a^4$. But the mass is K (volume of the hemisphere) $= \frac{2}{3}\pi K a^3$, so the centroid is $\left(0,0,\frac{3}{8}a\right)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x:

$$I_x = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a\left(K\rho^2\sin\phi\right)\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)d\rho\,d\phi\,d\theta$$
$$= K\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3\phi\sin^2\theta+\sin\phi\cos^2\phi\right)\left(\tfrac{1}{5}a^5\right)d\phi\,d\theta$$
$$= \tfrac{1}{5}Ka^5\int_0^{2\pi}\left[\sin^2\theta\left(-\cos\phi+\tfrac{1}{3}\cos^3\phi\right)+\left(-\tfrac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2}d\theta$$
$$= \tfrac{1}{5}Ka^5\int_0^{2\pi}\left[\tfrac{2}{3}\sin^2\theta+\tfrac{1}{3}\right]d\theta = \tfrac{1}{5}Ka^5\left[\tfrac{2}{3}\left(\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right)+\tfrac{1}{3}\theta\right]_0^{2\pi}$$
$$= \tfrac{1}{5}Ka^5\left[\tfrac{2}{3}(\pi-0)+\tfrac{1}{3}(2\pi-0)\right] = \tfrac{4}{15}Ka^5\pi$$

28. Place the center of the base at $(0, 0, 0)$, then the density is $\rho(x, y, z) = Kz$, K a constant. Then

$$m = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho\cos\phi)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi K \int_0^{\pi/2} \cos\phi \sin\phi \cdot \tfrac{1}{4}a^4 \, d\phi$$
$$= \tfrac{1}{2}\pi K a^4 \left[-\tfrac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \tfrac{\pi}{4} K a^4$$

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^4 \cos^2\phi \sin\phi \, d\rho \, d\phi \, d\theta = \tfrac{2}{5}\pi K a^5 \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi$$
$$= \tfrac{2}{5}\pi K a^5 \left[-\tfrac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \tfrac{2}{15}\pi K a^5$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \tfrac{8}{15}a\right)$.

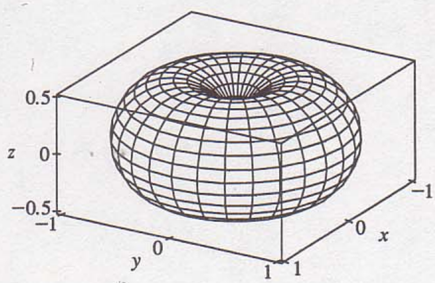
29. In spherical coordinates $z = \sqrt{x^2 + y^2}$ becomes $\cos\phi = \sin\phi$ or $\phi = \frac{\pi}{4}$. Then
$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \, d\phi \int_0^1 \rho^2 \, d\rho = \frac{1}{3}\pi\left(2 - \sqrt{2}\right)$,
$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = 2\pi \left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4} \left(\frac{1}{4}\right) = \frac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.
Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{3}{8\left(2 - \sqrt{2}\right)}\right)$.

30. Place the center of the sphere at $(0, 0, 0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by $V = \int_0^{\pi/6} \int_0^{\pi} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/6} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^a \rho^2 \, d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3$.

31. In cylindrical coordinates the paraboloid is given by $z = r^2$ and the plane by $z = 2r\sin\theta$ and they intersect in the circle $r = 2\sin\theta$. Then $\iiint_E z \, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} rz \, dz \, dr \, d\theta = \frac{5\pi}{6}$ (using a CAS).

32. (a) The region enclosed by the torus is $\{(\rho, \theta, \phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi, 0 \le \rho \le \sin\phi\}$, so its volume is
$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi} \frac{1}{3}\sin^4\phi \, d\phi = \frac{2}{3}\pi \left[\frac{3}{8}\phi - \frac{1}{4}\sin 2\phi + \frac{1}{16}\sin 4\phi\right]_0^{\pi} = \frac{1}{4}\pi^2$

(b) In Maple, we can plot the torus using the `plots[sphereplot]` command, or with the `coords=spherical` option in a regular `plot` command. In Mathematica, use `ParametricPlot3d`.



33. The region E of integration is the region above the paraboloid $z = x^2 + y^2$, or $z = r^2$, and below the paraboloid $z = 2 - x^2 - y^2$, or $z = 2 - r^2$. Also, we have $-1 \le x \le 1$ with $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$ which describes the unit circle in the xy-plane. Thus,

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} \left(x^2 + y^2\right)^{3/2} dz \, dy \, dx = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} \left(r^2\right)^{3/2} r \, dz \, dr \, d\theta$$
$$= \int_0^{2\pi} \int_0^1 \left[r^4 z\right]_{z=r^2}^{z=2-r^2} dr \, d\theta = \int_0^{2\pi} \int_0^1 \left(2r^4 - r^6 - r^6\right) dr \, d\theta = \int_0^{2\pi} \left(\tfrac{2}{5} - \tfrac{2}{7}\right) d\theta = \tfrac{8\pi}{35}$$

34. The region E of integration is the region above the paraboloid $z = x^2 + y^2 = r^2$ and below the cone $z = \sqrt{x^2+y^2} = r$. Also, we have $0 \le y \le 1$, $0 \le x \le \sqrt{1-y^2}$ which is equivalent to $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le 1$. Thus

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r} r^2 \cos\theta \sin\theta\, zr\,dz\,dr\,d\theta$$

$$= \tfrac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \cos\theta \sin\theta \left[z^2\right]_{z=r^2}^{z=r} dr\,d\theta = \tfrac{1}{2} \int_0^{\pi/2} \int_0^1 \left(r^5 - r^7\right) \cos\theta \sin\theta\,dr\,d\theta$$

$$= \tfrac{1}{2} \int_0^{\pi/2} \left[\tfrac{1}{6} r^6 - \tfrac{1}{8} r^8\right]_{r=0}^{r=1} \cos\theta \sin\theta\,d\theta = \tfrac{1}{2} \int_0^{\pi/2} \tfrac{1}{24} \cos\theta \sin\theta\,d\theta$$

$$= \tfrac{1}{48} \int_0^{\pi/2} \tfrac{1}{2} \sin 2\theta\,d\theta = \tfrac{1}{96} \left[-\tfrac{1}{2} \cos 2\theta\right]_0^{\pi/2} = \tfrac{1}{96}$$

35. The region of integration E is the top half of the sphere $x^2 + y^2 + z^2 = 9$. So

$$\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx = \iiint_E z\sqrt{x^2+y^2+z^2}\,dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \left(\rho^2 \cos\phi\right)\left(\rho^2 \sin\phi\right) d\rho\,d\phi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\phi \sin\phi\,d\phi \int_0^3 \rho^4\,d\rho$$

$$= \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2} \sin^2\phi\right]_0^{\pi/2} \left[\tfrac{1}{5}\rho^5\right]_0^3 = (2\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{243}{5}\right) = \tfrac{243}{5}\pi$$

36. The region of integration E is the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2 + y^2 + z^2 = 18$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4) and so $0 \le \phi \le \frac{\pi}{4}$. Also $0 \le \rho \le \sqrt{18} = 3\sqrt{2}$. So the integral becomes

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{3\sqrt{2}} \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/4} \sin\phi\,d\phi \int_0^{3\sqrt{2}} \rho^4\,d\rho$$

$$= \left[\theta\right]_0^{\pi/2} \left[-\cos\phi\right]_0^{\pi/4} \left[\tfrac{1}{5}\rho^5\right]_0^{3\sqrt{2}}$$

$$= \left(\tfrac{\pi}{2}\right)\left(1 - \tfrac{\sqrt{2}}{2}\right)\left(\tfrac{972\sqrt{2}}{5}\right) = 486\pi\left(\tfrac{\sqrt{2}-1}{5}\right)$$

37. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5}\sin 6\theta \sin 5\phi$, it can be described in spherical coordinates as $E = \left\{(\rho, \theta, \phi) \mid 0 \le \rho \le 1 + \frac{1}{5}\sin 6\theta \sin 5\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\right\}$. Its volume is given by $V(E) = \iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_0^{1+(\sin 6\theta \sin 5\phi)/5} \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi = \frac{136\pi}{99}$ (using a CAS).

38. The given integral is equal to

$\lim\limits_{R\to\infty} \int_0^{2\pi} \int_0^\pi \int_0^R \rho e^{-\rho^2} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \lim\limits_{R\to\infty} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^\pi \sin\phi\,d\phi\right)\left(\int_0^R \rho^3 e^{-\rho^2}\,d\rho\right)$. Now use integration

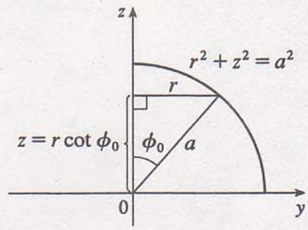
by parts with $u = \rho^2$, $dv = \rho e^{-\rho^2}\,d\rho$ to get

$$\lim_{R\to\infty} 2\pi\,(2)\left(\left[\rho^2\left(-\tfrac{1}{2}\right)e^{-\rho^2}\right]_0^R - \int_0^R 2\rho\left(-\tfrac{1}{2}\right)e^{-\rho^2}\,d\rho\right) = \lim_{R\to\infty} 4\pi\left(-\tfrac{1}{2}R^2 e^{-R^2} + \left[-\tfrac{1}{2}e^{-\rho^2}\right]_0^R\right)$$

$$= 4\pi \lim_{R\to\infty}\left[-\tfrac{1}{2}R^2e^{-R^2} - \tfrac{1}{2}e^{-R^2} + \tfrac{1}{2}\right] = 4\pi\left(\tfrac{1}{2}\right) = 2\pi$$

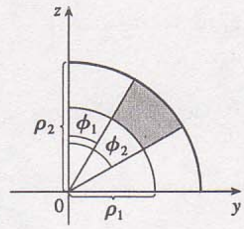
(Note that $R^2 e^{-R^2} \to 0$ as $R \to \infty$ by l'Hospital's Rule.)

39. (a) From the diagram, $z = r\cot\phi_0$ to $z = \sqrt{a^2 - r^2}$, $r = 0$ to $r = a\sin\phi_0$ (or use $a^2 - r^2 = r^2\cot^2\phi_0$). Thus



$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta \\
&= 2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2} - r^2\cot\phi_0\right)dr \\
&= \tfrac{2\pi}{3}\left[-\left(a^2-r^2\right)^{3/2} - r^3\cot\phi_0\right]_0^{a\sin\phi_0} \\
&= \tfrac{2\pi}{3}\left[-\left(a^2 - a^2\sin^2\phi_0\right)^{3/2} - a^3\sin^3\phi_0\cot\phi_0 + a^3\right] \\
&= \tfrac{2}{3}\pi a^3\left[1 - \left(\cos^3\phi_0 + \sin^2\phi_0\cos\phi_0\right)\right] = \tfrac{2}{3}\pi a^3\left(1 - \cos\phi_0\right)
\end{aligned}$$

(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$. Letting



V_{ij} = volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)

and letting V be the volume of the wedge, we have

$$\begin{aligned}
V &= (V_{22} - V_{21}) - (V_{12} - V_{11}) \\
&= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\rho_2^3(1 - \cos\phi_2) - \rho_2^3(1-\cos\phi_1) - \rho_1^3(1-\cos\phi_2) + \rho_1^3(1-\cos\phi_1)\right] \\
&= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(1-\cos\phi_2) - \left(\rho_2^3 - \rho_1^3\right)(1 - \cos\phi_1)\right] \\
&= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(\cos\phi_1 - \cos\phi_2)\right]
\end{aligned}$$

Or: Show that $V = \displaystyle\int_{\theta_1}^{\theta_2}\int_{\rho_1\sin\phi_1}^{\rho_2\sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta.$

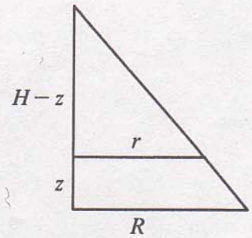
(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \le \tilde{\rho} \le \rho_2$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_1^3 - \rho_2^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly there exists ϕ with $\phi_1 \le \tilde{\phi} \le \phi_2$ such that $\cos\phi_2 - \cos\phi_1 = \left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives

$\Delta V = \left(\tilde{\rho}^2\Delta\rho\right)(\theta_2 - \theta_1)\left(\sin\tilde{\phi}\right)\Delta\phi = \tilde{\rho}^2\sin\tilde{\phi}\,\Delta\rho\Delta\phi\Delta\theta.$

40. (a) The mountain comprises a solid conical region C. The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)\,g(P)\,\Delta V$. Summing over the whole mountain we get $W = \iiint_C h(P)\,g(P)\,dV$.

(b) Here C is a solid right circular cone with radius $R = 62{,}000$ ft, height $H = 12{,}400$ ft, and density $g(P) = 200 \text{ lb/ft}^3$ at all points P in C. We use cylindrical coordinates:

$$\begin{aligned} W &= \int_0^{2\pi}\int_0^H\int_0^{R(1-z/H)} z\cdot 200r\,dr\,dz\,d\theta \\ &= 2\pi\int_0^H 200z\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)} dz \\ &= 400\pi\int_0^H z\frac{R^2}{2}\left(1-\frac{z}{H}\right)^2 dz \\ &= 200\pi R^2\int_0^H\left(z-\frac{2z^2}{H}+\frac{z^3}{H^2}\right)dz \\ &= 200\pi R^2\left[\frac{z^2}{2}-\frac{2z^3}{3H}+\frac{z^4}{4H^2}\right]_0^H \\ &= 200\pi R^2\left(\frac{H^2}{2}-\frac{2H^2}{3}+\frac{H^2}{4}\right) = \tfrac{50}{3}\pi R^2H^2 \\ &= \tfrac{50}{3}\pi\,(62{,}000)^2\,(12{,}400)^2 \approx 3.1\times 10^{19} \text{ ft-lb} \end{aligned}$$

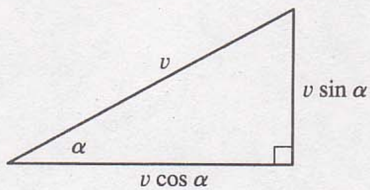


$$\frac{r}{R} = \frac{H-z}{H} = 1-\frac{z}{H}$$

Applied Project □ Roller Derby

1. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(m + I/r^2\right)v^2$, so $v^2 = \dfrac{2mgh}{m+I/r^2} = \dfrac{2gh}{1+I^*}$.

2. The vertical component of the speed is $v\sin\alpha$, so $\dfrac{dy}{dt} = \sqrt{\dfrac{2gy}{1+I^*}}\sin\alpha = \sqrt{\dfrac{2g}{1+I^*}}\sin\alpha\sqrt{y}$.



3. Solving the separable differential equation, we get $\dfrac{dy}{\sqrt{y}} = \sqrt{\dfrac{2g}{1+I^*}}\sin\alpha\,dt \;\Rightarrow$

$2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}(\sin\alpha)\,t + C$. But $y=0$ when $t=0$, so $C=0$ and we have $2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}(\sin\alpha)\,t$.

Solving for t when $y=h$ gives $T = \dfrac{2\sqrt{h}}{\sin\alpha}\sqrt{\dfrac{1+I^*}{2g}} = \sqrt{\dfrac{2h\,(1+I^*)}{g\sin^2\alpha}}$.

4. Assume that the length of each cylinder is ℓ. Then the density of the solid cylinder is $\dfrac{m}{\pi r^2\ell}$, and from Formulas 16.7.16 [ET 15.7.16], its moment of inertia (using cylindrical coordinates) is

$$I_z = \iiint \frac{m}{\pi r^2\ell}\left(x^2+y^2\right)dV = \int_0^\ell\int_0^{2\pi}\int_0^r \frac{m}{\pi r^2\ell}R^2\,R\,dR\,d\theta\,dz = \frac{m}{\pi r^2\ell}2\pi\ell\left[\tfrac{1}{4}R^4\right]_0^r = \frac{mr^2}{2}$$

and so $I^* = \dfrac{I_z}{mr^2} = \dfrac{1}{2}$.

For the hollow cylinder, we consider its entire mass to lie a distance r from the axis of rotation, so $x^2 + y^2 = r^2$ is a constant. We express the density in terms of mass per unit area as $\rho = \dfrac{m}{2\pi r\ell}$, and then the moment of inertia is calculated as a double integral:

$$I_z = \iint \left(x^2 + y^2\right) \frac{m}{2\pi r\ell}\, dA = \frac{mr^2}{2\pi r\ell} \iint dA = mr^2$$

so $I^* = \dfrac{I_z}{mr^2} = 1$.

5. The volume of such a ball is $\frac{4}{3}\pi\left(r^3 - a^3\right) = \frac{4}{3}\pi r^3\left(1 - b^3\right)$, and so its density is $\dfrac{m}{\frac{4}{3}\pi r^3\left(1 - b^3\right)}$.

Using Formula 16.8.4 [ET 15.8.4], we get

$$\begin{aligned}
I_z &= \iiint \left(x^2 + y^2\right) \frac{m}{\frac{4}{3}\pi r^3\left(1 - b^3\right)}\, dV \\
&= \frac{m}{\frac{4}{3}\pi r^3\left(1 - b^3\right)} \int_a^r \int_0^{2\pi} \int_0^{\pi} \left(\rho^2 \sin^2\phi\right)\left(\rho^2 \sin\phi\right) d\phi\, d\theta\, d\rho \\
&= \frac{m}{\frac{4}{3}\pi r^3\left(1 - b^3\right)} \cdot 2\pi \left[-\frac{\left(2 + \sin^2\phi\right)\cos\phi}{3}\right]_0^{\pi} \left[\frac{\rho^5}{5}\right]_a^r \qquad \text{(from the Table of Integrals)} \\
&= \frac{m}{\frac{4}{3}\pi r^3\left(1 - b^3\right)} \cdot 2\pi \cdot \frac{4}{3} \cdot \frac{r^5 - a^5}{5} = \frac{2mr^5\left(1 - b^5\right)}{5r^3\left(1 - b^3\right)} = \frac{2\left(1 - b^5\right)mr^2}{5\left(1 - b^3\right)}
\end{aligned}$$

Therefore $I^* = \dfrac{2\left(1 - b^5\right)}{5\left(1 - b^3\right)}$.

Since a represents the inner radius, $a \to 0$ corresponds to a solid ball, and $a \to r$ corresponds to a hollow ball.

6. For a solid ball, $a \to 0 \quad \Rightarrow \quad b \to 0$, so $I^* = \lim\limits_{b \to 0} \dfrac{2\left(1 - b^5\right)}{5\left(1 - b^3\right)} = \dfrac{2}{5}$. For a hollow ball, $a \to r \quad \Rightarrow \quad b \to 1$, so

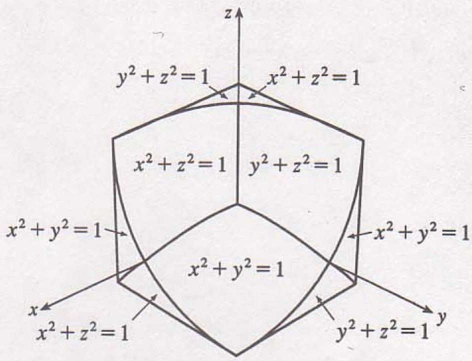
$I^* = \lim\limits_{b \to 1} \dfrac{2\left(1 - b^5\right)}{5\left(1 - b^3\right)} = \dfrac{2}{5} \lim\limits_{b \to 1} \dfrac{-5b^4}{-3b^2} = \dfrac{2}{5}\left(\dfrac{5}{3}\right) = \dfrac{2}{3}$ (by l'Hospital's Rule).

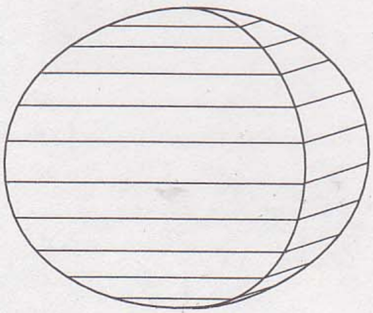
Note: We could instead have calculated $I^* = \lim\limits_{b \to 1} \dfrac{2\left(1 - b\right)\left(1 + b + b^2 + b^3 + b^4\right)}{5\left(1 - b\right)\left(1 + b + b^2\right)} = \dfrac{2 \cdot 5}{5 \cdot 3} = \dfrac{2}{3}$.

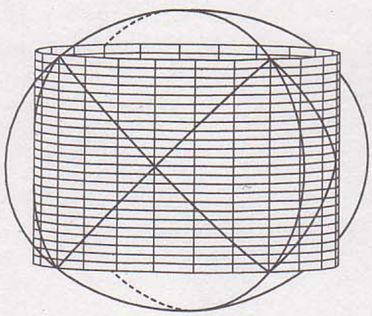
Thus the objects finish in the following order: solid ball ($I^* = \frac{2}{5}$), solid cylinder ($I^* = \frac{1}{2}$), hollow ball ($I^* = \frac{2}{3}$), hollow cylinder ($I^* = 1$).

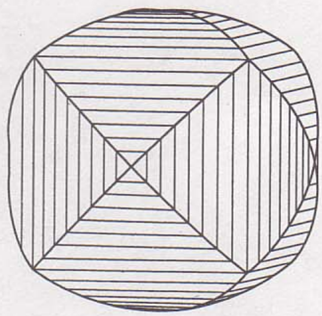
Discovery Project □ The Intersection of Three Cylinders

1. The three cylinders in the illustration in the text can be visualized as representing the surfaces $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Then we sketch the solid of intersection with the coordinate axes and equations indicated. To be more precise, we start by finding the bounding curves of the solid (shown in the first graph below) enclosed by the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$: $x = \pm y = \pm\sqrt{1 - z^2}$ are the symmetric equations, and these can be expressed parametrically as $x = s$, $y = \pm s$, $z = \pm\sqrt{1 - s^2}$, $-1 \le s \le 1$. Now the cylinder $x^2 + y^2 = 1$ intersects these curves at the eight points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. The resulting solid has twelve curved faces bounded by "edges" which are arcs of circles, as shown in the third diagram. Each cylinder defines four of the twelve faces.







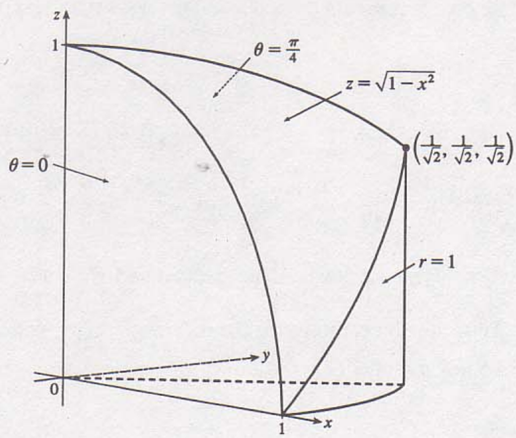


2. To find the volume, we split the solid into sixteen congruent pieces, one of which lies in the part of the first octant with $0 \le \theta \le \frac{\pi}{4}$. (Naturally, we use cylindrical coordinates!) This piece is described by

$\left\{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{4}, 0 \le z \le \sqrt{1 - x^2}\right\}$,

and so, substituting $x = r\cos\theta$, the volume of the entire solid is

$$\begin{aligned} V &= 16 \int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{1-x^2}} r \, dz \, dr \, d\theta \\ &= 16 \int_0^{\pi/4} \int_0^1 r\sqrt{1 - r^2\cos^2\theta} \, dr \, d\theta \\ &= 16 - 8\sqrt{2} \approx 4.6863 \end{aligned}$$

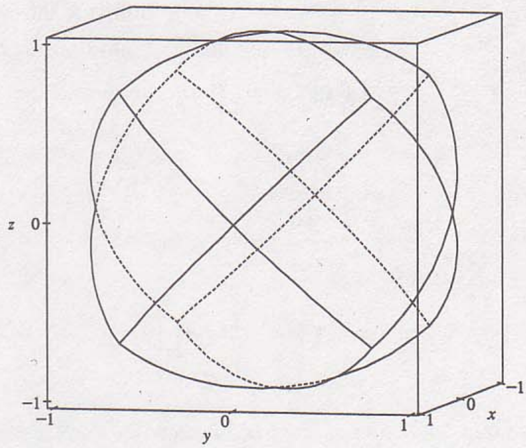


3. To graph the edges of the solid, we use parametrized curves similar to those found in Problem 1 for the intersection of two cylinders. We must restrict the parameter intervals so that each arc extends exactly to the desired vertex. One possible set of parametric equations (with all sign choices allowed) is

$x = r, y = \pm r, z = \pm\sqrt{1-r^2}, -\frac{1}{\sqrt{2}} \le r \le \frac{1}{\sqrt{2}}$;

$x = \pm s, y = \pm\sqrt{1-s^2}, z = s, -\frac{1}{\sqrt{2}} \le s \le \frac{1}{\sqrt{2}}$;

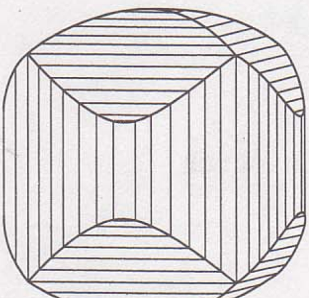
$x = \pm\sqrt{1-t^2}, y = t, z = \pm t, -\frac{1}{\sqrt{2}} \le t \le \frac{1}{\sqrt{2}}$.



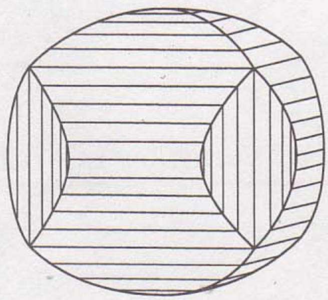
4. Let the three cylinders be $x^2 + y^2 = a^2$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

If $a < 1$, then the four faces defined by the cylinder $x^2 + y^2 = 1$ in Problem 1 collapse into a single face, as in the first graph. If $1 < a < \sqrt{2}$, then each pair of vertically opposed faces, defined by one of the other two cylinders, collapse into a single face, as in the second graph. If $a \ge \sqrt{2}$, then the vertical cylinder encloses the solid of intersection of the other two cylinders completely, so the solid of intersection coincides with the solid of intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, as illustrated in Problem 1.

If we were to vary b or c instead of a, we would get solids with the same shape, but differently oriented.



$a = 0.95, b = c = 1$

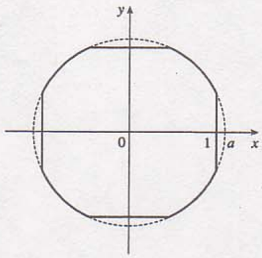


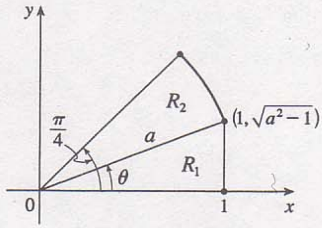
$a = 1.1, b = c = 1$

5. If $a < 1$, the solid looks similar to the first graph in Problem 4. As in Problem 2, we split the solid into sixteen congruent pieces, one of which can be described as the solid above the polar region $\{(r,\theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{4}\}$ in the xy-plane and below the surface $z = \sqrt{1-x^2} = \sqrt{1-r^2\cos^2\theta}$. Thus, the total volume is

$$V = 16\int_0^{\pi/4}\int_0^a \sqrt{1-r^2\cos^2\theta}\, r\, dr\, d\theta$$

If $a > 1$ and $a < \sqrt{2}$, we have a solid similar to the second graph in Problem 4. Its intersection with the xy-plane is graphed below. Again we split the solid into sixteen congruent pieces, one of which is the solid above the region shown in the second figure and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2 \cos^2 \theta}$.





We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = 1$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\frac{1}{a}$ in polar coordinates, or $y = \sqrt{a^2 - 1}\,x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1 - x^2}\,dy\,dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 16.6.22 [ET 15.6.22]. Its volume is

$$V = 16\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$$

16.9 Change of Variables in Multiple Integrals ET 15.9

1. $x = u + 4v$, $y = 3u - 2v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = 1(-2) - 4(3) = -14.$

2. $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4uv - (-4uv) = 8uv$$

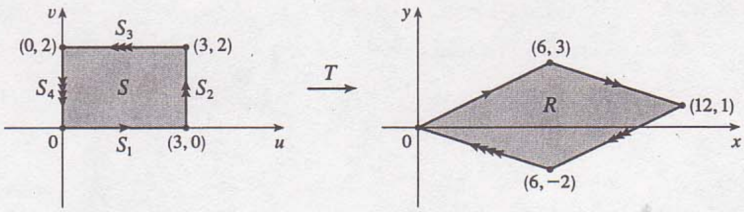
3. $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{v}{(u+v)^2} & -\dfrac{u}{(u+v)^2} \\ -\dfrac{v}{(u-v)^2} & \dfrac{u}{(u-v)^2} \end{vmatrix} = \frac{uv}{(u+v)^2(u-v)^2} - \frac{uv}{(u+v)^2(u-v)^2} = 0$$

4. $$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} \partial x/\partial\alpha & \partial x/\partial\beta \\ \partial y/\partial\alpha & \partial y/\partial\beta \end{vmatrix} = \begin{vmatrix} \sin\beta & \alpha\cos\beta \\ \cos\beta & -\alpha\sin\beta \end{vmatrix} = -\alpha\sin^2\beta - \alpha\cos^2\beta = -\alpha$$

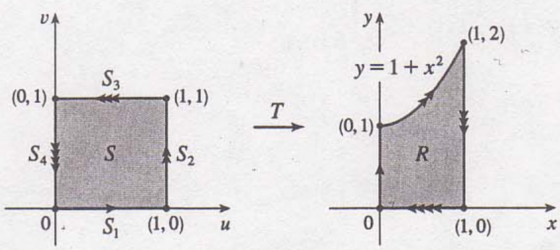
5. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix}$$
$$= v\begin{vmatrix} w & v \\ 0 & u \end{vmatrix} - u\begin{vmatrix} 0 & v \\ w & u \end{vmatrix} + 0\begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix} = v(uw-0) - u(0-vw) = 2uvw$$

6. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{vmatrix} = e^{u+v+w}\begin{vmatrix} e^{u-v} & -e^{u-v} \\ e^{u+v} & e^{u+v} \end{vmatrix}$$
$$= e^{u+v+w}\left(e^{u-v}e^{u+v} + e^{u-v}e^{u+v}\right) = e^{u+v+w}\left(2e^{2u}\right) = 2e^{3u+v+w}$$

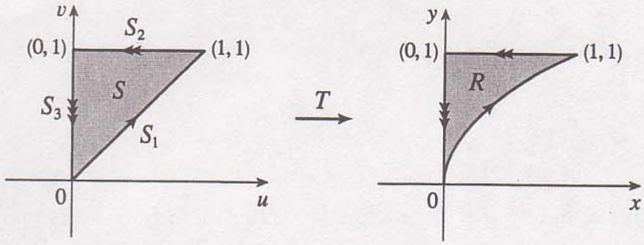
7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$ $(0 \le u \le 3)$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \le x \le 6$. S_2 is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \quad \Rightarrow \quad x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \quad \Rightarrow \quad x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \quad \Rightarrow \quad x = -3y$, $0 \le x \le 6$. The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



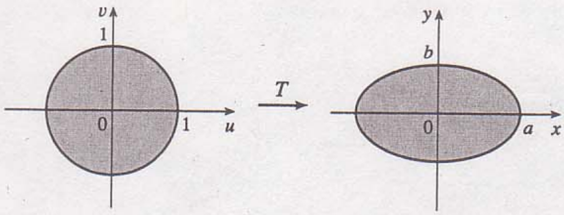
8. S_1 is the line segment $v = 0$, $0 \le u \le 1$, so $x = v = 0$ and $y = u\left(1 + v^2\right) = u$. Since $0 \le u \le 1$, the image is the line segment $x = 0$, $0 \le y \le 1$. S_2 is the segment $u = 1$, $0 \le v \le 1$, so $x = v$ and $y = u\left(1 + v^2\right) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \le x \le 1$. S_3 is the segment $v = 1$, $0 \le u \le 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \le y \le 2$. S_4 is described by $u = 0$, $0 \le v \le 1$, so $0 \le x = v \le 1$ and $y = u\left(1 + v^2\right) = 0$. The image is the line segment $y = 0$, $0 \le x \le 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x-axis, and the lines $x = 0$, $x = 1$.



9. S_1 is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. S_2 is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. S_3 is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \quad \Rightarrow \quad 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



10. Substituting $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ into $u^2 + v^2 \le 1$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, so the image of $u^2 + v^2 \le 1$ is the elliptical region $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$.



11. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = \frac{1}{3}$ and $3x + 4y = (u + v) + \frac{4}{3}(v - 2u) = \frac{1}{3}(7v - 5u)$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given tranformation: The line $y = x$ is the image of $\frac{1}{3}(v - 2u) = \frac{1}{3}(u + v)$ or $u = 0$, $y = x - 2 \quad \Rightarrow$ $\frac{1}{3}(v - 2u) = \frac{1}{3}(u + v) - 2$ or $u = 2$, $y = -2x \quad \Rightarrow \quad \frac{1}{3}(v - 2u) = -\frac{2}{3}(u + v)$ or $v = 0$, and $y = 3 - 2x \quad \Rightarrow \quad \frac{1}{3}(v - 2u) = 3 - \frac{2}{3}(u + v)$ or $v = 3$. Thus S is the rectangle $[0, 2] \times [0, 3]$ in the uv-plane and

$$\iint_R (3x + 4y)\, dA = \iint_S \frac{1}{3}(7v - 5u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \int_0^3 \int_0^2 \tfrac{1}{3}(7v - 5u)\left(\tfrac{1}{3}\right) du\, dv$$
$$= \tfrac{1}{9} \textstyle\int_0^3 (14v - 10)\, dv = \tfrac{1}{9}(33) = \tfrac{11}{3}$$

12. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -13$, $x + y = 5u + v$ and since $u = \dfrac{2x + 3y}{13}$ and $v = \dfrac{3x - 2y}{13}$, R is the image of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. Thus $\iint_R (x + y)\, dA = \int_0^1 \int_0^1 (5u + v)\, |-13|\, du\, dv = 13 \int_0^1 \left(\frac{5}{2} + v\right) dv = 13\,(3) = 39$.

13. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$.

Thus

$$\begin{aligned}\iint_R x^2\,dA &= \iint_{u^2+v^2\le 1} \left(4u^2\right)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 \left(24r^2\cos^2\theta\right) r\,dr\,d\theta \\ &= 24\int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 r^3\,dr = 24\left[\tfrac12 x + \tfrac14\sin 2x\right]_0^{2\pi}\left[\tfrac14 r^4\right]_0^1 \\ &= 24\,(\pi)\left(\tfrac14\right) = 6\pi\end{aligned}$$

14. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt2 & -\sqrt{2/3} \\ \sqrt2 & \sqrt{2/3} \end{vmatrix} = \frac{4}{\sqrt3}$, $x^2 - xy + y^2 = 2u^2 + 2v^2$ and the planar ellipse

$x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus

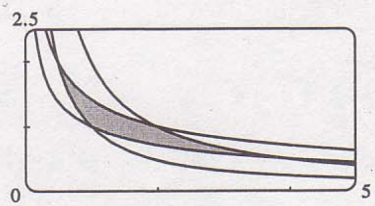
$\iint_R \left(x^2 - xy + y^2\right) dA = \iint_{u^2+v^2\le1} \left(2u^2 + 2v^2\right)\left(\frac{4}{\sqrt3}du\,dv\right) = \int_0^{2\pi}\int_0^1 \frac{8}{\sqrt3} r^3\,dr\,d\theta = \frac{4\pi}{\sqrt3}$.

15. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\begin{aligned}\iint_R xy\,dA &= \int_1^3\int_{\sqrt u}^{\sqrt{3u}} u\left(\frac1v\right) dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt u\right) du \\ &= \int_1^3 u\ln\sqrt3\,du = 4\ln\sqrt3 = 2\ln 3\end{aligned}$$

16. Here $y = \dfrac{v}{u}$, $x = \dfrac{u^2}{v}$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \dfrac1v$ and R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac1v\right) du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac34.$$



17. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. So

$\iiint_E dV = \iiint_{u^2+v^2+w^2\le1} abc\,du\,dv\,dw = (abc)\,(\text{volume of the ball}) = \frac43\pi abc$.

(b) If we approximate the surface of Earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of Earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac43\pi\,(6378)\,(6378)\,(6356) \approx 1.083 \times 10^{12}\ \text{km}^3$.

18. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2 \le 1$.

Now $x^2y = (a^2u^2)(bv)$, so

$$\begin{aligned}\iiint_E x^2y\,dV &= \iiint_{u^2+v^2+w^2 \le 1} (a^2bu^2v)(abc)\,du\,dv\,dw \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (a^3b^2c)(\rho^2\sin^2\phi\cos^2\theta)(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= a^3b^2c\int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho^5\sin^4\phi\cos^2\theta\sin\theta)\,d\rho\,d\phi\,d\theta \\ &= a^3b^2c\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta\int_0^{\pi}\sin^4\phi\,d\phi\int_0^1\rho^5\,d\rho \\ &= 0 \text{ since } \int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta = 0.\end{aligned}$$

19. Letting $u = 2x-y$ and $v = 3x+y$, we have $x = \frac{1}{5}(u+v)$, $y = \frac{1}{5}(2v-3u)$. Then

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{vmatrix} = \dfrac{1}{5}$ and

$$\begin{aligned}\iint_R xy\,dA &= \int_{-2}^{1}\int_{-3}^{1}\frac{(u+v)(2v-3u)}{25}\left(\frac{1}{5}\right)du\,dv = \tfrac{1}{125}\int_{-2}^{1}\int_{-3}^{1}(2v^2-uv-3u^2)\,du\,dv \\ &= \tfrac{1}{125}\int_{-2}^{1}(8v^2+4v-28)\,dv = -\tfrac{66}{125}\end{aligned}$$

20. Let $u = x-y$, $v = x+2y$, so $y = \frac{1}{3}(v-u)$ and $x = \frac{1}{3}(2u+v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{vmatrix} = \dfrac{1}{3}$ and

$$\begin{aligned}\iint_R \frac{x+2y}{\cos(x-y)}\,dA &= \frac{1}{3}\int_0^1\int_0^2 \frac{v}{\cos u}\,dv\,du = \tfrac{2}{3}\int_0^1 \sec u\,du = \tfrac{2}{3}\left[\ln|\sec u+\tan u|\right]_0^1 \\ &= \tfrac{2}{3}\left[\ln(\sec 1+\tan 1)-\ln 1\right] = \tfrac{2}{3}\ln(\sec 1+\tan 1)\end{aligned}$$

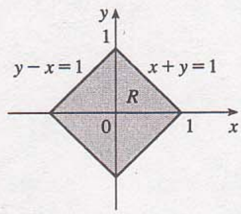
21. Letting $u = y-x$, $v = y+x$, we have $y = \frac{1}{2}(u+v)$, $x = \frac{1}{2}(v-u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$

and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\begin{aligned}\iint_R \cos\frac{y-x}{y+x}\,dA &= \int_1^2\int_{-v}^{v}\cos\frac{u}{v}\left|-\frac{1}{2}\right|du\,dv = \frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v}dv \\ &= \tfrac{1}{2}\int_1^2 2v\sin(1)\,dv = \tfrac{3}{2}\sin 1\end{aligned}$$

22. Letting $u = 3x$, $v = 2y$, we have $9x^2+4y^2 = u^2+v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{6}$ and R is the image of the quarter-disk D given by $u^2+v^2 \le 1$, $u \ge 0$, $v \ge 0$. Thus

$$\begin{aligned}\iint_R \sin(9x^2+4y^2)\,dA &= \iint_D \tfrac{1}{6}\sin(u^2+v^2)\,du\,dv = \int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\,r\,dr\,d\theta \\ &= \tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1 = \tfrac{\pi}{24}(1-\cos 1)\end{aligned}$$



23. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \;\Rightarrow\; y = \frac{1}{2}(u+v)$ and $u - v = 2x \;\Rightarrow\; x = \frac{1}{2}(u-v)$. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now

$|u| = |x+y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le u \le 1$, and

$|v| = |-x+y| \le |x| + |y| \le 1 \Rightarrow\; -1 \le v \le 1$.

R is the image of the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. So $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^{1}\left[v\right]_{-1}^{1} = e - e^{-1}$.

24. Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ and R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus

$\iint_R f(x+y)\,dA = \int_0^1\int_0^u (1)\,f(u)\,dv\,du = \int_0^1 f(u)\,[v]_{v=0}^{v=u}\,du = \int_0^1 uf(u)\,du$ as desired.

16 Review — ET 15

CONCEPT CHECK

1. (a) A double Riemann sum of f is $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$, where ΔA is the area of each subrectangle and $f\left(x_{ij}^*, y_{ij}^*\right)$ is a sample point in each subrectangle. If $f(x,y) \ge 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) $\iint_R f(x,y)\,dA = \lim\limits_{m,n\to\infty}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

(c) If $f(x,y) \ge 0$, $\iint_R f(x,y)\,dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$. If f takes on both positive and negative values, $\iint_R f(x,y)\,dA$ is the difference of the volume above R but below the surface $z = f(x,y)$ and the volume below R but above the surface $z = f(x,y)$.

(d) We usually evaluate $\iint_R f(x,y)\,dA$ as an iterated integral according to Fubini's Theorem (see Theorem 16.2.4 [ET 15.2.4]).

(e) The Midpoint Rule for Double Integrals says that we approximate the double integral $\iint_R f(x,y)\,dA$ by the double Riemann sum $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$ where the sample points $\left(\overline{x}_i, \overline{y}_j\right)$ are the centers of the subrectangles.

(f) $f_{\text{ave}} = \dfrac{1}{A(R)}\displaystyle\iint_R f(x,y)\,dA$ where $A(R)$ is the area of R.

2. (a) See (1) and (2) and the accompanying discussion in Section 16.3 [ET 15.3].

(b) See (1) and (2) and the accompanying discussion in Section 16.3 [ET 15.3].

(c) See (5) and the preceding discussion in Section 16.3 [ET 15.3].

(d) See (6)–(11) in Section 16.3 [ET 15.3].

3. We may want to change from rectangular to polar coordinates in a double integral if the region R of integration is more easily described in polar coordinates. To accomplish this, we use $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ where R is given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$.

4. (a) $m = \iint_D \rho(x,y)\,dA$

(b) $M_x = \iint_D y\rho(x,y)\,dA$, $M_y = \iint_D x\rho(x,y)\,dA$

(c) The center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$.

(d) $I_x = \iint_D y^2\rho(x,y)\,dA$, $I_y = \iint_D x^2\rho(x,y)\,dA$, $I_0 = \iint_D (x^2+y^2)\,\rho(x,y)\,dA$

5. (a) $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

(b) $f(x,y) \ge 0$ and $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$.

(c) The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\,dA$; the expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} y f(x,y)\,dA$.

6. $A(S) = \displaystyle\iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$

7. (a) $\iiint_B f(x,y,z)\,dV = \lim\limits_{l,m,n\to\infty} \sum\limits_{i=1}^{l} \sum\limits_{j=1}^{m} \sum\limits_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$

(b) We usually evaluate $\iiint_B f(x,y,z)\,dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals (see Theorem 16.7.4 [ET 15.7.4]).

(c) See the paragraph following Example 16.7.1 [ET 15.7.1].

(d) See (5) and (6) and the accompanying discussion in Section 16.7 [ET 15.7].

(e) See (10) and the accompanying discussion in Section 16.7 [ET 15.7].

(f) See (11) and the preceding discussion in Section 16.7 [ET 15.7].

8. (a) $m = \iiint_E \rho(x,y,z)\,dV$

(b) $M_{yz} = \iiint_E x\rho(x,y,z)\,dV$, $M_{xz} = \iiint_E y\rho(x,y,z)\,dV$, $M_{xy} = \iiint_E z\rho(x,y,z)\,dV$.

(c) The center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$.

(d) $I_x = \iiint_E (y^2+z^2)\,\rho(x,y,z)\,dV$, $I_y = \iiint_E (x^2+z^2)\,\rho(x,y,z)\,dV$,
$I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV$.

9. (a) See Formula 16.8.2 [ET 15.8.2] and the accompanying discussion.

(b) See Formula 16.8.4 [ET 15.8.4] and the accompanying discussion.

(c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

10. (a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 16.9 [ET 15.9].

(c) See (13) and the accompanying discussion in Section 16.9 [ET 15.9].

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. $\int_{-1}^{1}\int_0^1 e^{x^2+y^2}\sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^1 e^{y^2}\sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2}\sin y$ is an odd function. Therefore the statement is true.

3. True:

$\iint_D \sqrt{4-x^2-y^2}\,dA =$ the volume under the surface $x^2+y^2+z^2=4$ and above the xy-plane

$= \frac{1}{2}\left(\text{the volume of the sphere } x^2+y^2+z^2=4\right) = \frac{1}{2}\cdot\frac{4}{3}\pi(2)^3 = \frac{16}{3}\pi$

4. This statement is true because in the given region, $(x^2+\sqrt{y})\sin(x^2y^2) \le (1+2)(1) = 3$, so $\int_1^4\int_0^1 (x^2+\sqrt{y})\sin(x^2y^2)\,dx\,dy \le \int_1^4\int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

5. The volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates, $V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \ne \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

6. True. The moment of inertia about the z-axis of a solid E with constant density k is $I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta$.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m=n=3$ and the sample points the upper right corners of each square, so

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A \\ &= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2) \\ &\qquad + f(2,3)+f(3,1)+f(3,2)+f(3,3)]\end{aligned}$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1\,[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m=n=3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(\overline{x}_i,\overline{y}_j)\,\Delta A \\ &= \Delta A\,[f(0.5,0.5)+(0.5,1.5)+(0.5,2.5)+(1.5,0.5)+f(1.5,1.5) \\ &\qquad + f(1.5,2.5)+(2.5,0.5)+f(2.5,1.5)+f(2.5,2.5)] \\ &\approx 1\,[1.2+2.5+5.0+3.2+4.5+7.1+5.2+6.5+9.0] = 44.2\end{aligned}$$

3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy = \int_1^2 \left[xy+x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y+4e^y)\,dy = \left[y^2+4e^y\right]_1^2$

$= 4+4e^2-1-4e = 4e^2-4e+3$

4. $\int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1)\,dy = \left[e^y - y\right]_0^1 = e - 2$

5. $\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)\,y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1 \int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x} - x^4)\,dx$

$= \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ (integrating by parts in the first term)

$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

7. $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi \int_0^1 \left[(y\sin x)\,z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi \int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$

$= \int_0^\pi \left[-\frac{1}{3}\left(1-y^2\right)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

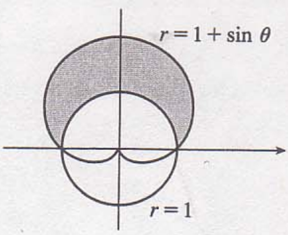
8. $\int_0^1 \int_{\sqrt{y}}^1 \int_0^y xy\,dz\,dx\,dy = \int_0^1 \int_{\sqrt{y}}^1 xy^2\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{x=\sqrt{y}}^{x=1} dy = \int_0^1 \left(\frac{1}{2}y^2 - \frac{1}{2}y^3\right) dy$

$= \left[\frac{1}{6}y^3 - \frac{1}{8}y^4\right]_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4, 0 \le \theta \le \pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^\pi \int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid y - 4 \le x \le 4 - y, 0 \le y \le 4\}$ and $\iint_R f(x,y)\,dA = \int_0^4 \int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

11.



The region whose area is given by $\int_0^\pi \int_1^{1+\sin\theta} r\,dr\,d\theta$ is $\{(r,\theta) \mid 0 \le \theta \le \pi, 1 \le r \le 1 + \sin\theta\}$, which is the region outside the circle $r = 1$ and inside the cardioid $r = 1 + \sin\theta$.

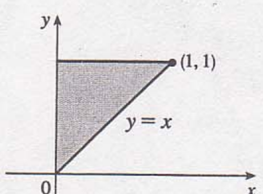
12. The solid is $\left\{(\rho,\theta,\phi) \mid 1 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{6}\right\}$, which lies inside the sphere $\rho = 3$, outside the sphere $\rho = 1$, and within the cone $\phi = \frac{\pi}{6}$.

$$\int_0^{2\pi} \int_0^{\pi/6} \int_1^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/6} \sin\phi\,d\phi \int_1^3 \rho^2\,d\rho$$

$$= \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/6} \left[\tfrac{1}{3}\rho^3\right]_1^3$$

$$= (2\pi)\left(1 - \tfrac{\sqrt{3}}{2}\right)\left(\tfrac{26}{3}\right) = \tfrac{26\pi}{3}\left(2 - \sqrt{3}\right)$$

13.



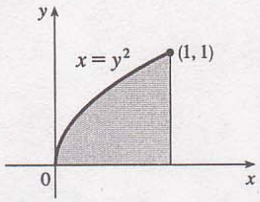
$$\int_0^1 \int_x^1 e^{x/y}\,dy\,dx = \int_0^1 \int_0^y e^{x/y}\,dx\,dy$$

$$= \int_0^1 \left[ye^{x/y}\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 (ey - y)\,dy = \left[\tfrac{e}{2}y^2 - \tfrac{1}{2}y^2\right]_0^1$$

$$= \tfrac{1}{2}(e - 1)$$

14.



$$\int_0^1 \int_{y^2}^1 y \sin\left(x^2\right) dx\, dy = \int_0^1 \int_0^{\sqrt{x}} y \sin\left(x^2\right) dy\, dx$$
$$= \int_0^1 \tfrac{1}{2} x \sin\left(x^2\right) dx$$
$$= \left[-\tfrac{1}{4}\cos\left(x^2\right)\right]_0^1 = \tfrac{1}{4}\left(1 - \cos 1\right)$$

15. $$\int_2^4 \int_0^1 \frac{1}{(x-y)^2}\, dx\, dy = \int_2^4 \left[-(x-y)^{-1}\right]_{x=0}^{x=1} dy = \int_2^4 \left(-\frac{1}{y} - \frac{1}{1-y}\right) dy$$
$$= \left[-\ln y + \ln|1-y|\right]_2^4 = -\ln 4 + \ln 3 + \ln 2 = \ln \tfrac{3}{2}$$

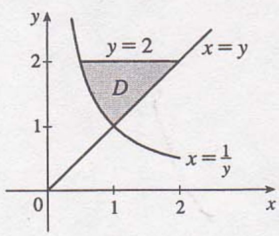
16. $\int_{-1}^1 \int_{x^2-1}^{x+1} x^3\, dy\, dx = \int_{-1}^1 \left(2x^3 + x^4 - x^5\right) dx = \left[\frac{1}{2}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6\right]_{-1}^1 = \frac{2}{5}$

17. The curves $y^2 = x^3$ and $y = x$ intersect when $x^3 = x$, that is when $x = 0$ and $x = 1$ (note that $x \neq -1$ since $x^3 = y^2 \;\Rightarrow\; x \geq 0$.) So $\int_0^1 \int_{x^{3/2}}^x xy\, dy\, dx = \int_0^1 \left[\frac{1}{2}x^3 - \frac{1}{2}x^4\right] dx = \left[\frac{1}{8}x^4 - \frac{1}{10}x^5\right]_0^1 = \frac{1}{40}$.

18. $\int_0^1 \int_0^{x^2} x e^y\, dy\, dx = \int_0^1 x\left(e^{x^2} - 1\right) dx = \frac{1}{2}\left(e^{x^2} - x^2\right)\Big]_0^1 = \dfrac{e-2}{2}$

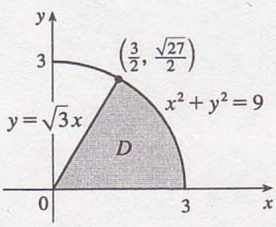
19. $$\int_0^1 \int_0^{1-y^2} (xy + 2x + 3y)\, dx\, dy = \int_0^1 \left[\tfrac{1}{2}x^2 y + x^2 + 3xy\right]_{x=0}^{x=1-y^2} dy$$
$$= \int_0^1 \left[\tfrac{1}{2}y\left(1-y^2\right)^2 + \left(1-y^2\right)^2 + 3y\left(1-y^2\right)\right] dy$$
$$= \tfrac{1}{2}y^5 + y^4 - 4y^3 - 2y^2 + \tfrac{7}{2}y + 1 = \tfrac{1}{5} + \tfrac{1}{12} - \tfrac{2}{3} + \tfrac{7}{4} = \tfrac{41}{30}$$

20.



$$\iint_D y\, dA = \int_1^2 \int_{1/y}^y y\, dx\, dy = \int_1^2 y\left(y - \frac{1}{y}\right) dy$$
$$= \int_1^2 \left(y^2 - 1\right) dy = \left[\tfrac{1}{3}y^3 - y\right]_1^2$$
$$= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{4}{3}$$

21.



$$\iint_D \left(x^2 + y^2\right)^{3/2} dA = \int_0^{\pi/3} \int_0^3 \left(r^2\right)^{3/2} r\, dr\, d\theta$$
$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\, dr = \left[\theta\right]_0^{\pi/3} \left[\frac{1}{5}r^5\right]_0^3$$
$$= \frac{\pi}{3}\frac{3^5}{5} = \frac{81\pi}{5}$$

22. The circle bounding the disk is given by $x^2 + (y-1)^2 = 1$ or $x^2 + y^2 = 2y$ and in polar coordinates $r = 2\sin\theta$. Thus $\iint_D \sqrt{x^2+y^2}\, dA = \int_0^\pi \int_0^{2\sin\theta} r^2\, dr\, d\theta = \int_0^\pi \frac{8}{3}\sin^3\theta\, d\theta = \frac{8}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^\pi = \frac{32}{9}$.

23. $\iiint_E x^2 z\, dV = \int_0^2 \int_0^{2x} \int_0^x x^2 z\, dz\, dy\, dx = \int_0^2 \int_0^{2x} \frac{1}{2}x^4\, dy\, dx = \int_0^2 x^5\, dx = \frac{1}{6}\cdot 2^6 = \frac{32}{3}$

24. $$\iiint_T y\, dV = \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} y\, dz\, dy\, dx = \int_0^1 \int_0^{2-2x} \left[(2-2x)\,y - y^2\right] dy\, dx$$
$$= \int_0^1 \left[\tfrac{1}{2}(2-2x)^3 - \tfrac{1}{3}(2-2x)^3\right] dx = \int_0^1 \tfrac{1}{6}(2-2x)^3\, dx = -\tfrac{1}{48}(2-2x)^4\Big]_0^1 = \tfrac{1}{3}$$

25. $\iiint_E y^2z^2\,dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2\left(1-y^2-z^2\right)dz\,dy$

$= \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\right)\left(r^2\sin^2\theta\right)\left(1-r^2\right)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{4}\sin^2 2\theta\left(r^5-r^7\right)dr\,d\theta$

$= \int_0^{2\pi}\frac{1}{8}\left(1-\cos 4\theta\right)\left[\frac{1}{6}r^6-\frac{1}{8}r^8\right]_{r=0}^{r=1}d\theta = \frac{1}{192}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{2\pi}{192} = \frac{\pi}{96}$

26. $\iiint_E z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^1\int_0^{\sqrt{1-y^2}}(2-y)\,z\,dz\,dy = \int_0^1 \frac{1}{2}(2-y)\left(1-y^2\right)dy$

$= \int_0^1 \frac{1}{2}\left(2-y-y^2+y^3\right)dy = \frac{11}{24}$

27. $\iiint_E yz\,dV = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^y yz\,dz\,dy\,dx = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\frac{1}{2}y^3\,dy\,dx = \int_0^{\pi}\int_0^2 \frac{1}{2}r^3\sin^3\theta\,r\,dr\,d\theta$

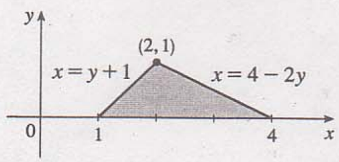
$= \frac{16}{5}\int_0^{\pi}\sin^3\theta\,d\theta = \frac{16}{5}\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{64}{15}$

28. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \left(\rho^3\cos^3\phi\right)\rho\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi}d\theta\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\int_0^1\rho^6\,d\rho = 2\pi\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}\left(\frac{1}{7}\right) = \frac{\pi}{14}$

29. $V = \int_0^2\int_1^4\left(x^2+4y^2\right)dy\,dx = \int_0^2\left[x^2y+\frac{4}{3}y^3\right]_{y=1}^{y=4}dx = \int_0^2\left(3x^2+84\right)dx = 176$

30.



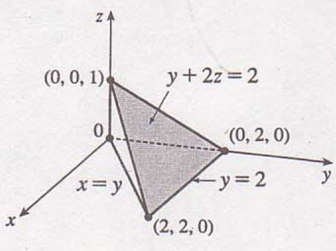
$V = \int_0^1\int_{y+1}^{4-2y}\int_0^{x^2y}dz\,dx\,dy = \int_0^1\int_{y+1}^{4-2y}x^2y\,dx\,dy$

$= \int_0^1\frac{1}{3}\left[(4-2y)^3y-(y+1)^3y\right]dy$

$= \int_0^1 3\left(-y^4+5y^3-11y^2+7y\right)dy$

$= 3\left(-\frac{1}{5}+\frac{5}{4}-\frac{11}{3}+\frac{7}{2}\right) = \frac{53}{20}$

31.



$V = \int_0^2\int_0^y\int_0^{(2-y)/2}dz\,dx\,dy = \int_0^2\int_0^y\left(1-\frac{1}{2}y\right)dx\,dy$

$= \int_0^2\left(y-\frac{1}{2}y^2\right)dy = \frac{2}{3}$

32. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta}r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\left(3r-r^2\sin\theta\right)dr\,d\theta = \int_0^{2\pi}\left[6-\frac{8}{3}\sin\theta\right]d\theta$

$= 6\theta\big]_0^{2\pi}+0 = 12\pi$

33. Using the wedge above the plane $z=0$ and below the plane $z=mx$ and noting that we have the same volume for $m<0$ as for $m>0$ (so use $m>0$), we have

$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}}mx\,dx\,dy = 2\int_0^{a/3}\frac{1}{2}m\left(a^2-9y^2\right)dy = m\left[a^2y-3y^3\right]_0^{a/3}$

$= m\left(\frac{1}{3}a^3-\frac{1}{9}a^3\right) = \frac{2}{9}ma^3$

34. The paraboloid and the half-cone intersect when $x^2 + y^2 = \sqrt{x^2 + y^2}$, that is when $x^2 + y^2 = 1$ or 0. So

$$V = \iint_{x^2+y^2\le 1} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r^2 - r^3\right) dr\,d\theta$$
$$= \int_0^{2\pi} \left(\tfrac{1}{3} - \tfrac{1}{4}\right) d\theta = \tfrac{1}{12}\left(2\pi\right) = \tfrac{\pi}{6}$$

35. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 \left(y - y^3\right) dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac{1}{2}y\left(1-y^2\right)^2 dy = -\frac{1}{12}\left(1-y^2\right)^3\Big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 \left(y^2 - y^4\right) dy = \frac{2}{15}$. Hence $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{8}{15}\right)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 \left(y^3 - y^5\right) dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac{1}{3}y\left(1-y^2\right)^3 dy = -\frac{1}{24}\left(1-y^2\right)^4\Big]_0^1 = \frac{1}{24}$, $I_0 = I_x + I_y = \frac{1}{8}$,

$\overline{\overline{y}}^2 = \frac{1/12}{1/4} = \frac{1}{3} \quad\Rightarrow\quad \overline{\overline{y}} = \frac{1}{\sqrt{3}}$, and $\overline{\overline{x}}^2 = \frac{1/24}{1/4} = \frac{1}{6} \quad\Rightarrow\quad \overline{\overline{x}} = \frac{1}{\sqrt{6}}$.

36. (a) $m = \frac{1}{4}\pi K a^2$ where K is constant,

$M_y = \iint_{x^2+y^2\le a^2} Kx\,dA = K\int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{1}{3}Ka^3\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}a^3K$, and

$M_x = K\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{1}{3}a^3K$ (by symmetry $M_y = M_x$). Hence the centroid is

$(\overline{x}, \overline{y}) = \left(\frac{4}{3\pi}a, \frac{4}{3\pi}a\right)$.

(b) $m = \int_0^{\pi/2}\int_0^a r^4\cos\theta\sin^2\theta\,dr\,d\theta = \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}\left(\frac{1}{5}a^5\right) = \frac{1}{15}a^5$,

$M_y = \int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{96}\pi a^6$, and

$M_x = \int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta = \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{24}a^6$. Hence $(\overline{x}, \overline{y}) = \left(\frac{5}{32}\pi a, \frac{5}{8}a\right)$.

37. (a) The equation of the cone with the suggested orientation is $(h - z) = \frac{h}{a}\sqrt{x^2 + y^2}$, $0 \le z \le h$. Then $V = \frac{1}{3}\pi a^2 h$ is the volume of one frustum of a cone; by symmetry $M_{yz} = M_{xz} = 0$; and

$$M_{xy} = \iint_{x^2+y^2\le a^2} \int_0^{h-(h/a)\sqrt{x^2+y^2}} z\,dz\,dA = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} rz\,dz\,dr\,d\theta$$
$$= \pi\int_0^a r\frac{h^2}{a^2}\left(a-r\right)^2 dr = \frac{\pi h^2}{a^2}\int_0^a \left(a^2 r - 2ar^2 + r^3\right) dr = \frac{\pi h^2}{a^2}\left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{\pi h^2 a^2}{12}$$

Hence the centroid is $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{4}h\right)$.

(b) $$I_z = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} r^3\,dz\,dr\,d\theta = 2\pi\int_0^a \frac{h}{a}\left(ar^3 - r^4\right) dr = \frac{2\pi h}{a}\left(\frac{a^5}{4} - \frac{a^5}{5}\right) = \frac{\pi a^4 h}{10}$$

38. $1 \le z^2 \le 4 \quad\Rightarrow\quad 1/a^2 \le x^2 + y^2 \le 4/a^2$. Let $D = \left\{(x, y) \mid 1/a^2 \le x^2 + y^2 \le 4/a^2\right\}$.

$z = f(x, y) = a\sqrt{x^2 + y^2}$, so $f_x(x, y) = ax\left(x^2 + y^2\right)^{-1/2}$, $f_y(x, y) = ay\left(x^2 + y^2\right)^{-1/2}$, and

$$A(S) = \iint_D \sqrt{\frac{a^2x^2 + a^2y^2}{x^2 + y^2} + 1}\,dA = \iint_D \sqrt{a^2 + 1}\,dA = \sqrt{a^2 + 1}\,A(D)$$
$$= \sqrt{a^2 + 1}\left[\pi\left(\frac{2}{a}\right)^2 - \pi\left(\frac{1}{a}\right)^2\right] = \frac{3\pi}{a^2}\sqrt{a^2 + 1}$$

39. Let D represent the given triangle; then D can be described as the area enclosed by the x- and y-axes and the line $y = 2 - 2x$, or equivalently $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x\}$. We want to find the surface area of the part of the graph of $z = x^2 + y$ that lies over D, so using Equation 16.6.3 [ET 15.6.3] we have

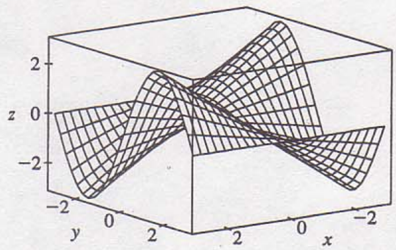
$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (1)^2}\, dA \\ &= \textstyle\int_0^1 \int_0^{2-2x} \sqrt{2+4x^2}\, dy\, dx = \int_0^1 \sqrt{2+4x^2}\, [y]_0^{2-2x}\, dx = \int_0^1 (2-2x)\sqrt{2+4x^2}\, dx \\ &= \textstyle\int_0^1 2\sqrt{2+4x^2}\, dx - \int_0^1 2x\sqrt{2+4x^2}\, dx \end{aligned}$$

Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2\,dx$, we have $\int 2\sqrt{2+4x^2}\, dx = x\sqrt{2+4x^2} + \ln\left(2x + \sqrt{2+4x^2}\right)$. If we substitute $u = 2 + 4x^2$ in the second integral, then $du = 8x\, dx$ and $\int 2x\sqrt{2+4x^2}\, dx = \frac{1}{4}\int \sqrt{u}\, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} = \frac{1}{6}\left(2+4x^2\right)^{3/2}$. Thus

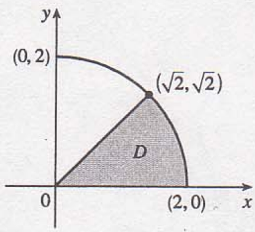
$$\begin{aligned} A(S) &= \left[x\sqrt{2+4x^2} + \ln\left(2x + \sqrt{2+4x^2}\right) - \tfrac{1}{6}\left(2+4x^2\right)^{3/2}\right]_0^1 \\ &= \sqrt{6} + \ln\left(2 + \sqrt{6}\right) - \tfrac{1}{6}(6)^{3/2} - \ln\sqrt{2} + \tfrac{\sqrt{2}}{3} \\ &= \ln\tfrac{2+\sqrt{6}}{\sqrt{2}} + \tfrac{\sqrt{2}}{3} = \ln\left(\sqrt{2} + \sqrt{3}\right) + \tfrac{\sqrt{2}}{3} \approx 1.6176 \end{aligned}$$

40. Using Formula 16.6.3 [ET 15.6.3] with $\partial z/\partial x = \sin y$, $\partial z/\partial y = x\cos y$, we get

$$\begin{aligned} S &= \textstyle\int_{-\pi}^{\pi} \int_{-3}^{3} \sqrt{\sin^2 y + x^2\cos^2 y + 1}\, dx\, dy \\ &\approx 62.9714 \end{aligned}$$



41.

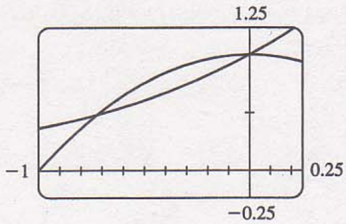


$$\begin{aligned} \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy &= \int_0^{\pi/4} \int_0^2 \frac{1}{1+r^2} r\, dr\, d\theta \\ &= \int_0^{\pi/4} d\theta \int_0^2 \frac{r}{1+r^2}\, dr \\ &= [\theta]_0^{\pi/4} \left[\tfrac{1}{2}\ln\left|1+r^2\right|\right]_0^2 \\ &= \tfrac{\pi}{4}\left(\tfrac{1}{2}\ln 5\right) = \tfrac{\pi}{8}\ln 5 \end{aligned}$$

42. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \left(x^2+y^2+z^2\right)^2 dz\, dy\, dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left(\rho^2\right)^2 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

$= \int_0^{\pi/2} \sin\phi\, d\phi \int_0^{\pi/2} d\theta \int_0^1 \rho^6\, d\rho = [-\cos\phi]_0^{\pi/2}\, [\theta]_0^{\pi/2} \left[\frac{1}{7}\rho^7\right]_0^1 = 1 \cdot \frac{\pi}{2} \cdot \frac{1}{7} = \frac{\pi}{14}$

43. From the graph, it appears that $1 - x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1 - x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is



$$\iint_D y^2 dA \approx \int_{-0.71}^{0} \int_{e^x}^{1-x^2} y^2\,dy\,dx$$
$$= \tfrac{1}{3} \int_{-0.71}^{0} \left[\left(1 - x^2\right)^3 - e^{3x}\right] dx$$
$$= \tfrac{1}{3} \left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{1}{3}e^{3x}\right]_{-0.71}^{0} \approx 0.0512$$

44. Let the tetrahedron be called T. The front face of T is given by the plane $x + \frac{1}{2}y + \frac{1}{3}z = 1$, or $z = 3 - 3x - \frac{3}{2}y$, which intersects the xy-plane in the line $y = 2 - 2x$. So the total mass is $m = \iiint_T \rho(x,y,z)\,dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} \left(x^2 + y^2 + z^2\right) dz\,dy\,dx = \frac{7}{5}$. The center of mass is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(m^{-1} \iiint_T x\rho(x,y,z)\,dV, m^{-1} \iiint_T y\rho(x,y,z)\,dV, m^{-1} \iiint_T z\rho(x,y,z)\,dV\right)$$
$$= \left(\tfrac{4}{21}, \tfrac{11}{21}, \tfrac{8}{7}\right)$$

45. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,3] \times [0,2]$, we can say

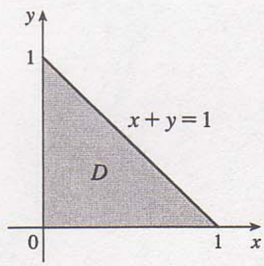
$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^3 \int_0^2 C(x+y)\,dy\,dx$$
$$= C \int_0^3 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=2} dx = C \int_0^3 (2x+2)\,dx = C\left[x^2 + 2x\right]_0^3 = 15C$$

Then $15C = 1 \;\Rightarrow\; C = \frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^{2} \int_{1}^{\infty} f(x,y)\,dy\,dx = \int_0^2 \int_1^2 \frac{1}{15}(x+y)\,dy\,dx = \frac{1}{15} \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2} dx$

$$= \tfrac{1}{15} \int_0^2 \left(x + \tfrac{3}{2}\right) dx = \tfrac{1}{15}\left[\tfrac{1}{2}x^2 + \tfrac{3}{2}x\right]_0^2 = \tfrac{1}{3}$$

(c) $P(X + Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X + Y \le 1) = \iint_D f(x,y)\,dA = \int_0^1 \int_0^{1-x} \tfrac{1}{15}(x+y)\,dy\,dx$$
$$= \tfrac{1}{15} \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$
$$= \tfrac{1}{15} \int_0^1 \left[x(1-x) + \tfrac{1}{2}(1-x)^2\right] dx$$
$$= \tfrac{1}{30} \int_0^1 \left(1 - x^2\right) dx = \tfrac{1}{30}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{45}$$

46. Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800}e^{-t/800} & \text{if } t \ge 0 \end{cases}$$

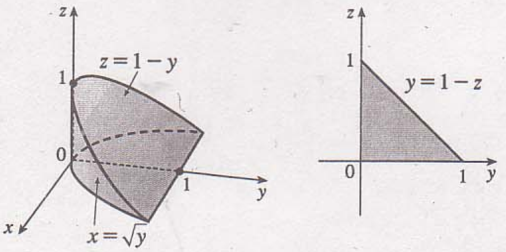
If X, Y, and Z are the lifetimes of the individual bulbs, then X, Y, and Z are independent, so the joint density function is the product of the individual density functions:

$$f(x,y,z) = \begin{cases} \frac{1}{800^3}e^{-(x+y+z)/800} & \text{if } x \ge 0, y \ge 0, z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P\,(X+Y+Z\le 1000)$, or equivalently $P\,((X,Y,Z)\in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1000$. The plane $x+y+z=1000$ meets the xy-plane in the line $x+y=1000$, so we have

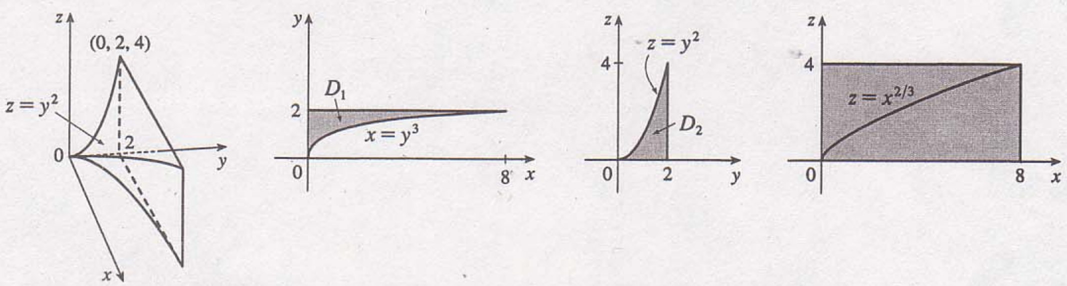
$$\begin{aligned}
P\,(X+Y+Z\le 1000) &= \iiint_E f\,(x,y,z)\;dV \\
&= \int_0^{1000}\int_0^{1000-x}\int_0^{1000-x-y} \tfrac{1}{800^3}e^{-(x+y+z)/800}\,dz\,dy\,dx \\
&= \tfrac{1}{800^3}\int_0^{1000}\int_0^{1000-x} -800\left[e^{-(x+y+z)/800}\right]_{z=0}^{z=1000-x-y}\,dy\,dx \\
&= \tfrac{-1}{800^2}\int_0^{1000}\int_0^{1000-x}\left[e^{-5/4}-e^{-(x+y)/800}\right]dy\,dx \\
&= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}y+800e^{-(x+y)/800}\right]_{y=0}^{y=1000-x}dx \\
&= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}\,(1800-x)-800e^{-x/800}\right]dx \\
&= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}\,(1800-x)^2+800^2e^{-x/800}\right]_0^{1000} \\
&= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}\,(800)^2+800^2e^{-5/4}+\tfrac{1}{2}e^{-5/4}\,(1800)^2-800^2\right] \\
&= 1-\tfrac{97}{32}e^{-5/4}\approx 0.1315
\end{aligned}$$

47.



$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f\,(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f\,(x,y,z)\,dx\,dy\,dz$

48.



$\int_0^2\int_0^{y^3}\int_0^{y^2} f\,(x,y,z)\,dz\,dx\,dy = \iiint_E f\,(x,y,z)\,dV$ where $E=\left\{(x,y,z)\mid 0\le y\le 2, 0\le x\le y^3, 0\le z\le y^2\right\}$. If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1=\left\{(x,y)\mid 0\le y\le 2, 0\le x\le y^3\right\}=\{(x,y)\mid 0\le x\le 8,\ \sqrt[3]{x}\le y\le 2\}$,

$D_2 = \{(y,z) \mid 0 \le z \le 4, \sqrt{z} \le y \le 2\} = \{(y,z) \mid 0 \le y \le 2, 0 \le z \le y^2\}$,
$D_3 = \{(x,z) \mid 0 \le x \le 8, 0 \le z \le 4\}$.
Therefore we have

$$\begin{aligned}\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\,dz\,dx\,dy &= \int_0^8 \int_{\sqrt[3]{x}}^2 \int_0^{y^2} f(x,y,z)\,dz\,dy\,dx = \int_0^4 \int_{\sqrt{z}}^2 \int_0^{y^3} f(x,y,z)\,dx\,dy\,dz\\ &= \int_0^2 \int_0^{y^2} \int_0^{y^3} f(x,y,z)\,dx\,dz\,dy\\ &= \int_0^8 \int_0^{x^{2/3}} \int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dz\,dx + \int_0^8 \int_{x^{2/3}}^4 \int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dz\,dx\\ &= \int_0^4 \int_0^{z^{3/2}} \int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dx\,dz + \int_0^4 \int_{z^{3/2}}^8 \int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

49. Since $u = x - y$ and $v = x + y$, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$. Thus $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$ and

$$\iint_R \frac{x-y}{x+y}\,dA = \int_2^4 \int_{-2}^0 \frac{u}{v}\left(\frac{1}{2}\right) du\,dv = -\int_2^4 \frac{dv}{v} = -\ln 2.$$

50. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$, so

$$\begin{aligned} V = \iiint_E dV &= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw\,dw\,dv\,du = \int_0^1 \int_0^{1-u} 4uv\,(1-u-v)^2\,du\\ &= \int_0^1 \int_0^{1-u} \left[4u(1-u)^2 v - 8u(1-u)v^2 + 4uv^3\right] dv\,du\\ &= \int_0^1 \left[2u(1-u)^4 - \tfrac{8}{3}u(1-u)^4 + u(1-u)^4\right] du = \int_0^1 \tfrac{1}{3}u(1-u)^4\,du\\ &= \int_0^1 \tfrac{1}{3}\left[(1-u)^4 - (1-u)^5\right] du = \tfrac{1}{3}\left[-\tfrac{1}{5}(1-u)^5 + \tfrac{1}{6}(1-u)^6\right]_0^1 = \tfrac{1}{3}\left(-\tfrac{1}{6} + \tfrac{1}{5}\right) = \tfrac{1}{90}\end{aligned}$$

51. Let $u = y - x$ and $v = y + x$ so $x = y - u = (v - x) - u \;\Rightarrow\; x = \frac{1}{2}(v-u)$ and
$y = v - \frac{1}{2}(v-u) = \frac{1}{2}(v+u)$. $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}\right| = \left|-\frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$. R is the image under this transformation of the square with vertices $(u,v) = (0,0)$, $(-2,0)$, $(0,2)$, and $(-2,2)$. So

$$\begin{aligned}\iint_R xy\,dA &= \int_0^2 \int_{-2}^0 \frac{v^2-u^2}{4}\left(\frac{1}{2}\right) du\,dv = \tfrac{1}{8}\int_0^2 \left[v^2u - \tfrac{1}{3}u^3\right]_{u=-2}^{u=0} dv = \tfrac{1}{8}\int_0^2 \left(2v^2 - \tfrac{8}{3}\right) dv\\ &= \tfrac{1}{8}\left[\tfrac{2}{3}v^3 - \tfrac{8}{3}v\right]_0^2 = 0\end{aligned}$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

52. By the Extreme Value Theorem (15.7.8 [ET 14.7.8]), f has an absolute minimum value m and an absolute maximum value M in D. Then by Property 16.3.11 [ET 15.3.11], $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$. Dividing through by the positive number $A(D)$, we get $m \le \dfrac{1}{A(D)}\displaystyle\iint_D f(x,y)\,dA \le M$. This says that the average value of f over D lies between m and M. But f is continuous on D and takes on the values m and M, and so by the Intermediate Value Theorem must take on all values between m and M. Specifically, there exists a point (x_0, y_0) in D such that $f(x_0,y_0) = \dfrac{1}{A(D)}\displaystyle\iint_D f(x,y)\,dA$ or equivalently $\iint_D f(x,y)\,dA = f(x_0,y_0)\,A(D)$.

53. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for Double Integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \dfrac{1}{\pi r^2} \iint_{D_r} f(x,y)\,dA$. But $\lim\limits_{r\to 0+} (x_r, y_r) = (a,b)$, so $\lim\limits_{r\to 0+} \dfrac{1}{\pi r^2} \iint_{D_r} f(x,y)\,dA = \lim\limits_{r\to 0+} f(x_r, y_r) = f(a,b)$ by the continuity of f.

54. (a) $$\iint_D \frac{1}{(x^2+y^2)^{n/2}}\,dA = \int_0^{2\pi}\int_r^R \frac{1}{(t^2)^{n/2}}\,t\,dt\,d\theta = 2\pi \int_r^R t^{1-n}\,dt$$

$$= \begin{cases} \left.\dfrac{2\pi}{2-n} t^{2-n}\right]_r^R = \dfrac{2\pi}{2-n}\left(R^{2-n} - r^{2-n}\right) & \text{if } n \neq 2 \\[2ex] 2\pi \ln(R/r) & \text{if } n = 2 \end{cases}$$

(b) The integral in part (a) has a limit as $r \to 0^+$ for all values of n such that $2 - n > 0 \quad\Leftrightarrow\quad n < 2$.

(c) $$\iiint_E \frac{1}{(x^2+y^2+z^2)^{n/2}}\,dV = \int_r^R \int_0^\pi \int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}} \rho^2 \sin\phi\,d\theta\,d\phi\,d\rho$$

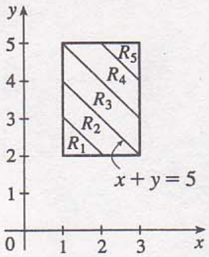
$$= 2\pi \int_r^R \int_0^\pi \rho^{2-n} \sin\phi\,d\phi\,d\rho$$

$$= \begin{cases} \left.\dfrac{4\pi}{3-n}\rho^{3-n}\right]_r^R = \dfrac{4\pi}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{if } n \neq 3 \\[2ex] 4\pi \ln(R/r) & \text{if } n = 3 \end{cases}$$

(d) As $r \to 0^+$, the above integral has a limit, provided that $3 - n > 0 \quad\Leftrightarrow\quad n < 3$.

Problems Plus

1.



Let $R = \bigcup_{i=1}^{5} R_i$, where

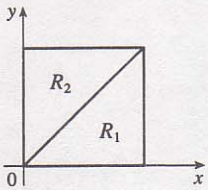
$R_i = \{(x, y) \mid x + y \geq i + 2, x + y < i + 3, 1 \leq x \leq 3, 2 \leq y \leq 5\}$.

$\iint_R [\![x + y]\!]\, dA = \sum_{i=1}^{5} \iint_{R_i} [\![x + y]\!]\, dA = \sum_{i=1}^{5} [\![x + y]\!] \iint_{R_i} dA$, since

$[\![x + y]\!] = \text{constant} = i + 2$ for $(x, y) \in R_i$. Therefore

$$\begin{aligned}\iint_R [\![x + y]\!]\, dA &= \textstyle\sum_{i=1}^{5} (i + 2)\, [A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\tfrac{1}{2}\right) + 4\left(\tfrac{3}{2}\right) + 5(2) + 6\left(\tfrac{3}{2}\right) + 7\left(\tfrac{1}{2}\right) = 30\end{aligned}$$

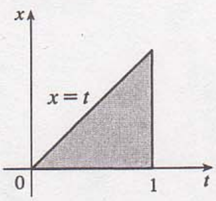
2.



Let $R = \{(x, y) \mid 0 \leq x, y \leq 1\}$. For $x, y \in R$, $\max\{x^2, y^2\} = x^2$ if $x \geq y$, and $\max\{x^2, y^2\} = y^2$ if $x \leq y$. Therefore we divide R into two regions: $R = R_1 \cup R_2$, where $R_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ and $R_2 = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$. Now $\max\{x^2, y^2\} = x^2$ for $(x, y) \in R_1$, and $\max\{x^2, y^2\} = y^2$ for $(x, y) \in R_2$ $\Rightarrow$

$$\begin{aligned}\int_0^1\!\int_0^1 e^{\max\{x^2, y^2\}}\, dy\, dx &= \iint_R e^{\max\{x^2, y^2\}}\, dA = \iint_{R_1} e^{\max\{x^2, y^2\}}\, dA + \iint_{R_2} e^{\max\{x^2, y^2\}}\, dA \\ &= \int_0^1\!\int_0^x e^{x^2}\, dy\, dx + \int_0^1\!\int_0^y e^{y^2}\, dx\, dy = \int_0^1 x e^{x^2}\, dx + \int_0^1 y e^{y^2}\, dy \\ &= e^{x^2}\Big]_0^1 = e - 1\end{aligned}$$

3. $$\begin{aligned}f_{\text{ave}} &= \frac{1}{b - a} \int_a^b f(x)\, dx = \frac{1}{1 - 0} \int_0^1 \left[\int_x^1 \cos(t^2)\, dt\right] dx \\ &= \int_0^1\!\int_x^1 \cos(t^2)\, dt\, dx \\ &= \int_0^1\!\int_0^t \cos(t^2)\, dx\, dt \quad \text{(changing the order of integration)} \\ &= \int_0^1 t \cos(t^2)\, dt = \tfrac{1}{2} \sin(t^2)\Big]_0^1 = \tfrac{1}{2} \sin 1\end{aligned}$$



4. Let $u = \mathbf{a} \cdot \mathbf{r}$, $v = \mathbf{b} \cdot \mathbf{r}$, $w = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$. Under this change of variables, E corresponds to the rectangular box $0 \leq u \leq \alpha$, $0 \leq v \leq \beta$, $0 \leq w \leq \gamma$. So, by Formula 16.9.3 [ET 15.9.3], $\int_0^\gamma\!\int_0^\beta\!\int_0^\alpha uvw\, du\, dv\, dw = \iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r}) \left|\dfrac{\partial(u, v, w)}{\partial(x, y, z)}\right| dV$. But

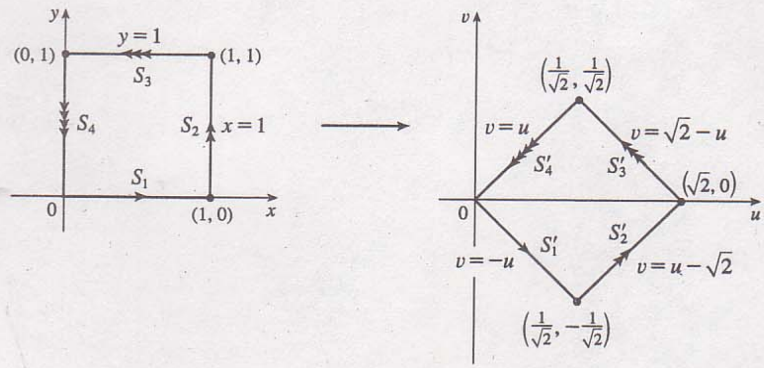
$$\left|\frac{\partial(u, v, w)}{\partial(x, y, z)}\right| = \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\| = |\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}| \quad \Rightarrow$$

$$\begin{aligned}\iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r})\, dV &= \frac{1}{|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|} \int_0^\gamma\!\int_0^\beta\!\int_0^\alpha uvw\, du\, dv\, dw \\ &= \frac{1}{|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|} \left(\frac{\alpha^2}{2}\right)\left(\frac{\beta^2}{2}\right)\left(\frac{\gamma^2}{2}\right) = \frac{(\alpha\beta\gamma)^2}{8\,|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|}\end{aligned}$$

5. Since $|xy| < 1$, except at $(1, 1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xy} = \displaystyle\sum_{n=0}^{\infty} (xy)^n$, so

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n \; dx\, dy = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^n \; dx\, dy = \sum_{n=0}^{\infty} \left[\int_0^1 x^n\, dx\right] \left[\int_0^1 y^n\, dy\right]$$
$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Let $x = \dfrac{u-v}{\sqrt{2}}$ and $y = \dfrac{u+v}{\sqrt{2}}$. We know the region of integration in the xy-plane, so to find its image in the uv-plane we get u and v in terms of x and y, and then use the methods of Section 16.9 [ET 15.9].
$x + y = \dfrac{u-v}{\sqrt{2}} + \dfrac{u+v}{\sqrt{2}} = \sqrt{2}u$, so $u = \dfrac{x+y}{\sqrt{2}}$, and similarly $v = \dfrac{y-x}{\sqrt{2}}$. S_1 is given by $y = 0$, $0 \le x \le 1$, so from the equations derived above, the image of S_1 is S_1': $u = \frac{1}{\sqrt{2}}x$, $v = -\frac{1}{\sqrt{2}}x$, $0 \le x \le 1$, that is, $v = -u$, $0 \le u \le \frac{1}{\sqrt{2}}$. Similarly, the image of S_2 is S_2': $v = u - \sqrt{2}$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, the image of S_3 is S_3': $v = \sqrt{2} - u$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, and the image of S_4 is S_4': $v - u$, $0 \le u \le \frac{1}{\sqrt{2}}$.



The Jacobian of the transformation is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$. From the diagram, we see that we must evaluate two integrals: one over the region $\left\{(u,v) \mid 0 \le u \le \frac{1}{\sqrt{2}}, -u \le v \le u\right\}$ and the other over $\left\{(u,v) \mid \frac{1}{\sqrt{2}} \le u \le \sqrt{2}, -\sqrt{2} + u \le v \le \sqrt{2} - u\right\}$. So

$$\int_0^1 \int_0^1 \frac{dx\, dy}{1-xy} = \int_0^{\sqrt{2}/2} \int_{-u}^{u} \frac{dv\, du}{1 - \left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]}$$
$$+ \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{dv\, du}{1 - \left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]}$$
$$= \int_0^{\sqrt{2}/2} \int_{-u}^{u} \frac{2\, dv\, du}{2 - u^2 + v^2} + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{2\, dv\, du}{2 - u^2 + v^2}$$

$$= 2\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}} \left[\arctan \frac{v}{\sqrt{2-u^2}}\right]_{-u}^{u} du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \left[\arctan \frac{v}{\sqrt{2-u^2}}\right]_{-\sqrt{2}+u}^{\sqrt{2}-u} du\right]$$

$$= 4\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}} \arctan \frac{u}{\sqrt{2-u^2}}\, du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}}\, du\right]$$

Now let $u = \sqrt{2}\sin\theta$, so $du = \sqrt{2}\cos\theta\, d\theta$ and the limits change to 0 and $\frac{\pi}{6}$ (in the first integral) and $\frac{\pi}{6}$ and $\frac{\pi}{2}$ (in the second integral). Continuing:

$$\int_0^1 \int_0^1 \frac{dx\, dy}{1-xy} = 4\left[\int_0^{\pi/6} \frac{1}{\sqrt{2-2\sin^2\theta}} \arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\, d\theta\right)\right.$$
$$\left. + \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2-2\sin^2\theta}} \arctan\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\, d\theta\right)\right]$$
$$= 4\left[\int_0^{\pi/6} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta} \arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) d\theta + \int_{\pi/6}^{\pi/2} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta} \arctan\left(\frac{\sqrt{2}\,(1-\sin\theta)}{\sqrt{2}\cos\theta}\right) d\theta\right]$$
$$= 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\, d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\frac{1-\sin\theta}{\cos\theta}\right) d\theta\right]$$

But (following the hint)

$$\frac{1-\sin\theta}{\cos\theta} = \frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{1-\left[1-2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right]}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} \quad \text{(half-angle formulas)}$$
$$= \frac{2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} = \tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)$$

Continuing:

$$\int_0^1 \int_0^1 \frac{dx\, dy}{1-xy} = 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\, d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right) d\theta\right]$$
$$= 4\left[\int_0^{\pi/6} \theta\, d\theta + \int_{\pi/6}^{\pi/2} \left[\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right] d\theta\right]$$
$$= 4\left(\left[\frac{\theta^2}{2}\right]_0^{\pi/6} + \left[\frac{\pi\theta}{4} - \frac{\theta^2}{4}\right]_{\pi/6}^{\pi/2}\right) = 4\left(\frac{3\pi^2}{72}\right) = \frac{\pi^2}{6}$$

7. (a) Since $|xyz| < 1$ except at $(1, 1, 1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum\limits_{n=0}^{\infty} (xyz)^n$, so

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyz}\, dx\, dy\, dz = \int_0^1 \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xyz)^n\, dx\, dy\, dz = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \int_0^1 (xyz)^n\, dx\, dy\, dz$$
$$= \sum_{n=0}^{\infty} \left[\int_0^1 x^n\, dx\right]\left[\int_0^1 y^n\, dy\right]\left[\int_0^1 z^n\, dz\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{n+1} \cdot \frac{1}{n+1}$$
$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

(b) Since $|-xyz| < 1$, except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1+xyz} = \displaystyle\sum_{n=0}^{\infty} (-xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty} (-xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty} \int_0^1\int_0^1\int_0^1 (-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

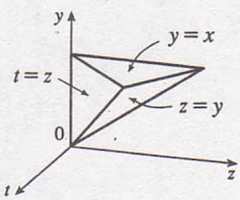
To evaluate this sum, we first write out a few terms: $s = 1 - \dfrac{1}{2^3} + \dfrac{1}{3^3} - \dfrac{1}{4^3} + \dfrac{1}{5^3} - \dfrac{1}{6^3} \approx 0.8998$. Notice that $a_7 = \dfrac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 12.5 [ET 11.5], we have $|s - s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

8. $$\int_0^{\infty} \frac{\arctan \pi x - \arctan x}{x}\,dx = \int_0^{\infty} \left[\frac{\arctan yx}{x}\right]_{y=1}^{y=\pi} dx = \int_0^{\infty}\int_1^{\pi} \frac{1}{1+y^2x^2}\,dy\,dx$$

$$= \int_1^{\pi}\int_0^{\infty} \frac{1}{1+y^2x^2}\,dx\,dy = \int_1^{\pi} \lim_{t\to\infty} \left[\frac{\arctan yx}{y}\right]_{x=0}^{x=t} dy$$

$$= \int_1^{\pi} \frac{\pi}{2y}\,dy = \tfrac{\pi}{2}\left[\ln y\right]_1^{\pi} = \tfrac{\pi}{2}\ln \pi$$

9. $\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where $E = \{(t,z,y) \mid 0 \le t \le z, 0 \le z \le y, 0 \le y \le x\}$. If we let D be the projection of E on the yt-plane then $D = \{(y,t) \mid 0 \le t \le x, t \le y \le x\}$. And we see from the diagram that $E = \{(t,z,y) \mid t \le z \le y, t \le y \le x, 0 \le t \le x\}$. So



$$\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \int_0^x \int_t^x \int_t^y f(t)\,dz\,dy\,dt = \int_0^x \left[\int_t^x (y-t)\,f(t)\,dy\right] dt$$

$$= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right) f(t)\right]_{y=t}^{y=x} dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right] f(t)\,dt$$

$$= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right] f(t)\,dt = \int_0^x \tfrac{1}{2}\left(x^2 - 2tx + t^2\right) f(t)\,dt$$

$$= \tfrac{1}{2}\int_0^x (x-t)^2 f(t)\,dt$$

10. (a) Consider a polar division of the disk, similar to that in Figure 16.4.4 [ET 15.4.4], where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$, $0 = r_1 < r_2 < \cdots < r_m = R$, and where the polar subrectangle R_{ij}, as well as r_i^*, θ_j^*, Δr and $\Delta\theta$ are the same as in that figure. Thus $\Delta A_i = r_i^*\,\Delta r\,\Delta\theta$. The mass of R_{ij} is $\rho\,\Delta A_i$, and its distance from m is $s_{ij} \approx \sqrt{(r_i^*)^2 + d^2}$. According to Newton's Law of Gravitation, the force of attraction experienced by m due to this polar subrectangle is in the direction from m towards R_{ij} and has

magnitude $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}$. The symmetry of the lamina with respect to the x- and y-axes and the position of m are such that all horizontal components of the gravitational force cancel, so that the total force is simply in the z-direction. Thus, we need only be concerned with the components of this vertical force; that is, $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}\sin\alpha$, where α is the angle between the origin, r_i^* and the mass m. Thus $\sin\alpha = \dfrac{d}{s_{ij}}$ and the previous result becomes $\dfrac{Gm\rho d\,\Delta A_i}{s_{ij}^3}$. The total attractive force is just the Riemann sum $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d\,\Delta A_i}{s_{ij}^3} = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d\,(r_i^*)\,\Delta r\,\Delta\theta}{\left[(r_i^*)^2+d^2\right]^{3/2}}$ which becomes $\displaystyle\int_0^R\int_0^{2\pi}\frac{Gm\rho d}{(r^2+d^2)^{3/2}}\,r\,d\theta\,dr$ as $m\to\infty$ and $n\to\infty$. Therefore,

$$F = 2\pi Gm\rho d\int_0^R\frac{r}{(r^2+d^2)^{3/2}}\,dr = 2\pi Gm\rho d\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^R = 2\pi Gm\rho d\left(\frac{1}{d}-\frac{1}{\sqrt{R^2+d^2}}\right)$$

(b) This is just the result of part (a) in the limit as $R\to\infty$. In this case $\dfrac{1}{\sqrt{R^2+d^2}}\to 0$, and we are left with

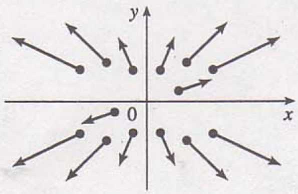
$$F = 2\pi Gm\rho d\left(\frac{1}{d}-0\right) = 2\pi Gm\rho.$$

17 Vector Calculus ET 16

17.1 Vector Fields ET 16.1

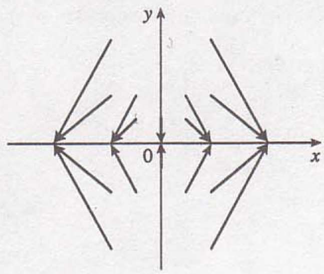
1. $\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$

The length of the vector $x\,\mathbf{i} + y\,\mathbf{j}$ is the distance from $(0, 0)$ to (x, y). Each vector points away from the origin.



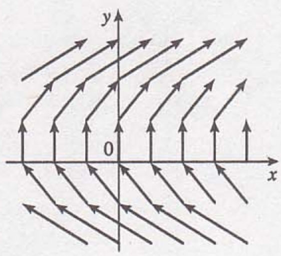
2. $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$

The length of the vector $x\,\mathbf{i} - y\,\mathbf{j}$ is the distance from $(0, 0)$ to (x, y). For each (x, y), $\mathbf{F}(x, y)$ terminates on the x-axis at the point $(2x, 0)$.



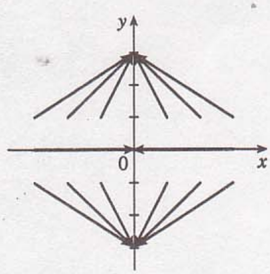
3. $\mathbf{F}(x, y) = y\,\mathbf{i} + \mathbf{j}$

The length of the vector $y\,\mathbf{i} + \mathbf{j}$ is $\sqrt{y^2 + 1}$. Vectors are tangent to parabolas opening about the x-axis.



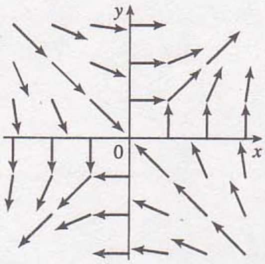
4. $\mathbf{F}(x, y) = -x\,\mathbf{i} + 2y\,\mathbf{j}$

The length of the vector $-x\,\mathbf{i} + 2y\,\mathbf{j}$ is $\sqrt{x^2 + 4y^2}$. $\mathbf{F}(x, y)$ terminates on the y-axis at the point $(0, 3y)$.



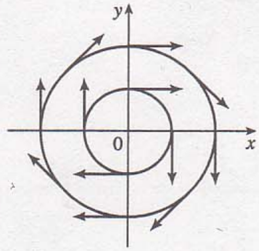
5. $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

The length of the vector $\dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2+y^2}}$ is 1.



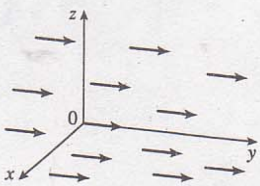
6. $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

All the vectors $\mathbf{F}(x,y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2+y^2}$.



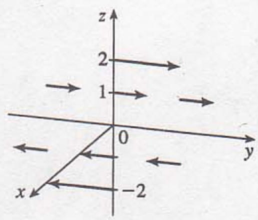
7. $\mathbf{F}(x,y,z) = \mathbf{j}$

All vectors in this field are parallel to the y-axis and have length 1.



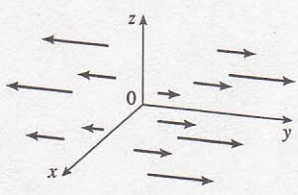
8. $\mathbf{F}(x,y,z) = z\,\mathbf{j}$

At each point (x,y,z), $\mathbf{F}(x,y,z)$ is a vector of length $|z|$. For $z > 0$, all point in the direction of the positive y-axis while for $z < 0$, all are in the direction of the negative y-axis.



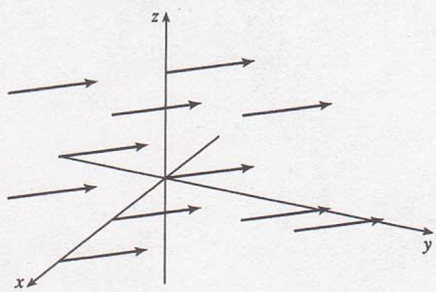
9. $\mathbf{F}(x,y,z) = y\,\mathbf{j}$

The length of $\mathbf{F}(x,y,z)$ is $|y|$. No vectors emanate from the xz-plane since $y = 0$ there. In each plane $y = b$, all the vectors are identical.



10. $\mathbf{F}(x,y,z) = \mathbf{j} - \mathbf{i}$

All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xy-plane.



11. $\mathbf{F}(x,y) = \langle y, x\rangle$ corresponds to graph III, since in the first quadrant all the vectors have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components.

12. $\mathbf{F}(x,y) = \langle 2x-3y, 2x+3y\rangle$ corresponds to graph IV, since as we move to the right (so x increases and y is constant), both the x- and the y-components of the vectors get larger, and as we move upward (so y increases and x is constant), the x-components decrease, while the y-components increase.

13. $\mathbf{F}(x,y) = \langle \sin x, \sin y\rangle$ corresponds to graph II, since the vector field is the same on each square of the form $[2n\pi, 2(n+1)\pi] \times [2m\pi, 2(m+1)\pi]$, m, n any integers.

14. $\mathbf{F}(x,y) = \langle \ln(1+x^2+y^2), x\rangle$ corresponds to graph I, since $\ln(1+x^2+y^2)$ is always positive, so all vectors point to the right.

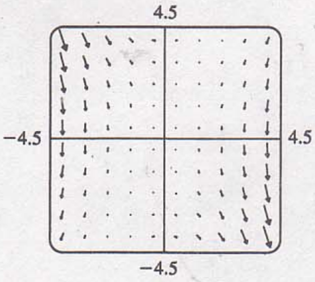
15. $\mathbf{F}(x,y,z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

16. $\mathbf{F}(x,y,z) = \mathbf{i} + 2\mathbf{j} + z\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy-plane point generally upward while the vectors below the xy-plane point generally downward.

17. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 3\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy-plane is $x\mathbf{i} + y\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z-components are all 3.

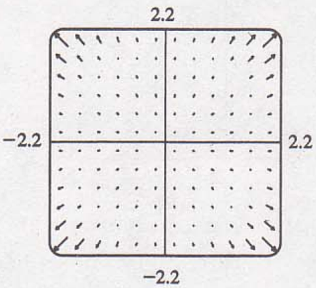
18. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x,y,z)$ has the same length and direction as the position vector of the point (x,y,z), and therefore the vectors all point directly away from the origin.

19.



The vector field seems to have very short vectors near the line $y = 2x$. For $\mathbf{F}(x,y) = \langle 0, 0\rangle$ we must have $y^2 - 2xy = 0$ and $3xy - 6x^2 = 0$. The first equation holds if $y = 0$ or $y = 2x$, and the second holds if $x = 0$ or $y = 2x$. So both equations hold [and thus $\mathbf{F}(x,y) = \mathbf{0}$] along the line $y = 2x$.

20.



From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|\mathbf{x}| = 2$ and near the origin. Note that $\mathbf{F}(\mathbf{x}) = 0 \Leftrightarrow r(r-2) = 0 \Leftrightarrow r = 0$ or 2, so as we suspected, $\mathbf{F}(\mathbf{x}) = 0$ for $|\mathbf{x}| = 2$ and for $|\mathbf{x}| = 0$. Note that where $r^2 - r < 0$, the vectors point towards the origin, and where $r^2 - r > 0$, they point away from the origin.

21. $\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} = \dfrac{1}{x+2y}\,\mathbf{i} + \dfrac{2}{x+2y}\,\mathbf{j}$

22. $\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} = \left[x^\alpha\left(-\beta e^{-\beta x}\right) + \alpha x^{\alpha-1}e^{-\beta x}\right]\mathbf{i} + 0\,\mathbf{j} = (\alpha - \beta x)\,x^{\alpha-1}e^{-\beta x}\,\mathbf{i}$

23.

$$\begin{aligned}\nabla f(x,y,z) &= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}\end{aligned}$$

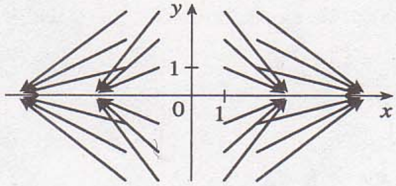
24. $\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}$

$$= \left(\cos\frac{y}{z}\right)\mathbf{i} - x\left(\sin\frac{y}{z}\right)\left(\frac{1}{z}\right)\mathbf{j} - x\left(\sin\frac{y}{z}\right)\left(-\frac{y}{z^2}\right)\mathbf{k}$$

$$= \left(\cos\frac{y}{z}\right)\mathbf{i} - \frac{x}{z}\left(\sin\frac{y}{z}\right)\mathbf{j} + \frac{xy}{z^2}\left(\sin\frac{y}{z}\right)\mathbf{k}$$

25. $f(x,y) = x^2 - \frac{1}{2}y^2$, $\nabla f(x,y) = 2x\,\mathbf{i} - y\,\mathbf{j}$

The length of $\nabla f(x,y)$ is $\sqrt{4x^2+y^2}$, and $\nabla f(x,y)$ terminates on the x-axis at the point $(3x, 0)$.

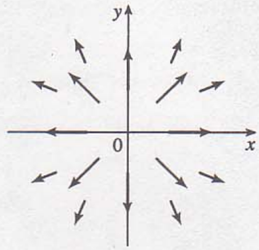


26. $f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln\left(x^2+y^2\right) \Rightarrow$

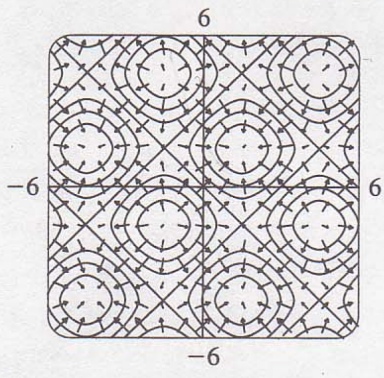
$$\nabla f = \tfrac{1}{2}\nabla \ln\left(x^2+y^2\right)$$

$$= \frac{x}{x^2+y^2}\,\mathbf{i} + \frac{y}{x^2+y^2}\,\mathbf{j} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$$

The length of ∇f decreases as x and/or y increase and all the vectors "flow out" away from the origin.

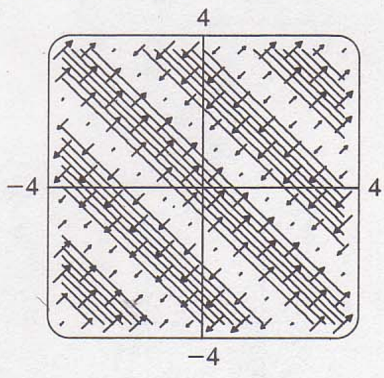


27. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

28. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

29. $f(x,y) = xy \Rightarrow \nabla f(x,y) = y\mathbf{i} + x\mathbf{j}$. In the first quadrant, both components of each vector are positive, while in the third quadrant both components are negative. However, in the second quadrant each vector's x-component is positive while its y-component is negative (and vice versa in the fourth quadrant). Thus, ∇f is graph IV.

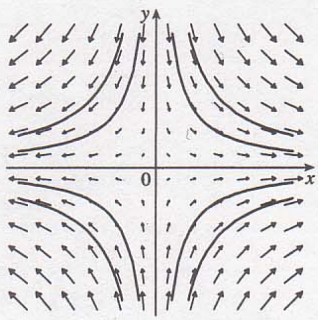
30. $f(x,y) = x^2 - y^2 \quad\Rightarrow\quad \nabla f(x,y) = 2x\mathbf{i} - 2y\mathbf{j}$. In the first quadrant, the x-component of each vector is positive while the y-component is negative. The other three quadrants are similar, where the x-component of each vector has the same sign as the x-value of its initial point, and the y-component has sign opposite that of the y-value of the initial point. Thus, ∇f is graph III.

31. $f(x,y) = x^2 + y^2 \quad\Rightarrow\quad \nabla f(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$. Thus, each vector $\nabla f(x,y)$ has the same direction and twice the length of the position vector of the point (x,y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.

32. $f(x,y) = \sqrt{x^2+y^2} \quad\Rightarrow\quad \nabla f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j}$. Then

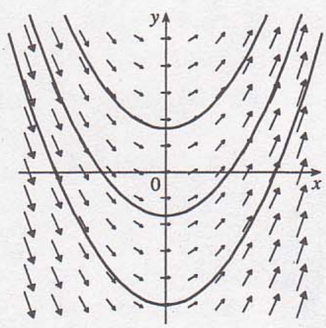
$|\nabla f(x,y)| = \dfrac{1}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2} = 1$, so all vectors are unit vectors. In addition, each vector $\nabla f(x,y)$ has the same direction as the position vector of the point (x,y), so the vectors all point directly away from the origin. Hence, ∇f is graph I.

33. (a) We sketch the vector field $\mathbf{F}(x,y) = x\mathbf{i} - y\mathbf{j}$ along with several approximate flow lines.The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x,y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = x\mathbf{i} - y\mathbf{j} \quad\Rightarrow\quad dx/dt = x,\ dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \quad\Rightarrow\quad dx/x = dt \quad\Rightarrow\quad \ln|x| = t + C \quad\Rightarrow\quad x = \pm e^{t+C} = Ae^t$ for some constant A, and $dy/dt = -y \quad\Rightarrow\quad dy/y = -dt \quad\Rightarrow\quad \ln|y| = -t + K \quad\Rightarrow\quad y = \pm e^{-t+K} = Be^{-t}$ for some constant B. Therefore $xy = Ae^tBe^{-t} = AB =$ constant. If the flow line passes through $(1,1)$ then $(1)(1) = \text{constant} = 1 \quad\Rightarrow\quad xy = 1 \quad\Rightarrow\quad y = 1/x,\ x > 0$.

34. (a) We sketch the vector field $\mathbf{F}(x,y) = \mathbf{i} + x\mathbf{j}$ along with several approximate flow lines.The flow lines appear to be parabolas.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = \mathbf{i} + x\mathbf{j} \;\Rightarrow\; \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = x$. Thus $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{x}{1} = x$.

(c) From part (b), $dy/dx = x$. Integrating, we have $y = \frac{1}{2}x^2 + c$. Since the particle starts at the origin, we know $(0, 0)$ is on the curve, so $0 = 0 + c \;\Rightarrow\; c = 0$ and the path the particle follows is $y = \frac{1}{2}x^2$.

17.2 Line Integrals — ET 16.2

1. $x = t^2$ and $y = t$, $0 \le t \le 2$, so by Formula 3

$$\int_C y\,ds = \int_0^2 t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 t\sqrt{(2t)^2 + (1)^2}\,dt$$
$$= \int_0^2 t\sqrt{4t^2+1}\,dt = \tfrac{1}{12}\left(4t^2+1\right)^{3/2}\Big]_0^2 = \tfrac{1}{12}\left(17\sqrt{17} - 1\right)$$

2. $$\int_C \frac{y}{x}\,ds = \int_0^1 \frac{t^3}{t^4}\sqrt{(4t^3)^2 + (3t^2)^2}\,dt = \int_0^1 \frac{1}{t}\sqrt{16t^6 + 9t^4}\,dt = \int_0^1 t\sqrt{16t^2+9}\,dt$$
$$= \tfrac{1}{48}\left(16t^2+9\right)^{3/2}\Big]_0^1 = \tfrac{1}{48}\left(25^{3/2} - 9^{3/2}\right) = \tfrac{49}{24}$$

3. Parametric equatons for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4 \sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt$$
$$= \int_{-\pi/2}^{\pi/2} 4^5 \cos t \sin^4 t\sqrt{16\left(\sin^2 t + \cos^2 t\right)}\,dt$$
$$= 4^5 \int_{-\pi/2}^{\pi/2} \left(\sin^4 t\cos t\right)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \tfrac{2\cdot 4^6}{5} = 1638.4$$

4. Parametric equations for C are $x = 1 + 3t$, $y = 2 + 5t$, $0 \le t \le 1$. Then

$$\int_C ye^x\,ds = \int_0^1 (2+5t)\,e^{1+3t}\sqrt{3^2+5^2}\,dt = \sqrt{34}\int_0^1 (2+5t)\,e^{1+3t}\,dt$$

Integrating by parts with $u = 2 + 5t \;\Rightarrow\; du = 5\,dt$, $dv = e^{1+3t} \;\Rightarrow\; v = \frac{1}{3}e^{1+3t}\,dt$ gives

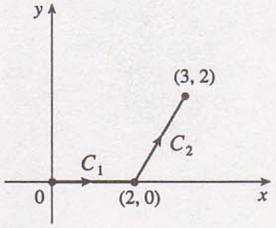
$$\int_C ye^x\,ds = \sqrt{34}\left[\tfrac{1}{3}(2+5t)\,e^{1+3t} - \tfrac{5}{9}e^{1+3t}\right]_0^1$$
$$= \sqrt{34}\left[\left(\tfrac{7}{3} - \tfrac{5}{9}\right)e^4 - \left(\tfrac{2}{3} - \tfrac{5}{9}\right)e\right] = \tfrac{\sqrt{34}}{9}\left(16e^4 - e\right)$$

5. If we choose x as the parameter, parametric equations for C are $x = x$, $y = x^2$ for $1 \le x \le 3$ and

$$\int_C (xy + \ln x)\,dy = \int_1^3 \left(x\cdot x^2 + \ln x\right)2x\,dx = \int_1^3 2\left(x^4 + x\ln x\right)dx$$
$$= 2\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_1^3 \quad \text{(by integrating by parts in the second term)}$$
$$= 2\left(\tfrac{243}{5} + \tfrac{9}{2}\ln 3 - \tfrac{9}{4} - \tfrac{1}{5} + \tfrac{1}{4}\right) = \tfrac{464}{5} + 9\ln 3$$

6. Choosing y as the parameter, we have $x = y^4$, $y = y$, $-1 \le y \le 1$. Then
$\int_C \sin x\,dx = \int_{-1}^{1} \left(\sin y^4\right)\left(4y^3\right) dy = -\cos y^4\Big]_{-1}^{1} = 0.$

7.



$C = C_1 + C_2$

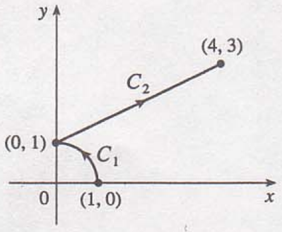
On C_1: $x = x, y = 0 \quad\Rightarrow\quad dy = 0\,dx, 0 \le x \le 2$.

On C_2: $x = x, y = 2x - 4 \quad\Rightarrow\quad dy = 2\,dx, 2 \le x \le 3$.

Then

$$\begin{aligned}\int_C xy\,dx + (x-y)\,dy &= \int_{C_1} xy\,dx + (x-y)\,dy + \int_{C_2} xy\,dx + (x-y)\,dy \\ &= \int_0^2 (0+0)\,dx + \int_2^3 \left[\left(2x^2 - 4x\right) + (-x+4)(2)\right] dx \\ &= \int_2^3 \left(2x^2 - 6x + 8\right) dx = \tfrac{17}{3}\end{aligned}$$

8.



On C_1: $x = \cos t \quad\Rightarrow\quad dx = -\sin t\,dt, y = \sin t \quad\Rightarrow$
$y = \cos t\,dt, 0 \le t \le \frac{\pi}{2}$.

On C_2: $x = 4t \quad\Rightarrow\quad dx = 4\,dt, y = 2t + 1 \quad\Rightarrow$
$dy = 2\,dt, 0 \le t \le 1$.

Then $\quad \int_C x\sqrt{y}\,dx + 2y\sqrt{x}\,dy$

$$\begin{aligned}&= \int_{C_1} x\sqrt{y}\,dx + 2y\sqrt{x}\,dy + \int_{C_2} x\sqrt{y}\,dx + 2y\sqrt{x}\,dy \\ &= \int_0^{\pi/2} \left[-\cos t\,(\sin t)^{3/2} + 2\sin t\,(\cos t)^{3/2}\right] dt + \int_0^1 \left[16t\sqrt{2t+1} + 8\,(2t+1)\sqrt{t}\right] dt \\ &= \left[-\tfrac{2}{5}(\sin t)^{5/2} - \tfrac{4}{5}(\cos t)^{5/2}\right]_0^{\pi/2} + \left[\tfrac{16}{3}t\,(2t+1)^{3/2} - \tfrac{16}{15}(2t+1)^{5/2} + 8\left(\tfrac{4}{5}t^{5/2} + \tfrac{2}{3}t^{3/2}\right)\right]_0^1 \\ &= \tfrac{2}{5} + \tfrac{16}{3}\cdot 3\sqrt{3} - \tfrac{16}{15}\cdot 3^2\cdot\sqrt{3} + \tfrac{16}{15} + 8\left(\tfrac{4}{5} + \tfrac{2}{3}\right) = \tfrac{32\sqrt{3}+66}{5}\end{aligned}$$

9. $x = 4\sin t$, $y = 4\cos t$, $z = 3t$, $0 \le t \le \frac{\pi}{2}$. Then by Formula 9,

$$\begin{aligned}\int_C xy^3\,ds &= \int_0^{\pi/2} (4\sin t)(4\cos t)^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_0^{\pi/2} 4^4\cos^3 t\sin t\sqrt{(4\cos t)^2 + (-4\sin t)^2 + (3)^2}\,dt \\ &= \int_0^{\pi/2} 256\cos^3 t\sin t\sqrt{16\left(\cos^2 t + \sin^2 t\right) + 9}\,dt \\ &= 1280\int_0^{\pi/2} \cos^3 t\sin t\,dt = -320\cos^4 t\Big]_0^{\pi/2} = 320\end{aligned}$$

10. Parametric equations for C are $x = 4t$, $y = 6 - 5t$, $z = -1 + 6t$, $0 \le t \le 1$. Then

$$\begin{aligned}\int_C x^2 z\,ds &= \int_0^1 (4t)^2(6t-1)\sqrt{4^2 + (-5)^2 + 6^2}\,dt = \sqrt{77}\int_0^1 \left(96t^3 - 16t^2\right) dt \\ &= \sqrt{77}\left[96\cdot\frac{t^4}{4} - 16\cdot\frac{t^3}{3}\right]_0^1 = \tfrac{56}{3}\sqrt{77}\end{aligned}$$

11. Parametric equations for C are $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt$$
$$= \sqrt{14}\left[\tfrac{1}{12}e^{6t^2}\right]_0^1 = \tfrac{\sqrt{14}}{12}\left(e^6-1\right)$$

12. $\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2} = \sqrt{36+36\,(2)\,t^2+36t^4} = 6\sqrt{(t^2+1)^2} = 6\left(t^2+1\right)$. Then $\int_C xz\,ds = \int_0^1 72t^4\left(t^2+1\right)dt = 72\left[\tfrac17 t^7+\tfrac15 t^5\right]_0^1 = 72\left(\tfrac{12}{35}\right) = \tfrac{864}{35}$.

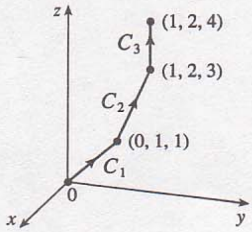
13. $\int_C x^3y^2z\,dz = \int_0^1 (2t)^3\left(t^2\right)^2\left(t^2\right)(2t)\,dt = \int_0^1 \left(8t^3\right)\left(t^4\right)\left(t^2\right)(2t)\,dt = \int_0^1 16t^{10}\,dt = \tfrac{16}{11}t^{11}\big]_0^1 = \tfrac{16}{11}$

14. $\int_C yz\,dy + xy\,dz = \int_0^1 (t)\left(t^2\right)dt + \int_0^1 \sqrt{t}\,(t)\,2t\,dt = \int_0^1\left(t^3+2t^{5/2}\right)dt = \left[\tfrac14 t^4+\tfrac47 t^{7/2}\right]_0^1 = \tfrac{23}{28}$

15. On C_1: $x = 0 \;\Rightarrow\; dx = 0\,dt$, $y = t \;\Rightarrow\; dy = dt$, $z = t \;\Rightarrow$ $dz = dt$, $0 \le t \le 1$.

On C_2: $x = t \;\Rightarrow\; dx = dt$, $y = t+1 \;\Rightarrow\; dy = dt$, $z = 2t+1 \;\Rightarrow$ $dz = 2\,dt$, $0 \le t \le 1$.

On C_3: $x = 1 \;\Rightarrow\; dx = 0\,dt$, $y = 2 \;\Rightarrow\; dy = 0\,dt$, $z = t+3 \;\Rightarrow$ $dz = dt$, $0 \le t \le 1$.



Then

$\int_C z^2\,dx - z\,dy + 2y\,dz$
$$= \int_0^1 (0-t+2t)\,dt + \int_0^1\left[(2t+1)^2-(2t+1)+2\,(t+1)\,(2)\right]dt + \int_0^1 (0+0+4)\,dt$$
$$= \tfrac12 + \left[\tfrac43 t^3+3t^2+4t\right]_0^1 + 4 = \tfrac{77}{6}$$

16. C_1: $(0,0,0)$ to $(2,0,0)$: $x = 2t$, $y = z = 0$, $0 \le t \le 1$.
C_2: $(2,0,0)$ to $(1,3,-1)$: $x = -t+2$, $y = 3t$, $z = -t$, $0 \le t \le 1$.
C_3: $(1,3,-1)$ to $(1,3,0)$: $x = 1$, $y = 3$, $z = t-1$, $0 \le t \le 1$.
Then

$$\int_C yz\,dx + xz\,dy + xy\,dz = 0 + \int_0^1\left[\left(3t^2\right)+3\left(t^2-2t\right)-3\left(2t-t^2\right)\right]dt + \int_0^1 3\,dt$$
$$= \left[3t^3-6t^2\right]_0^1 + 3 = 0.$$

17. (a) Along the line $x = -3$, the vectors of $\mathbf{F}$ have positive y-components, so since the path goes upward, the integrand $\mathbf{F}\cdot\mathbf{T}$ is always positive. Therefore $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot\mathbf{T}\,ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F}\cdot\mathbf{T}$ is negative, and therefore $\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot\mathbf{T}\,ds$ is negative.

18. Vectors starting on C_1 point in roughly the same direction as C_1, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. Then $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot\mathbf{T}\,ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2, while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot\mathbf{T}\,ds$ to be negative.

19. $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) = \left(t^2\right)^2\left(-t^3\right)^3\mathbf{i} - \left(-t^3\right)\sqrt{t^2}\,\mathbf{j} = -t^{13}\,\mathbf{i} + t^4\,\mathbf{j}$ and $\mathbf{r}'(t) = 2t\,\mathbf{i} - 3t^2\,\mathbf{j}$. Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1\left(-2t^{14}-3t^6\right)dt = \left[-\tfrac{2}{15}t^{15}-\tfrac37 t^7\right]_0^1 = -\tfrac{59}{105}$$

20. $\mathbf{F}(\mathbf{r}(t)) = \left(t^2\right)\left(t^3\right)\mathbf{i} + (t)\left(t^3\right)\mathbf{j} + (t)\left(t^2\right)\mathbf{k} = t^5\,\mathbf{i} + t^4\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^2\left(t^5+2t^5+3t^5\right)dt = t^6\big]_0^2 = 64$$

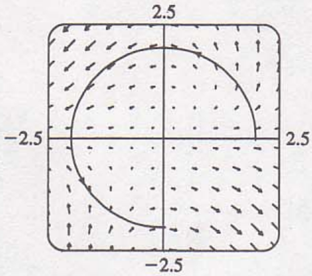
21. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle \sin t^3, \cos(-t^2), t^4\rangle \cdot \langle 3t^2, -2t, 1\rangle\, dt$

$= \int_0^1 \left(3t^2 \sin t^3 - 2t\cos t^2 + t^4\right) dt = \left[-\cos t^3 - \sin t^2 + \frac{1}{5}t^5\right]_0^1 = \frac{6}{5} - \cos 1 - \sin 1$

22. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \langle \sin^2 t, \sin t\cos t, t^4\rangle \cdot \langle \cos t, -\sin t, 2t\rangle\, dt$

$= \int_0^{\pi/2} \left(\sin^2 t\cos t - \sin^2 t\cos t + 2t^5\right) dt = \left[\frac{1}{3}t^6\right]_0^{\pi/2} = \dfrac{\pi^6}{192}$

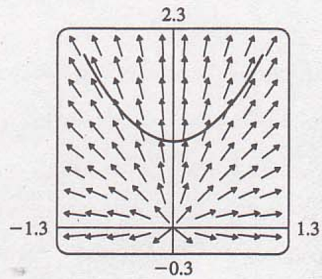
23. We graph $\mathbf{F}(x,y) = (x-y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ to be positive.



To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le \frac{3\pi}{2}$, so $\mathbf{F}(\mathbf{r}(t)) = (2\cos t - 2\sin t)\,\mathbf{i} + 4\cos t\sin t\,\mathbf{j}$ and $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt\\ &= \int_0^{3\pi/2} \left[-2\sin t\,(2\cos t - 2\sin t) + 2\cos t\,(4\cos t\sin t)\right] dt\\ &= 4\int_0^{3\pi/2} \left(\sin^2 t - \sin t\cos t + 2\sin t\cos^2 t\right) dt\\ &= 3\pi + \tfrac{2}{3} \quad \text{(using a CAS)}\end{aligned}$$

24. We graph $\mathbf{F}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$ and the curve C. In the first quadrant, each vector starting on C points in roughly the same direction as C, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. In the second quadrant, each vector starting on C points in roughly the direction opposite to C, so $\mathbf{F}\cdot\mathbf{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants counteract each other, so it seems reasonable to guess that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is zero. To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = t\,\mathbf{i} + \left(1+t^2\right)\mathbf{j}$, $-1 \le t \le 1$, so

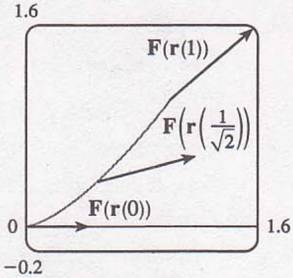


$\mathbf{F}(\mathbf{r}(t)) = \dfrac{t}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{i} + \dfrac{1+t^2}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{j}$ and $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{-1}^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_{-1}^1 \left(\frac{t}{\sqrt{t^2+(1+t^2)^2}} + \frac{2t\left(1+t^2\right)}{\sqrt{t^2+(1+t^2)^2}}\right) dt\\ &= \int_{-1}^1 \frac{t\left(3+2t^2\right)}{\sqrt{t^4+3t^2+1}}\,dt = 0 \quad \text{(since the integrand is an odd function)}\end{aligned}$$

25. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle e^{t^2-1}, t^5\right\rangle \cdot \langle 2t, 3t^2\rangle\,dt = \int_0^1 \left(2te^{t^2-1} + 3t^7\right)dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - 1/e$

(b) $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{F}(\mathbf{r}(0)) = \langle e^{-1}, 0\rangle$; $\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}}\right\rangle$; $\mathbf{r}(1) = \langle 1, 1\rangle$, $\mathbf{F}(\mathbf{r}(1)) = \langle 1, 1\rangle$.



In order to generate the graph with Maple, we use the `PLOT` command (not to be confused with the `plot` command) to define each of the vectors. For example,

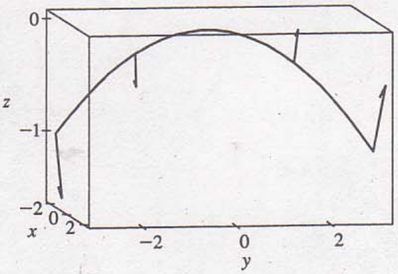
```
v1:=PLOT(CURVES([[0,0],[evalf(1/exp(1)),0]]));
```

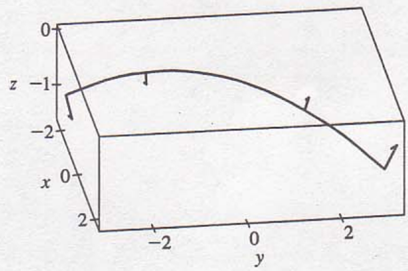
generates the vector from the vector field at the point $(0, 0)$ (but without an arrowhead) and gives it the name `v1`. To show everything on the same screen, we use the `display` command.

In Mathematica, we use `ListPlot` (with the `PlotJoined -> True` option) to generate the vectors, and then `Show` to show everything on the same screen.

26. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^1 \langle 2t, t^2, 3t\rangle \cdot \langle 2, 3, -2t\rangle\,dt = \int_{-1}^1 \left(4t + 3t^2 - 6t^2\right)dt = \left[2t^2 - t^3\right]_{-1}^1 = -2.$

(b) Now $\mathbf{F}(\mathbf{r}(t)) = \langle 2t, t^2, 3t\rangle$, so $\mathbf{F}(\mathbf{r}(-1)) = \langle -2, 1, -3\rangle$, $\mathbf{F}\left(\mathbf{r}\left(-\frac{1}{2}\right)\right) = \left\langle -1, \frac{1}{4}, -\frac{3}{2}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{2}\right)\right) = \left\langle 1, \frac{1}{4}, \frac{3}{2}\right\rangle$, and $\mathbf{F}(\mathbf{r}(1)) = \langle 2, 1, 3\rangle$.





27. The part of the astroid that lies in the quadrant is parametrized by $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \frac{\pi}{2}$. Now $\dfrac{dx}{dt} = 3\cos^2 t\,(-\sin t)$ and $\dfrac{dy}{dt} = 3\sin^2 t\cos t$, so

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} = 3\cos t\sin t\sqrt{\cos^2 t + \sin^2 t} = 3\cos t\sin t.$$

Therefore $\int_C x^3y^5\,ds = \int_0^{\pi/2} \cos^9 t\sin^{15} t\,(3\cos t\sin t)\,dt = \frac{945}{16{,}777{,}216}\pi$.

28. We parametrize the line as $\mathrm{r}(t) = \langle 1, 2, 1\rangle + t\left(\langle 6, 4, 5\rangle - \langle 1, 2, 1\rangle\right) = (1 + 5t)\,\mathbf{i} + (2 + 2t)\,\mathbf{j} + (1 + 4t)\,\mathbf{k}$, $0 \le t \le 1$. Using a CAS, we calculate

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle (1+5t)^4 e^{2+2t}, \ln(1+4t), \sqrt{(2+2t)^2 + (1+4t)^2}\right\rangle \cdot \langle 5, 2, 4\rangle\,dt$$

$$= \frac{5235e^4}{4} - \frac{6285e^2}{4} + \frac{9\sqrt{5}\sinh^{-1}\left(\frac{14}{3}\right)}{25} - \frac{9\sqrt{5}\sinh^{-1}\left(\frac{4}{3}\right)}{25} + \frac{5\ln 5}{2} + \frac{14\sqrt{41}}{5} - \frac{4\sqrt{5}}{5} - 2$$

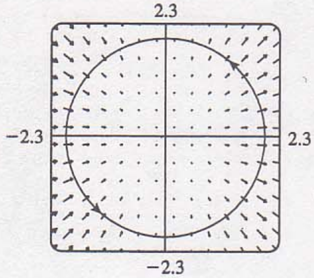
$$= \frac{5235e^4}{4} - \frac{6285e^2}{4} - \frac{18\sqrt{5}\ln 3}{25} + \frac{9\sqrt{5}\ln\left(14 + \sqrt{205}\right)}{25} + \frac{5\ln 5}{2} + \frac{14\sqrt{41} - 4\sqrt{5}}{5} - 2$$

The first answer is the one given by Maple. The two answers are equivalent by Equation 7.6.3 [ET 3.9.3].

29. A calculator or CAS gives $\int_C x \sin y \, ds = \int_1^2 \ln t \sin\left(e^{-t}\right) \sqrt{(1/t)^2 + (-e^{-t})^2} \, dt \approx 0.052$.

30. (a) We parametrize the circle C as $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. So $\mathbf{F}(\mathbf{r}(t)) = \left\langle 4\cos^2 t, 4\cos t \sin t \right\rangle$, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t \rangle$, and $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-8\cos^2 t \sin t + 8\cos^2 t \sin t\right) dt = 0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C, $\mathbf{F} \cdot \mathbf{T} = 0$, and so certainly $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

31. We use the parametrization $x = 2\cos t$, $y = 2\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

Then $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} \, dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2} \, dt = 2\,dt$, so

$m = \int_C k \, ds = 2k \int_{-\pi/2}^{\pi/2} dt = 2k(\pi)$, $\overline{x} = \frac{1}{2\pi k} \int_C xk \, ds = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (2\cos t) \, 2 \, dt = \frac{1}{2\pi} \left[4\sin t\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi}$,

$\overline{y} = \frac{1}{2\pi k} \int_C yk \, ds = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (2\sin t) \, 2 \, dt = 0$. Hence $(\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, 0\right)$.

32. We use the parametrization $x = r\cos t$, $y = r\sin t$, $0 \le t \le \frac{\pi}{2}$.

Then $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} \, dt = \sqrt{(-r\sin t)^2 + (r\cos t)^2} \, dt = r\,dt$, so

$m = \int_C (x + y) \, ds = \int_0^{\pi/2} (r\cos t + r\sin t) \, r \, dt = r^2 \left[\sin t - \cos t\right]_0^{\pi/2} = 2r^2$,

$$\overline{x} = \frac{1}{2r^2} \int_C x(x+y) \, ds = \frac{1}{2r^2} \int_0^{\pi/2} \left(r^2\cos^2 t + r^2 \cos t \sin t\right) r \, dt = \frac{r}{2} \left[\frac{t}{2} + \frac{\sin 2t}{4} - \frac{\cos 2t}{4}\right]_0^{\pi/2}$$

$$= \frac{r(\pi + 2)}{8}, \text{ and}$$

$$\overline{y} = \frac{1}{2r^2} \int_C y(x+y) \, ds = \frac{1}{2r^2} \int_0^{\pi/2} \left(r^2 \sin t \cos t + r^2 \sin^2 t\right) r \, dt$$

$$= \frac{r}{2} \left[-\frac{\cos 2t}{4} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi/2} = \frac{r(\pi + 2)}{8}.$$

Therefore $(\overline{x}, \overline{y}) = \left(\dfrac{r(\pi + 2)}{8}, \dfrac{r(\pi + 2)}{8}\right)$.

33. (a) $\overline{x} = \dfrac{1}{m} \displaystyle\int_C x\rho(x,y,z) \, ds$, $\overline{y} = \dfrac{1}{m} \displaystyle\int_C y\rho(x,y,z) \, ds$, $\overline{z} = \dfrac{1}{m} \displaystyle\int_C z\rho(x,y,z) \, ds$ where $m = \int_C \rho(x,y,z) \, ds$.

(b) $m = \int_C k \, ds = k \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t + 9} \, dt = k\sqrt{13} \int_0^{2\pi} dt = 2\pi k\sqrt{13}$,

$$\overline{x} = \frac{1}{2\pi k\sqrt{13}} \int_0^{2\pi} k2\sqrt{13} \sin t \, dt = 0, \quad \overline{y} = \frac{1}{2\pi k\sqrt{13}} \int_0^{2\pi} k2\sqrt{13} \cos t \, dt = 0,$$

$$\overline{z} = \frac{1}{2\pi k\sqrt{13}} \int_0^{2\pi} \left(k\sqrt{13}\right)(3t) \, dt = \frac{3}{2\pi}\left(2\pi^2\right) = 3\pi. \text{ Hence } (\overline{x}, \overline{y}, \overline{z}) = (0, 0, 3\pi).$$

34. $m = \int_C \left(x^2 + y^2 + z^2\right) ds = \int_0^{2\pi} \left(t^2 + 1\right) \sqrt{(1)^2 + (-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} \left(t^2 + 1\right) \sqrt{2}\, dt$

$= \sqrt{2}\left(\frac{8}{3}\pi^3 + 2\pi\right),$

$$\overline{x} = \frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3 + 2\pi\right)} \int_0^{2\pi} \sqrt{2}\left(t^3 + t\right) dt = \frac{4\pi^4 + 2\pi^2}{\frac{8}{3}\pi^3 + 2\pi} = \frac{3\pi\left(2\pi^2 + 1\right)}{4\pi^2 + 3},$$

$$\overline{y} = \frac{3}{2\sqrt{2}\pi\left(4\pi^2 + 3\right)} \int_0^{2\pi} \left(\sqrt{2}\cos t\right)\left(t^2 + 1\right) dt = 0, \text{ and}$$

$$\overline{z} = \frac{3}{2\sqrt{2}\pi\left(4\pi^2 + 3\right)} \int_0^{2\pi} \left(\sqrt{2}\sin t\right)\left(t^2 + 1\right) dt = 0. \text{ Hence } (\overline{x}, \overline{y}, \overline{z}) = \left(\frac{3\pi\left(2\pi^2 + 1\right)}{4\pi^2 + 3}, 0, 0\right).$$

35. From Example 3, $\rho(x, y) = k(1 - y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \le t \le \pi \Rightarrow$

$$I_x = \int_C y^2 \rho(x, y)\, ds = \int_0^{\pi} \sin^2 t\,[k(1 - \sin t)]\, dt = k\int_0^{\pi} \left(\sin^2 t - \sin^3 t\right) dt$$

$$= \tfrac{1}{2}k\int_0^{\pi} (1 - \cos 2t)\, dt - k\int_0^{\pi} \left(1 - \cos^2 t\right) \sin t\, dt$$

(Let $u = \cos t$, $du = -\sin t\, dt$ in the second integral)

$$= k\left[\tfrac{\pi}{2} + \int_1^{-1} \left(1 - u^2\right) du\right] = k\left(\tfrac{\pi}{2} - \tfrac{4}{3}\right)$$

$$I_y = \int_C x^2 \rho(x, y)\, ds = k\int_0^{\pi} \cos^2 t\,(1 - \sin t)\, dt = \tfrac{k}{2}\int_0^{\pi} (1 + \cos 2t)\, dt - k\int_0^{\pi} \cos^2 t \sin t\, dt$$

$= k\left(\frac{\pi}{2} - \frac{2}{3}\right)$, using the same substitution as above.

36. The wire is given as $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \le t \le 2\pi$ with $\rho(x, y, z) = k$. Then

$ds = \sqrt{(2\cos t)^2 + (-2\sin t)^2 + 3^2} = \sqrt{4\left(\cos^2 t + \sin^2 t\right) + 9} = \sqrt{13}$ and

$$I_x = \int_C \left(y^2 + z^2\right) \rho(x, y, z)\, ds = \int_0^{2\pi} \left(4\cos^2 t + 9t^2\right)(k)\sqrt{13}\, dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) + 3t^3\right]_0^{2\pi}$$

$$= \sqrt{13}\,k\left(4\pi + 24\pi^3\right) = 4\sqrt{13}\,\pi k\left(1 + 6\pi^2\right)$$

$$I_y = \int_C \left(x^2 + z^2\right) \rho(x, y, z)\, ds = \int_0^{2\pi} \left(4\sin^2 t + 9t^2\right)(k)\sqrt{13}\, dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 3t^3\right]_0^{2\pi}$$

$$= \sqrt{13}\,k\left(4\pi + 24\pi^3\right) = 4\sqrt{13}\,\pi k\left(1 + 6\pi^2\right)$$

$$I_z = \int_C \left(x^2 + y^2\right) \rho(x, y, z)\, ds = \int_0^{2\pi} \left(4\sin^2 t + 4\cos^2 t\right)(k)\sqrt{13}\, dt = 4\sqrt{13}\,k\int_0^{2\pi} dt = 8\pi\sqrt{13}\,k$$

37. $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle t - \sin t, 3 - \cos t\rangle \cdot \langle 1 - \cos t, \sin t\rangle\, dt$

$$= \int_0^{2\pi} (t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\, dt$$

$$= \int_0^{2\pi} (t - t\cos t + 2\sin t)\, dt = \left[\tfrac{1}{2}t^2 - (t\sin t + \cos t) - 2\cos t\right]_0^{2\pi}$$

(by integrating by parts in the second term)

$= 2\pi^2$

38. $x = x$, $y = x^2$, $-1 \le x \le 2$,

$$W = \int_{-1}^{2} \left\langle x\sin x^2, x^2\right\rangle \cdot \langle 1, 2x\rangle\, dx = \int_{-1}^{2} \left(x\sin x^2 + 2x^3\right) dx = \left[-\tfrac{1}{2}\cos x^2 + \tfrac{1}{2}x^4\right]_{-1}^{2}$$

$$= \tfrac{1}{2}(15 + \cos 1 - \cos 4)$$

39. $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t^6, -t^5, -t^7 \rangle \cdot \langle 2t, -3t^2, 4t^3 \rangle \, dt = \int_0^1 (5t^7 - 4t^{10}) \, dt = \frac{5}{8} - \frac{4}{11} = \frac{23}{88}$

40. $\mathbf{r}(t) = 2\,\mathbf{i} + t\,\mathbf{j} + 5t\,\mathbf{k}$, $0 \le t \le 1$. Therefore

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{K \langle 2, t, 5t \rangle}{(4 + 26t^2)^{3/2}} \cdot \langle 0, 1, 5 \rangle \, dt = K \int_0^1 \frac{26t}{(4 + 26t^2)^{3/2}} \, dt$$

$$= K \left[-\left(4 + 26t^2\right)^{-1/2} \right]_0^1 = K \left(\tfrac{1}{2} - \tfrac{1}{\sqrt{30}} \right)$$

41. Let $\mathbf{F} = 185\,\mathbf{k}$. To parametrize the staircase, let
$x = 20 \cos t$, $y = 20 \sin t$, $z = \frac{90}{6\pi} t = \frac{15}{\pi} t$, $0 \le t \le 6\pi \Rightarrow$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \langle 0, 0, 185 \rangle \cdot \left\langle -20 \sin t, 20 \cos t, \tfrac{15}{\pi} \right\rangle dt = (185) \tfrac{15}{\pi} \int_0^{6\pi} dt = (185)(90)$$

$$\approx 1.67 \times 10^4 \text{ ft-lb}$$

42. This time m is a function of t: $m = 185 - \frac{9}{6\pi} t = 185 - \frac{3}{2\pi} t$. So let $\mathbf{F} = \left(185 - \frac{3}{2\pi} t\right) \mathbf{k}$. To parametrize the staircase, let $x = 20 \cos t$, $y = 20 \sin t$, $z = \frac{90}{6\pi} t = \frac{15}{\pi} t$, $0 \le t \le 6\pi$. Therefore

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \left\langle 0, 0, 185 - \tfrac{3}{2\pi} t \right\rangle \cdot \left\langle -20 \sin t, 20 \cos t, \tfrac{15}{\pi} \right\rangle dt = \tfrac{15}{\pi} \int_0^{6\pi} \left(185 - \tfrac{3}{2\pi} t\right) dt$$

$$= \tfrac{15}{\pi} \left[185t - \tfrac{3}{4\pi} t^2 \right]_0^{6\pi} = 90 \left(185 - \tfrac{9}{2}\right) \approx 1.62 \times 10^4 \text{ ft-lb}$$

43. The work done in moving the object is $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s = 2$ and approximating $\mathbf{F} \cdot \mathbf{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F} \cdot \mathbf{T}$ is the scalar projection of each force vector onto C. If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds \approx \sum_{i=1}^{7} \left[\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*) \right] \Delta s$$

$$= [2 + 2 + 2 + 2 + 1 + 1 + 1](2) = 22$$

Thus, we estimate the work done to be approximately 22 J.

44. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\,\mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r \cos\theta$, $y = r \sin\theta$. Thus $\mathbf{B} = |\mathbf{B}| \langle -\sin\theta, \cos\theta \rangle$. Then
$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}| \langle -\sin\theta, \cos\theta \rangle \cdot \langle -r \sin\theta, r \cos\theta \rangle \, d\theta = \int_0^{2\pi} |\mathbf{B}|\, r \, d\theta = 2\pi r\, |\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I / (2\pi r)$.

17.3 The Fundamental Theorem for Line Integrals ET 16.3

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

2. C is represented by the vector function $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^3 + t)\,\mathbf{j}$, $0 \le t \le 1$, so $\mathbf{r}'(t) = 2t\,\mathbf{i} + (3t^2 + 1)\,\mathbf{j}$. Since $3t^2 + 1 \ne 0$, we have $\mathbf{r}'(t) \ne \mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2, 2) - f(1, 0) = 9 - 3 = 6$.

3. $\partial(6x+5y)/\partial y = 5 = \partial(5x+4y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = 6x+5y$ and $f_y(x,y) = 5x+4y$. But $f_x(x,y) = 6x+5y$ implies $f(x,y) = 3x^2+5xy+g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = 5x+g'(y)$. Thus $5x+4y = 5x+g'(y)$ so $g'(y) = 4y$ and $g(y) = 2y^2+K$ where K is a constant. Hence $f(x,y) = 3x^2+5xy+2y^2+K$ is a potential function for $\mathbf{F}$.

4. $\partial(x^3+4xy)/\partial y = 4x$, $\partial(4xy-y^3)/\partial x = 4y$. Since these are not equal, $\mathbf{F}$ is not conservative.

5. $\partial(xe^y)/\partial y = xe^y$, $\partial(ye^x)/\partial x = ye^x$. Since these are not equal, $\mathbf{F}$ is not conservative.

6. $\partial(e^y)/\partial y = e^y = \partial(xe^y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = e^y$ implies $f(x,y) = xe^y + g(y)$ and $f_y(x,y) = xe^y + g'(y)$. But $f_y(x,y) = xe^y$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$. Then $f(x,y) = xe^y + K$ is a potential function for $\mathbf{F}$.

7. $\partial(2x\cos y - y\cos x)/\partial y = -2x\sin y - \cos x = \partial(-x^2\sin y - \sin x)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 2x\cos y - y\cos x$ implies $f(x,y) = x^2\cos y - y\sin x + g(y)$ and $f_y(x,y) = -x^2\sin y - \sin x + g'(y)$. But $f_y(x,y) = -x^2\sin y - \sin x$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$. Then $f(x,y) = x^2\cos y - y\sin x + K$ is a potential function for $\mathbf{F}$.

8. $\partial(1+2xy+\ln x)/\partial y = 2x = \partial(x^2)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid x>0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 1+2xy+\ln x$ implies $f(x,y) = x + x^2y + x\ln x - x + g(y)$ and $f_y(x,y) = x^2 + g'(y)$. But $f_y(x,y) = x^2$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$. Then $f(x,y) = x^2y + x\ln x + K$ is a potential function for $\mathbf{F}$.

9. $\partial(ye^x+\sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. But $f_y(x,y) = e^x + x\cos y$ so $g(y) = K$ and $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

10. $\partial(ye^{xy}+4x^3y)/\partial y = e^{xy}(yx+1) + 4x^3 = \partial(xe^{xy}+x^4)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Thus $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^{xy} + 4x^3y$ implies $f(x,y) = e^{xy} + x^4y + g(y)$ and $f_y(x,y) = xe^{yx} + x^4 + g'(y)$. But $f_y(x,y) = xe^{xy} + x^4$ so $g(y) = K$ and $f(x,y) = e^{xy} + x^4y + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}(2xy) = 2x = \dfrac{\partial}{\partial x}(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F}\cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to x, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \quad\Rightarrow\quad g'(y) = 0 \quad\Rightarrow\quad g(y) = K$, a constant. Thus $f(x,y) = x^2y + K$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F}\cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

12. (a) $f_x(x,y) = y$ implies $f(x,y) = xy + g(y)$ and $f_y(x,y) = x + g'(y)$. But $f_y(x,y) = x + 2y$ so $g'(y) = 2y \quad\Rightarrow\quad g(y) = y^2 + K$. We can take $K = 0$, so $f(x,y) = xy + y^2$.

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,1) - f(0,1) = 3 - 1 = 2$.

13. (a) $f_x(x,y) = x^3y^4$ implies $f(x,y) = \frac{1}{4}x^4y^4 + g(y)$ and $f_y(x,y) = x^4y^3 + g'(y)$. But $f_y(x,y) = x^4y^3$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$, a constant. We can take $K = 0$, so $f(x,y) = \frac{1}{4}x^4y^4$.

(b) The initial point of C is $\mathbf{r}(0) = (0,1)$ and the terminal point is $\mathbf{r}(1) = (1,2)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2) - f(0,1) = 4 - 0 = 4$.

14. (a) $f_x(x,y) = e^{2y}$ implies $f(x,y) = xe^{2y} + g(y)$ and $f_y(x,y) = 2xe^{2y} + g'(y)$. But $f_y(x,y) = 1 + 2xe^{2y}$ so $g'(y) = 1$ and $g(y) = y$ (setting $K = 0$). Thus $f(x,y) = xe^{2y} + y$.

(b) Since $\mathbf{r}(0) = \langle 0,1 \rangle$ and $\mathbf{r}(1) = \langle e,2 \rangle$, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(e,2) - f(0,1) = (e)\,e^4 + 2 - 1 = e^5 + 1$.

15. (a) $f_x(x,y,z) = y$ implies $f(x,y,z) = xy + g(y,z)$ and $f_y(x,y,z) = x + \partial g/\partial y$. But $f_y(x,y,z) = x + z$ so $\partial g/\partial y = z$ and $g(y,z) = yz + h(z)$. Thus $f(x,y,z) = xy + yz + h(z)$ and $f_z(x,y,z) = y + h'(z)$. But $f_z(x,y,z) = y$ so $h'(z) = 0$ or $h(z) = K$. Hence $f(x,y,z) = xy + yz$ (setting $K = 0$).

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(8,3,-1) - f(2,1,4) = 21 - 6 = 15$

16. (a) $f_x(x,y,z) = 2xy^3z^4$ implies $f(x,y,z) = x^2y^3z^4 + g(y,z)$ and $f_y(x,y,z) = 3x^2y^2z^4 + g_y(y,z)$. But $f_y(x,y,z) = 3x^2y^2z^4$, so $g_y(y,z) = h(z)$, and also $f(x,y,z) = x^2y^3z^4 + h(z)$, implying $f_z(x,y,z) = 4x^2y^3z^3 + h'(z)$. But $f_z(x,y,z) = 4x^2y^3z^3$, so $h'(z) = 0$. Hence $f(x,y,z) = x^2y^3z^4$.

(b) $\mathbf{r}(0) = \langle 0,0,0 \rangle$ and $\mathbf{r}(2) = \langle 2,4,8 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2,4,8) - f(0,0,0) = 2^2 \cdot 4^3 \cdot 8^4 = 2^{20}$.

17. (a) $f_x(x,y,z) = 2xz + \sin y$ implies $f(x,y,z) = x^2z + x\sin y + g(y,z)$ and $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$ so $g_y(y,z) = 0$ and $f(x,y,z) = x^2z + x\sin y + h(z)$. Thus $f_z(x,y,z) = x^2 + h'(z)$. But $f_z(x,y,z) = x^2$ so $h'(z) = 0$ and $f(x,y,z) = x^2z + x\sin y$ (setting $K = 0$).

(b) $\mathbf{r}(0) = \langle 1,0,0 \rangle$, $\mathbf{r}(2\pi) = \langle 1,0,2\pi \rangle$. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,0,2\pi) - f(1,0,0) = 2\pi$.

18. (a) $f_x(x,y,z) = 4xe^z$ implies $f(x,y,z) = 2x^2e^z + g(y,z)$ and $f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = \cos y$ so $g_y(y,z) = \cos y$ or $g(y,z) = \sin y + h(z)$. Thus $f(x,y,z) = 2x^2e^z + \sin y + h(z)$, and $f_z(x,y,z) = 2x^2e^z + h'(z)$. But $f_z(x,y,z) = 2x^2e^z$ so $h'(z) = 0$ and $f(x,y,z) = 2x^2e^z + \sin y$ (setting $K = 0$).

(b) $\mathbf{r}(0) = \langle 0,0,0 \rangle$, $\mathbf{r}(1) = \langle 1,1,1 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 2e + \sin 1$.

19. Here $\mathbf{F}(x,y) = (2x\sin y)\,\mathbf{i} + (x^2\cos y - 3y^2)\,\mathbf{j}$. Then $f(x,y) = x^2\sin y - y^3$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)\,dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(5,1) - f(-1,0) = 25\sin 1 - 1$.

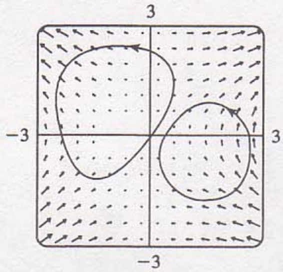
20. Here $\mathbf{F}(x,y) = (2y^2 - 12x^3y^3)\,\mathbf{i} + (4xy - 9x^4y^2)\,\mathbf{j}$. Then $f(x,y) = 2xy^2 - 3x^4y^3$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. Hence $\mathbf{F}$ is conservative and its line integral is independent of path. $\int_C (2y^2 - 12x^3y^3)\,dx + (4xy - 9x^4y^2)\,dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(3,2) - f(1,1) = -1920 - (-1) = -1919$.

21. $\mathbf{F}(x,y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(x^2y^3)/\partial y = 3x^2y^2 = \partial(x^3y^2)/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = x^2y^3 \quad \Rightarrow \quad f(x,y) = \frac{1}{3}x^3y^3 + g(y) \quad \Rightarrow \quad f_y = x^3y^2 + g'(y) \quad \Rightarrow$ $g'(y) = 0$, so we can take $f(x,y) = \frac{1}{3}x^3y^3$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2,1) - f(0,0) = \frac{1}{3}(2^3)(1^3) - 0 = \frac{8}{3}$.

22. $\mathbf{F}(x,y) = \dfrac{y^2}{x^2}\,\mathbf{i} - \dfrac{2y}{x}\,\mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\dfrac{\partial}{\partial y}\left(\dfrac{y^2}{x^2}\right) = \dfrac{2y}{x^2} = \dfrac{\partial}{\partial x}\left(-\dfrac{2y}{x}\right)$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = y^2/x^2 \quad \Rightarrow \quad f(x,y) = -y^2/x + g(y) \quad \Rightarrow \quad f_y = -2y/x + g'(y) \quad \Rightarrow$ $g'(y) = 0$, so we can take $f(x,y) = -y^2/x$ as a potential function for $\mathbf{F}$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(4,-2) - f(1,1) = -\left[(-2)^2/4\right] + (1/1) = 0$.

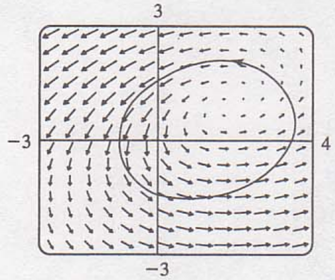
23. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be some circle centered at the origin, oriented counterclockwise. All of the field vectors along C oppose motion along C, so the integral around C will be negative. Therefore the field is not conservative.

24.



From the graph, it appears that $\mathbf{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate $\dfrac{\partial}{\partial y}(2xy + \sin y) = 2x + \cos y$, $\dfrac{\partial}{\partial x}(x^2 + x\cos y) = 2x + \cos y$. Thus $\mathbf{F}$ is conservative, by Theorem 6.

25.



From the graph, it appears that $\mathbf{F}$ is not conservative. For example, any closed curve containing the point $(2, 1)$ seems to have many field vectors pointing counterclockwise along it, and none pointing clockwise. So along this path the integral $\int \mathbf{F} \cdot d\mathbf{r} \neq 0$. To confirm our guess, we calculate

$$\frac{\partial}{\partial y}\left(\frac{x-2y}{\sqrt{1+x^2+y^2}}\right) = (x-2y)\left[\frac{-y}{(1+x^2+y^2)^{3/2}}\right] - \frac{2}{\sqrt{1+x^2+y^2}} = \frac{-2-2x^2-xy}{(1+x^2+y^2)^{3/2}},$$

$$\frac{\partial}{\partial x}\left(\frac{x-2}{\sqrt{1+x^2+y^2}}\right) = (x-2)\left[\frac{-x}{(1+x^2+y^2)^{3/2}}\right] + \frac{1}{\sqrt{1+x^2+y^2}} = \frac{1+y^2+2x}{(1+x^2+y^2)^{3/2}}.$$ These are not equal, so the field is not conservative, by Theorem 5.

26. $\nabla f(x, y) = \cos(x - 2y)\,\mathbf{i} - 2\cos(x - 2y)\,\mathbf{j}$

(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0, 0) = \sin 0 = 0$ and $f(\pi, \pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0, 0)$ to (π, π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j}$, $0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0, 0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2} t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0, 0)$ to $\left(\frac{\pi}{2}, 0\right)$.

27. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

28. Here $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 27, $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

29. $D = \{(x, y) \mid x > 0, y > 0\}$ = the first quadrant (excluding the axes).

(a) D is open because around every point in D we can put a disk that lies in D.

(b) D is connected because the straight line segment joining any two points in D lies in D.

(c) D is simply-connected because it's connected and has no holes.

30. $D = \{(x, y) \mid x \neq 0\}$ consists of all points in the xy-plane except for those on the y-axis.

(a) D is open.

(b) Points on opposite sides of the y-axis cannot be joined by a path that lies in D, so D is not connected.

(c) D is not simply-connected because it is not connected.

31. $D = \{(x, y) \mid 1 < x^2 + y^2 < 4\}$ = the annular region between the circles with center $(0, 0)$ and radii 1 and 2.

(a) D is open.

(b) D is connected.

(c) D is not simply-connected. For example, $x^2 + y^2 = (1.5)^2$ is simple and closed and lies within D but encloses points that are not in D. (Or we can say, D has a hole, so is not simply-connected.)

32. $D = \{(x, y) \mid x^2 + y^2 \leq 1 \text{ or } 4 \leq x^2 + y^2 \leq 9\}$ = the points on or inside the circle $x^2 + y^2 = 1$, together with the points on or between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

(a) D is not open because, for instance, no disk with center $(0, 2)$ lies entirely within D.

(b) D is not connected because, for example, $(0, 0)$ and $(0, 2.5)$ lie in D but cannot be joined by a path that lies entirely in D.

(c) D is not simply-connected because, for example, $x^2 + y^2 = 9$ is a simple closed curve in D but encloses points that are not in D.

33. (a) $P = -\dfrac{y}{x^2 + y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ and $Q = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x = \cos t$, $y = \sin t$, $0 \leq t \leq \pi$, C_2: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\, dt = \int_0^\pi dt = \pi \text{ and } \textstyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{\pi} dt = -\pi.$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

34. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 17.1 [ET 16.1].) Hence $\mathbf{F}$ is conservative and its line integral is independent of path.

Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$W = \textstyle\int_C \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = -\dfrac{c}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} + \dfrac{c}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = c\left(\dfrac{1}{d_1} - \dfrac{1}{d_2}\right).$$

(b) In this case, $c = -(mMG) \Rightarrow$

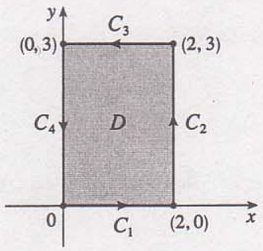
$$W = -mMG\left(\frac{1}{1.52\times10^8} - \frac{1}{1.47\times10^8}\right)$$
$$= -\left(5.97\times10^{24}\right)\left(1.99\times10^{30}\right)\left(6.67\times10^{-11}\right)\left(-2.2377\times10^{-10}\right) \approx 1.77\times10^{35}\text{ J}$$

(c) In this case, $c = \epsilon qQ \Rightarrow$

$$W = \epsilon qQ\left(\tfrac{1}{10^{-12}} - \tfrac{1}{5\times10^{-13}}\right) = \left(8.985\times10^{10}\right)(1)\left(-1.6\times10^{-19}\right)\left(-10^{12}\right) \approx 1.4\times10^4\text{ J.}$$

17.4 Green's Theorem ET 16.4

1. (a)



C_1: $x = t \Rightarrow dx = dt, y = 0 \Rightarrow dy = 0\,dt, 0 \le t \le 2.$

C_2: $x = 2 \Rightarrow dx = 0\,dt, y = t \Rightarrow dy = dt, 0 \le t \le 3.$

C_3: $x = 2 - t \Rightarrow dx = -dt, y = 3 \Rightarrow dy = 0\,dt, 0 \le t \le 2$

C_4: $x = 0 \Rightarrow dx = 0\,dt, y = 3 - t \Rightarrow dy = -dt, 0 \le t \le 3.$

Thus $\oint_C xy^2\,dx + x^3\,dy = \oint_{C_1+C_2+C_3+C_4} xy^2\,dx + x^3\,dy$

$$= \int_0^2 0\,dt + \int_0^3 8\,dt + \int_0^2 -9(2-t)\,dt + \int_0^3 0\,dt$$
$$= 0 + 24 - 18 + 0 = 6$$

(b) $\oint_C xy^2\,dx + x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(x^3\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right] dA = \int_0^2\int_0^3 \left(3x^2 - 2xy\right) dy\,dx$

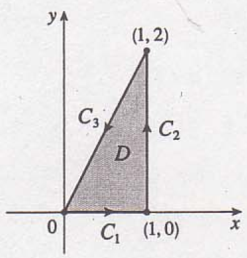
$$= \int_0^2 \left(9x^2 - 9x\right) dx = 24 - 18 = 6$$

2. (a) $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Then

$\oint_C y\,dx - x\,dy = \int_0^{2\pi} [\sin t\,(-\sin t) - \cos t\,(\cos t)]\,dt = -\int_0^{2\pi} dt = -2\pi.$

(b) $\oint_C y\,dx - x\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right] dA = -2\iint_D dA = -2A(D) = -2\pi(1)^2 = -2\pi$

3. (a)



C_1: $x = t \Rightarrow dx = dt, y = 0 \Rightarrow dy = 0\,dt, 0 \le t \le 1.$

C_2: $x = 1 \Rightarrow dx = 0\,dt, y = t \Rightarrow dy = dt, 0 \le t \le 2.$

C_3: $x = 1 - t \Rightarrow dx = -dt, y = 2 - 2t \Rightarrow dy = -2\,dt, 0 \le t \le 1.$

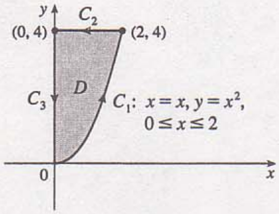
Thus $\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy$

$$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$$
$$= 0 + \left[\tfrac{1}{4}t^4\right]_0^2 + \left[\tfrac{2}{3}(1-t)^3 + \tfrac{8}{3}(1-t)^6\right]_0^1 = 4 - \tfrac{10}{3} = \tfrac{2}{3}$$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(x^2y^3\right) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1\int_0^{2x} \left(2xy^3 - x\right) dy\,dx$

$$= \int_0^1 \left[\tfrac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 \left(8x^5 - 2x^2\right) dx = \tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}$$

4. (a)



$$\oint_C (x^2 + y^2)\,dx + 2xy\,dy = \oint_{C_1 + C_2 + C_3} (x^2 + y^2)\,dx + 2xy\,dy$$
$$= \int_0^2 \left[(x^2 + x^4) + (2x^3)(2x)\right] dx + \int_2^0 (x^2 + 16)\,dx + \int_4^0 0\,dy$$
$$= \tfrac{8}{3} + 32 - \tfrac{8}{3} - 32 = 0$$

(b) $\oint_C (x^2 + y^2)\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2 + y^2)\right] dA$

$$= \iint_D (2y - 2y)\,dA = \iint_D (0)\,dA = 0$$

5. We can parametrize C as $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$. Then the line integral is $\oint_C P\,dx + Q\,dy = \int_0^{2\pi} \cos^4\theta \sin^5\theta\,(-\sin\theta)\,d\theta + \int_0^{2\pi} (-\cos^7\theta \sin^6\theta)\cos\theta\,d\theta = -\frac{29\pi}{1024}$, according to a CAS. The double integral is $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-7x^6y^6 - 5x^4y^4)\,dy\,dx = -\frac{29\pi}{1024}$, verifying Green's Theorem in this case.

6. Since $y = x^2$ along the first part of C and $y = x$ along the second part, the line integral is

$$\oint_C P\,dx + Q\,dy = \int_0^1 \left[x^4 \sin x + x^2 \sin(x^2)(2x)\right] dx + \int_1^0 (x^2 \sin x + x^2 \sin x)\,dx$$
$$= -16\cos 1 - 23\sin 1 + 28$$

according to a CAS. The double integral is

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_{x^2}^{x} (2x\sin y - 2y\sin x)\,dy\,dx = -16\cos 1 - 23\sin 1 + 28.$$

7. The region D enclosed by C is $[0, 1] \times [0, 1]$, so

$$\int_C e^y\,dx + 2xe^y\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xe^y) - \frac{\partial}{\partial y}(e^y)\right] dA = \int_0^1 \int_0^1 (2e^y - e^y)\,dy\,dx$$
$$= \int_0^1 dx \int_0^1 e^y\,dy = (1)(e^1 - e^0) = e - 1$$

8. The region D enclosed by C is given by $\{(x, y) \mid 0 \le x \le 1, 3x \le y \le 3\}$, so

$$\int_C x^2y^2\,dx + 4xy^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(4xy^3) - \frac{\partial}{\partial y}(x^2y^2)\right] dA = \int_0^1 \int_{3x}^{3} (4y^3 - 2x^2y)\,dy\,dx$$
$$= \int_0^1 \left[y^4 - x^2y^2\right]_{y=3x}^{y=3} dx = \int_0^1 (81 - 9x^2 - 72x^4)\,dx = 81 - 3 - \tfrac{72}{5} = \tfrac{318}{5}$$

9. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1)\,dx\,dy = \int_0^1 \left(y^{1/2} - y^2\right) dy = \tfrac{1}{3}$$

10. $\iint_D \left[\frac{\partial}{\partial x}(3x + \sin y) - \frac{\partial}{\partial y}(y^2 - \tan^{-1} x)\right] dA = \int_{-2}^{2} \int_{x^2}^{4} (3 - 2y)\,dy\,dx$

$$= \int_{-2}^{2} (-4 - 3x^2 + x^4)\,dx = -\tfrac{96}{5}$$

11. $\int_C y^3\,dx - x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right] dA = \iint_D (-3x^2 - 3y^2)\,dA = \int_0^{2\pi} \int_0^2 (-3r^2)\,r\,dr\,d\theta$

$$= -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3(2\pi)(4) = -24\pi$$

12. $\int_C \sin y\,dx + x\cos y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(\sin y)\right] dA = \iint_D (\cos y - \cos y)\,dA = \iint_D 0\,dA = 0$

13. The region D enclosed by C is given by $\{(x,y) \mid -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\}$ or, in polar coordinates, $\{(r,\theta) \mid 0 \le \theta \le \pi, 0 \le r \le 2\}$. Thus,

$$\int_C xy\,dx + 2x^2\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(2x^2) - \tfrac{\partial}{\partial y}(xy)\right] dA = \iint_D (4x - x)\,dA = \int_0^\pi \int_0^2 (3r\cos\theta)\,r\,dr\,d\theta$$

$$= 3\int_0^\pi \cos\theta\,d\theta \int_0^2 r^2\,dr = 3\,[\sin\theta]_0^\pi \left[\tfrac{1}{3}r^3\right]_0^2 = 3\,(0)\left(\tfrac{8}{3}\right) = 0$$

14. $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy = \iint_{1 \le x^2+y^2 \le 9} \left[\frac{\partial}{\partial x}(x^3+y^3) - \frac{\partial}{\partial y}(x^3-y^3)\right] dA$

$$= \iint_{1 \le x^2+y^2 \le 9} (3x^2 + 3y^2)\,dA$$

$$= 3\int_{-\pi}^{\pi} \int_1^3 r^3\,dr\,d\theta = 6\pi\left(\tfrac{81}{4} - \tfrac{1}{4}\right) = 120\pi$$

15. The region D enclosed by C is given, in polar coordinates, by $\left\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{4}, 0 \le r \le 2\right\}$. Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (y^2 - x^2 y)\,dx + xy^2\,dy = \iint_D (y^2 - 2y + x^2)\,dA$$

$$= \int_0^{\pi/4} \int_0^2 (r^2 - 2r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/4} \left[4 - \tfrac{16}{3}\sin\theta\right] d\theta$$

$$= \left[4\theta + \tfrac{16}{3}\cos\theta\right]_0^{\pi/4} = \pi + \tfrac{8}{3}\left(\sqrt{2} - 2\right)$$

16. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y^6\,dx + xy^5\,dy = \iint_D \left[\frac{\partial}{\partial x}(xy^5) - \frac{\partial}{\partial y}(y^6)\right] dA = \iint_D -5y^5\,dA = 0$ since $-5y^5$ is an odd function of y and D is symmetric with respect to the y-axis.

17. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dy\,dx$ where C is the path described in the question and D is the triangle bounded by C. So

$$W = \int_0^1 \int_0^{1-x} (y^2 - x)\,dy\,dx = \int_0^1 \left[\tfrac{1}{3}y^3 - xy\right]_{y=0}^{y=1-x} dx = \int_0^1 \left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right) dx$$

$$= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}$$

18. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C. Converting to polar coordinates, we have $W = 3\int_0^2 \int_0^\pi r^2 \cdot r\,d\theta\,dr = 3\pi\left[\frac{1}{4}r^4\right]_0^2 = 12\pi$.

19. $A = \oint_C x\,dy = \int_0^{2\pi} (\cos^3 t)(3\sin^2 t\cos t)\,dt = 3\int_0^{2\pi} (\cos^4 t\sin^2 t)\,dt$

$$= 3\left[-\tfrac{1}{6}(\sin t\cos^5 t) + \tfrac{1}{6}\left[\tfrac{1}{4}(\sin t\cos^3 t) + \tfrac{3}{8}(\cos t\sin t) + \tfrac{3}{8}t\right]\right]_0^{2\pi} = 3\left(\tfrac{1}{6}\right)\left(\tfrac{6}{8}\pi\right) = \tfrac{3}{8}\pi$$

Or: $3\int_0^{2\pi} (\cos^4 t\sin^2 t)\,dt = 3\int_0^{2\pi} \frac{1}{8}\left[\frac{1}{2}(1 - \cos 4t) + \sin^2 2t\cos 2t\right] dt = \frac{3}{8}\pi$

20. $A = \oint_C x\,dy = \int_0^{2\pi} (\cos t)(3\sin^2 t\cos t)\,dt = 3\int_0^{2\pi} \frac{1}{8}(1 - \cos 4t)\,dt = \frac{3}{4}\pi$

21. (a) Using Equation 17.2.8 [ET 16.2.8], we write parametric equations of the line segment as $x = (1-t)\,x_1 + tx_2$, $y = (1-t)\,y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$$\int_C x\,dy - y\,dx = \int_0^1 [(1-t)\,x_1 + tx_2]\,(y_2 - y_1)\,dt + [(1-t)\,y_1 + ty_2]\,(x_2 - x_1)\,dt$$

$$= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)])\,dt$$

$$= \int_0^1 (x_1 y_2 - x_2 y_1)\,dt = x_1 y_2 - x_2 y_1$$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

$$\text{area of polygon} = A(D) = \iint_D dA = \tfrac{1}{2}\int_C x\,dy - y\,dx$$
$$= \tfrac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right)$$

To evaluate these integrals we use the formula from (a) to get
$A(D) = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right]$.

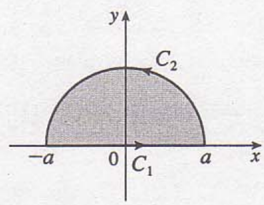
(c) $A = \frac{1}{2}\left[(0\cdot 1 - 2\cdot 0) + (2\cdot 3 - 1\cdot 1) + (1\cdot 2 - 0\cdot 3) + (0\cdot 1 - (-1)\cdot 2) + (-1\cdot 0 - 0\cdot 1)\right]$

$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and
$-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D (-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

23. Here $A = \frac{1}{2}(1)(1) = \frac{1}{2}$ and $C = C_1 + C_2 + C_3$, where C_1: $x = x$, $y = 0$, $0 \le x \le 1$;
C_2: $x = x$, $y = 1 - x$, $x = 1$ to $x = 0$; and C_3: $x = 0$, $y = 1$ to $y = 0$. Then
$\overline{x} = \frac{1}{2A}\int_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = 0 + \int_1^0 (x^2)(-dx) + 0 = \frac{1}{3}$. Similarly,
$\overline{y} = -\frac{1}{2A}\int_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = 0 + \int_1^0 (1-x)^2(-dx) + 0 = \frac{1}{3}$.
Therefore $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)$.

24. $A = \frac{\pi a^2}{2}$ so $\overline{x} = \frac{1}{\pi a^2}\oint_C x^2\,dy$ and $\overline{y} = -\frac{1}{\pi a^2}\oint_C y^2 dx$. Orienting the semicircular region as in the figure,



$\overline{x} = \frac{1}{\pi a^2}\oint_{C_1 + C_2} x^2\,dy = \frac{1}{\pi a^2}\left[0 + \int_0^\pi (a^2\cos^2 t)(a\cos t)\,dt\right] = 0$ and

$$\overline{y} = -\frac{1}{\pi a^2}\left[\int_{-a}^{a} 0\,dx + \int_0^\pi (a^2\sin^2 t)(-a\sin t)\,dt\right]$$
$$= \frac{a}{\pi}\int_0^\pi \sin^3 t\,dt = \frac{a}{\pi}\left[-\cos t + \tfrac{1}{3}(\cos^3 t)\right]_0^\pi = \frac{4a}{3\pi}$$

Thus $(\overline{x}, \overline{y}) = \left(0, \frac{4a}{3\pi}\right)$.

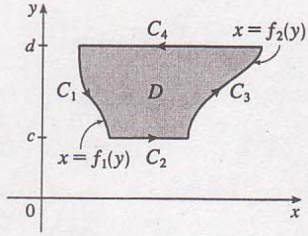
25. By Green's Theorem, $-\frac{1}{3}\rho\oint_C y^3\,dx = -\frac{1}{3}\rho\iint_D (-3y^2)\,dA = \iint_D y^2\rho\,dA = I_x$ and
$\frac{1}{3}\rho\oint_C x^3\,dy = \frac{1}{3}\rho\iint_D (3x^2)\,dA = \iint_D x^2\rho\,dA = I_y$.

26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$I_y = \tfrac{1}{3}\rho\oint_C x^3\,dy = \tfrac{1}{3}\rho\int_0^{2\pi}(a^4\cos^4 t)\,dt = \tfrac{1}{3}a^4\rho\int_0^{2\pi}\left[\tfrac{3}{8} + \tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t\right]dt$$
$$= \tfrac{1}{3}a^4\rho\cdot\tfrac{3(2\pi)}{8} = \tfrac{1}{4}\pi a^4\rho$$

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2 + y^2)$ and $Q = x/(x^2 + y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) [ET 16.3.33(a)], $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D 0\,dA = 0$.

28. We express D as a type II region: $D = \{(x, y) \mid f_1(y) \le x \le f_2(y), c \le y \le d\}$ where f_1 and f_2 are continuous functions. Then $\displaystyle\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(f_2(y), y) - Q(f_1(y), y)]\,dy$ by the Fundamental Theorem of Calculus. But referring to the figure, $\oint_C Q\,dy = \oint_{C_1 + C_2 + C_3 + C_4} Q\,dy$. Then $\int_{C_1} Q\,dy = \int_d^c Q(f_1(y), y)\,dy$, $\int_{C_2} Q\,dy = \int_{C_4} Q\,dy = 0$, and $\int_{C_3} Q\,dy = \int_c^d Q(f_2(y), y)\,dy$. Hence $\oint_C Q\,dy = \int_c^d [Q(f_2(y), y) - Q(f_1(y), y)]\,dy = \iint_D (\partial Q/\partial x)\,dA$.



29. Using the first part of (5), we have that $\iint_R dx\,dy = A(R) = \int_{\partial R} x\,dy$. But $x = g(u, v)$, and $dy = \dfrac{\partial h}{\partial u}\,du + \dfrac{\partial h}{\partial v}\,dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R, so

$$\begin{aligned}
\int_{\partial R} x\,dy &= \int_{\partial S} g(u, v)\left(\frac{\partial h}{\partial u}\,du + \frac{\partial h}{\partial v}\,dv\right) = \int_{\partial S} g(u, v)\,\frac{\partial h}{\partial u}\,du + g(u, v)\,\frac{\partial h}{\partial v}\,dv \\
&= \pm \iint_S \left[\frac{\partial}{\partial u}\left(g(u, v)\,\frac{\partial h}{\partial v}\right) - \frac{\partial}{\partial v}\left(g(u, v)\,\frac{\partial h}{\partial u}\right)\right] dA \quad \text{(using Green's Theorem in the } uv\text{-plane)} \\
&= \pm \iint_S \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} + g(u, v)\,\frac{\partial^2 h}{\partial u \partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} - g(u, v)\,\frac{\partial^2 h}{\partial v \partial u}\right) dA \quad \text{(using the Chain Rule)} \\
&= \pm \iint_S \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) dA \text{ (by the equality of mixed partials)} = \pm \iint_S \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv
\end{aligned}$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\dfrac{\partial(x, y)}{\partial(u, v)}$. Therefore $A(R) = \displaystyle\iint_R dx\,dy = \iint_S \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$.

17.5 Curl and Divergence ET 16.5

1. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix}$

$$\begin{aligned}
&= \left[\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial z}(xy)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right]\mathbf{k} \\
&= (0 - y)\,\mathbf{i} - (z - 0)\,\mathbf{j} + (0 - x)\,\mathbf{k} = -y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}
\end{aligned}$$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(xy) + \dfrac{\partial}{\partial y}(yz) + \dfrac{\partial}{\partial z}(zx) = y + z + x = x + y + z$

2. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x - 2z & x + y + z & x - 2y \end{vmatrix} = (-2 - 1)\,\mathbf{i} - (1 + 2)\,\mathbf{j} + (1 - 0)\,\mathbf{k} = -3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(x - 2z) + \dfrac{\partial}{\partial y}(x + y + z) + \dfrac{\partial}{\partial z}(x - 2y) = 1 + 1 + 0 = 2$

3. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 0 & -x^2 y \end{vmatrix} = (-x^2 - 0)\,\mathbf{i} - (-2xy - xy)\,\mathbf{j} + (0 - xz)\,\mathbf{k}$

$= -x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(xyz) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(-x^2 y) = yz + 0 + 0 = yz$

4. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xe^y & ye^z \end{vmatrix} = (e^z - 0)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (e^y - 0)\,\mathbf{k} = e^z\,\mathbf{i} + e^y\,\mathbf{k}$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(0) + \dfrac{\partial}{\partial y}(xe^y) + \dfrac{\partial}{\partial z}(ye^z) = xe^y + ye^z$

5. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y & e^x \cos y & z \end{vmatrix} = (0 - 0)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (e^x \cos y - e^x \cos y)\,\mathbf{k} = \mathbf{0}$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(e^x \sin y) + \dfrac{\partial}{\partial y}(e^x \cos y) + \dfrac{\partial}{\partial z}(z) = e^x \sin y - e^x \sin y + 1 = 1$

6. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{x^2 + y^2 + z^2} & \dfrac{y}{x^2 + y^2 + z^2} & \dfrac{z}{x^2 + y^2 + z^2} \end{vmatrix}$

$= \dfrac{1}{(x^2 + y^2 + z^2)^2}\left[(-2yz + 2yz)\,\mathbf{i} - (-2xz + 2xz)\,\mathbf{j} + (-2xy + 2xy)\,\mathbf{k}\right] = \mathbf{0}$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2 + y^2 + z^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2 + y^2 + z^2}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{x^2 + y^2 + z^2}\right)$

$= \dfrac{x^2 + y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^2} + \dfrac{x^2 + y^2 + z^2 - 2y^2}{(x^2 + y^2 + z^2)^2} + \dfrac{x^2 + y^2 + z^2 - 2z^2}{(x^2 + y^2 + z^2)^2}$

$= \dfrac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \dfrac{1}{x^2 + y^2 + z^2}$

7. (a) $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/z & y/z & -1/z \end{vmatrix} = \left(0 + \dfrac{y}{z^2}\right)\mathbf{i} - \left(0 + \dfrac{x}{z^2}\right)\mathbf{j} + (0 - 0)\,\mathbf{k} = \dfrac{y}{z^2}\,\mathbf{i} - \dfrac{x}{z^2}\,\mathbf{j}$

(b) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{z}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{z}\right) + \dfrac{\partial}{\partial z}\left(-\dfrac{1}{z}\right) = \dfrac{1}{z} + \dfrac{1}{z} + \dfrac{1}{z^2} = \dfrac{2z + 1}{z^2}$

8. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{yz} & ye^{xz} & ze^{xy} \end{vmatrix}$

$$= (xze^{xy} - xye^{xz})\,\mathbf{i} - (yze^{xy} - xye^{yz})\,\mathbf{j} + (yze^{xz} - xze^{yz})\,\mathbf{k}$$
$$= x\,(ze^{xy} - ye^{xz})\,\mathbf{i} + y\,(xe^{yz} - ze^{xy})\,\mathbf{j} + z\,(ye^{xz} - xe^{yz})\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(xe^{yz}) + \dfrac{\partial}{\partial y}(ye^{xz}) + \dfrac{\partial}{\partial z}(ze^{xy}) = e^{yz} + e^{xz} + e^{xy}$

9. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the x-component of each vector of $\mathbf{F}$ is 0, so $P = 0$, hence $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\dfrac{\partial Q}{\partial y} < 0$, but Q doesn't change in the x- or z-directions, so $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + \dfrac{\partial Q}{\partial y} + 0 < 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

10. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, P and Q don't vary in the z-direction, so $\dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial z} = \dfrac{\partial Q}{\partial z} = 0$. As x increases, the x-component of each vector of $\mathbf{F}$ increases while the y-component remains constant, so $\dfrac{\partial P}{\partial x} > 0$ and $\dfrac{\partial Q}{\partial x} = 0$. Similarly, as y increases, the y-component of each vector increases while the x-component remains constant, so $\dfrac{\partial Q}{\partial y} > 0$ and $\dfrac{\partial P}{\partial y} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + 0 > 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k}$

$$= (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$$

11. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the y-component of each vector of $\mathbf{F}$ is 0, so $Q = 0$, hence $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial y} = \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. P increases as y increases, so $\dfrac{\partial P}{\partial y} > 0$, but P doesn't change in the x- or z-directions, so $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + 0 + 0 = 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k}$

$$= (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + \left(0 - \frac{\partial P}{\partial y}\right)\mathbf{k} = -\frac{\partial P}{\partial y}\mathbf{k}$$

Since $\dfrac{\partial P}{\partial y} > 0$, $-\dfrac{\partial P}{\partial y}\mathbf{k}$ is a vector pointing in the negative z-direction.

12. (a) curl $f = \nabla \times f$ is meaningless because f is a scalar field.

(b) grad f is a vector field.

(c) div $\mathbf{F}$ is a scalar field.

(d) curl (grad f) is a vector field.

(e) grad $\mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) grad (div $\mathbf{F}$) is a vector field.

(g) div (grad f) is a scalar field.

(h) grad (div f) is meaningless because f is a scalar field.

(i) curl (curl $\mathbf{F}$) is a vector field.

(j) div (div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(k) (grad f) $\times$ (div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(l) div (curl (grad f)) is a scalar field.

13. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x - x)\,\mathbf{i} - (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = \mathbf{0}$ and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions which have continuous partial derivatives, so by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_x(x, y, z) = yz$ implies $f(x, y, z) = xyz + g(y, z)$ and $f_y(x, y, z) = xz + g_y(y, z)$. But $f_y(x, y, z) = xz$, so $g(y, z) = h(z)$ and $f(x, y, z) = xyz + h(z)$. Thus $f_z(x, y, z) = xy + h'(z)$ but $f_z(x, y, z) = xy$ so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x, y, z) = xyz + K$.

14. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = (0 - 0)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = \mathbf{0}$, $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = x$ implies $f(x, y, z) = \frac{1}{2}x^2 + g(y, z)$ and $f_y(x, y, z) = g_y(y, z)$. But $f_y(x, y, z) = y$, so $g(y, z) = \frac{1}{2}y^2 + h(z)$ and $f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + h(z)$. Thus $f_z(x, y, z) = h'(z)$ but $f_z(x, y, z) = z$ so $h(z) = \frac{1}{2}z^2 + K$ and $f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + K$.

15. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 + 2yz & y^2 \end{vmatrix} = (2y - 2y)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (2x - 2x)\,\mathbf{k} = \mathbf{0}$, $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = 2xy$ implies $f(x, y, z) = x^2y + g(y, z)$ and $f_y(x, y, z) = x^2 + g_y(y, z)$. But $f_y(x, y, z) = x^2 + 2yz$, so $g(y, z) = y^2z + h(z)$ and $f(x, y, z) = x^2y + y^2z + h(z)$. Thus $f_z(x, y, z) = y^2 + h'(z)$ but $f_z(x, y, z) = y^2$ so $h(z) = K$ and $f(x, y, z) = x^2y + y^2z + K$.

16. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2z^3 & 2x^2yz^3 & 3x^2y^2z^2 \end{vmatrix}$

$= \left(6x^2yz^2 - 6x^2yz^2\right)\mathbf{i} - \left(6xy^2z^2 - 3xy^2z^2\right)\mathbf{j} + \left(4xyz^3 - 2xyz^3\right)\mathbf{k} \neq \mathbf{0}$,

so $\mathbf{F}$ isn't conservative.

17. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & e^z & e^y \end{vmatrix} = (e^y - e^z)\,\mathbf{i} - (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} \neq \mathbf{0}$, so $\mathbf{F}$ isn't conservative.

18. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yze^{xz} & e^{xz} & xye^{xz} \end{vmatrix} =$

$(xe^{xz} - xe^{xz})\,\mathbf{i} - [(xyze^{xz} + ye^{xz}) - (xyze^{xz} + ye^{xz})]\,\mathbf{j} + (ze^{xz} - ze^{xz})\,\mathbf{k} = \mathbf{0}$, $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = yze^{xz}$ implies $f(x,y,z) = ye^{xz} + g(y,z)$ and $f_y(x,y,z) = e^{xz} + g_y(y,z)$. But $f_y(x,y,z) = e^{xz}$, so $g(y,z) = h(z)$ and $f(x,y,z) = ye^{xz} + h(z)$. Thus $f_z(x,y,z) = xye^{xz} + h'(z)$ but $f_z(x,y,z) = xye^{xz}$ so $h(z) = K$ and $f(x,y,z) = ye^{xz} + K$.

19. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = y^2 + z^2 + x^2 \neq 0$, which contradicts Theorem 11.

20. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = xz \neq 0$ which contradicts Theorem 11.

21. $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$. Hence $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$ is irrotational.

22. $\operatorname{div}\mathbf{F} = \dfrac{\partial(f(y,z))}{\partial x} + \dfrac{\partial(g(x,z))}{\partial y} + \dfrac{\partial(h(x,y))}{\partial z} = 0$ so $\mathbf{F}$ is incompressible.

For Exercises 23–29, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

23. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \dfrac{\partial(P_1+P_2)}{\partial x} + \dfrac{\partial(Q_1+Q_2)}{\partial y} + \dfrac{\partial(R_1+R_2)}{\partial z}$

$$= \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_3}{\partial z}\right) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$$

24.
$$\begin{aligned}
\operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G} &= \left[\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\mathbf{k}\right] \\
&\quad + \left[\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\mathbf{k}\right] \\
&= \left[\frac{\partial(R_1+R_2)}{\partial y} - \frac{\partial(Q_1+Q_2)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(P_1+P_2)}{\partial z} - \frac{\partial(R_1+R_2)}{\partial x}\right]\mathbf{j} \\
&\quad + \left[\frac{\partial(Q_1+Q_2)}{\partial x} - \frac{\partial(P_1+P_2)}{\partial y}\right]\mathbf{k} = \operatorname{curl}(\mathbf{F}+\mathbf{G})
\end{aligned}$$

25. $\operatorname{div}(f\mathbf{F}) = \dfrac{\partial(fP_1)}{\partial x} + \dfrac{\partial(fQ_1)}{\partial y} + \dfrac{\partial(fR_1)}{\partial z}$

$$= \left(f\frac{\partial P_1}{\partial x} + P_1\frac{\partial f}{\partial x}\right) + \left(f\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial f}{\partial y}\right) + \left(f\frac{\partial R_1}{\partial z} + R_1\frac{\partial f}{\partial z}\right)$$

$$= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$$

26. $\operatorname{curl}(f\mathbf{F}) = \left[\dfrac{\partial(fR_1)}{\partial y} - \dfrac{\partial(fQ_1)}{\partial z}\right]\mathbf{i} + \left[\dfrac{\partial(fP_1)}{\partial z} - \dfrac{\partial(fR_1)}{\partial x}\right]\mathbf{j} + \left[\dfrac{\partial(fQ_1)}{\partial x} - \dfrac{\partial(fP_1)}{\partial y}\right]\mathbf{k}$

$$= \left[f\frac{\partial R_1}{\partial y} + R_1\frac{\partial f}{\partial y} - f\frac{\partial Q_1}{\partial z} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[f\frac{\partial P_1}{\partial z} + P_1\frac{\partial f}{\partial z} - f\frac{\partial R_1}{\partial x} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j}$$

$$+ \left[f\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial f}{\partial x} - f\frac{\partial P_1}{\partial y} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right]\mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right]\mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right]\mathbf{k}$$

$$+ \left[R_1\frac{\partial f}{\partial y} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[P_1\frac{\partial f}{\partial z} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} + \left[Q_1\frac{\partial f}{\partial x} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$$

27. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(\mathbf{F}\times\mathbf{G}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \dfrac{\partial}{\partial x}\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \dfrac{\partial}{\partial y}\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \dfrac{\partial}{\partial z}\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$

$$= \left[Q_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial Q_1}{\partial x} - Q_2\frac{\partial R_1}{\partial x} - R_1\frac{\partial Q_2}{\partial x}\right]$$

$$- \left[P_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial P_1}{\partial y} - P_2\frac{\partial R_1}{\partial y} - R_1\frac{\partial P_2}{\partial y}\right]$$

$$+ \left[P_1\frac{\partial Q_2}{\partial z} + Q_2\frac{\partial P_1}{\partial z} - P_2\frac{\partial Q_1}{\partial z} - Q_1\frac{\partial P_2}{\partial z}\right]$$

$$= \left[P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right]$$

$$- \left[P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right]$$

$$= \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

28. $\operatorname{div}(\nabla f\times\nabla g) = \nabla g\cdot\operatorname{curl}(\nabla f) - \nabla f\cdot\operatorname{curl}(\nabla g)$ (by Exercise 27) $= 0$ (by Theorem 3)

29. curl curl $\mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$

$$= \left(\frac{\partial^2 Q_1}{\partial y \partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z \partial x}\right)\mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z \partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x \partial y}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial x \partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y \partial z}\right)\mathbf{k}$$

Now let's consider grad div $\mathbf{F} - \nabla^2\mathbf{F}$ and compare with the above.
(Note that $\nabla^2\mathbf{F}$ is defined on page 1113 [ET 1079].)

$$\text{grad div}\,\mathbf{F} - \nabla^2\mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y \partial z}\right)\mathbf{j}\right.$$
$$\left. + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}\right.$$
$$\left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$= \left(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 R_1}{\partial y \partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right)\mathbf{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have curl curl $\mathbf{F} = \text{grad div}\,\mathbf{F} - \nabla^2\mathbf{F}$ as desired.

30. (a) $\nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 1 + 1 + 1 = 3$

(b) $\nabla \cdot (r\mathbf{r}) = \nabla \cdot \sqrt{x^2+y^2+z^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

$$= \left(\frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right) + \left(\frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$
$$+ \left(\frac{z^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$
$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(4x^2+4y^2+4z^2\right) = 4\sqrt{x^2+y^2+z^2} = 4r$$

Another Method:

By Exercise 25, $\nabla \cdot (r\mathbf{r}) = \operatorname{div}(r\mathbf{r}) = r \operatorname{div}\mathbf{r} + \mathbf{r} \cdot \nabla r = 3r + \mathbf{r} \cdot \frac{\mathbf{r}}{r}$ [see Exercise 31(a) below] $= 4r$.

(c) $\nabla^2 r^3 = \nabla^2 \left(x^2+y^2+z^2\right)^{3/2}$

$$= \frac{\partial}{\partial x}\left[\tfrac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}(2x)\right] + \frac{\partial}{\partial y}\left[\tfrac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}(2y)\right] + \frac{\partial}{\partial z}\left[\tfrac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}(2z)\right]$$

$$= 3\left[\tfrac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2x)(x) + \left(x^2+y^2+z^2\right)^{1/2}\right] + 3\left[\tfrac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2y)(y) + \left(x^2+y^2+z^2\right)^{1/2}\right] + 3\left[\tfrac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2z)(z) + \left(x^2+y^2+z^2\right)^{1/2}\right]$$

$$= 3\left(x^2+y^2+z^2\right)^{-1/2}\left(4x^2+4y^2+4z^2\right) = 12\left(x^2+y^2+z^2\right)^{1/2}$$

$$= 12r$$

Another Method: $\dfrac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{3/2} = 3x\sqrt{x^2+y^2+z^2} \;\Rightarrow\; \nabla r^3 = 3r\left(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}\right) = 3r\,\mathbf{r}$, so $\nabla^2 r^3 = \nabla\cdot\nabla r^3 = \nabla\cdot(3r\,\mathbf{r}) = 3\,(4r) = 12r$ by part (b).

31. (a) $\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$

$$= \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r}$$

(b) $\nabla\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\mathbf{k} = \mathbf{0}$$

(c) $\nabla\left(\dfrac{1}{r}\right) = \nabla\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$= \frac{-\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\left(x^2+y^2+z^2\right)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d) $\nabla\ln r = \nabla\ln\left(x^2+y^2+z^2\right)^{1/2} = \tfrac{1}{2}\nabla\ln\left(x^2+y^2+z^2\right)$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

32. $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad \Rightarrow \quad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{k}$$

Then $\dfrac{\partial}{\partial x}\dfrac{x}{(x^2+y^2+z^2)^{p/2}} = \dfrac{(x^2+y^2+z^2) - px^2}{(x^2+y^2+z^2)^{1+p/2}} = \dfrac{r^2 - px^2}{r^{p+2}}$. Similarly,
$\dfrac{\partial}{\partial y}\dfrac{y}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2 - py^2}{r^{p+2}}$ and $\dfrac{\partial}{\partial z}\dfrac{z}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2 - pz^2}{r^{p+2}}$. Thus

$$\begin{aligned} \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{r^2 - px^2}{r^{p+2}} + \frac{r^2 - py^2}{r^{p+2}} + \frac{r^2 - pz^2}{r^{p+2}} = \frac{3r^2 - px^2 - py^2 - pz^2}{r^{p+2}} \\ &= \frac{3r^2 - p\left(x^2+y^2+z^2\right)}{r^{p+2}} = \frac{3r^2 - pr^2}{r^{p+2}} = \frac{3-p}{r^p} \end{aligned}$$

Consequently, if $p = 3$ we have $\operatorname{div}\mathbf{F} = 0$.

33. By (13), $\oint_C f\,(\nabla g)\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}(f\nabla g)\,dA = \iint_D [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dA$ by Exercise 25. But $\operatorname{div}(\nabla g) = \nabla^2 g$. Hence $\iint_D f\nabla^2 g\,dA = \oint_C f\,(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$.

34. By Exercise 33, $\iint_D f\nabla^2 g\,dA = \oint_C f\,(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$ and
$\iint_D g\nabla^2 f\,dA = \oint_C g\,(\nabla f)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$. Hence

$$\begin{aligned} \iint_D \left(f\nabla^2 g - g\nabla^2 f\right)dA &= \oint_C [f\,(\nabla g)\cdot\mathbf{n} - g\,(\nabla f)\cdot\mathbf{n}]\,ds + \iint_D (\nabla f\cdot\nabla g - \nabla g\cdot\nabla f)\,dA \\ &= \oint_C [f\nabla g - g\nabla f]\cdot\mathbf{n}\,ds \end{aligned}$$

35. (a) We know that $\omega = v/d$, and from the diagram $\sin\theta = d/r \quad \Rightarrow \quad v = d\omega = (\sin\theta)\,r\omega = |\mathbf{w}\times\mathbf{r}|$. But $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, so that $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) From (a), $\mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0\cdot z - \omega y)\,\mathbf{i} + (\omega x - 0\cdot z)\,\mathbf{j} + (0\cdot y - x\cdot 0)\,\mathbf{k} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$

(c) $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

$$\begin{aligned} &= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right]\mathbf{k} \\ &= [\omega - (-\omega)]\,\mathbf{k} = 2\omega\,\mathbf{k} = 2\mathbf{w} \end{aligned}$$

36. Let $\mathbf{H} = \langle h_1, h_2, h_3 \rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$.

(a) $$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (\operatorname{curl} \mathbf{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix}$$

$$= -\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y \partial t} - \frac{\partial^2 h_2}{\partial z \partial t}\right)\mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z \partial t} - \frac{\partial^2 h_3}{\partial x \partial t}\right)\mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x \partial t} - \frac{\partial^2 h_1}{\partial y \partial t}\right)\mathbf{k}\right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\mathbf{k}\right]$$

(assuming that the partial derivatives are continuous so that the order of differentiation does not matter)

$$= -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl} \mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

(b) $$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times (\operatorname{curl} \mathbf{H}) = \nabla \times \left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t \end{vmatrix}$$

$$= \frac{1}{c}\left[\left(\frac{\partial^2 E_3}{\partial y \partial t} - \frac{\partial^2 E_2}{\partial z \partial t}\right)\mathbf{i} + \left(\frac{\partial^2 E_1}{\partial z \partial t} - \frac{\partial^2 E_3}{\partial x \partial t}\right)\mathbf{j} + \left(\frac{\partial^2 E_2}{\partial x \partial t} - \frac{\partial^2 E_1}{\partial y \partial t}\right)\mathbf{k}\right]$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)\mathbf{k}\right]$$

(assuming that the partial derivatives are continuous so that the order of differentiation does not matter)

$$= \frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl} \mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

(c) Using Exercise 29, we have that $\operatorname{curl}\operatorname{curl} \mathbf{E} = \operatorname{grad}\operatorname{div} \mathbf{E} - \nabla^2 \mathbf{E} \;\Rightarrow$

$$\nabla^2 \mathbf{E} = \operatorname{grad}\operatorname{div} \mathbf{E} - \operatorname{curl}\operatorname{curl} \mathbf{E} = \operatorname{grad} 0 + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \text{ [from part (a)]} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2 \mathbf{H} = \operatorname{grad}\operatorname{div} \mathbf{H} - \operatorname{curl}\operatorname{curl} \mathbf{H} = \operatorname{grad} 0 + \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$ [using part (b)] $= \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$.

17.6 Parametric Surfaces and Their Areas ET 16.6

1. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = u^2$. For any point (x,y,z) on the surface, we have $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z$. Since no restrictions are placed on the parameters, the surface is $z = x^2 + y^2$, which we recognize as a circular paraboloid opening upward whose axis is the z-axis.

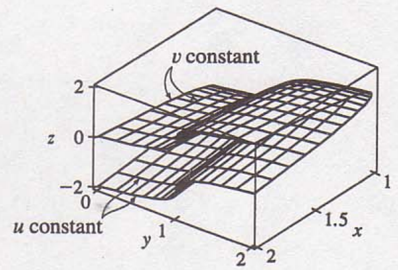
2. $\mathbf{r}(u,v) = (1+2u)\,\mathbf{i} + (-u+3v)\,\mathbf{j} + (2+4u+5v)\,\mathbf{k} = \langle 1,0,2\rangle + u\,\langle 2,-1,4\rangle + v\,\langle 0,3,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(1,0,2)$ and containing vectors $\mathbf{a} = \langle 2,-1,4\rangle$ and $\mathbf{b} = \langle 0,3,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix} = -17\,\mathbf{i} - 10\,\mathbf{j} + 6\,\mathbf{k}$ and an equation of the plane is $-17(x-1) - 10(y-0) + 6(z-2) = 0$ or $-17x - 10y + 6z = -5$.

3. $\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$, so the corresponding parametric equations for the surface are $x = x$, $y = \cos\theta$, $z = \sin\theta$. For any point (x,y,z) on the surface, we have $y^2 + z^2 = \cos^2\theta + \sin^2\theta = 1$, so any vertical trace in $x = k$ is the circle $y^2 + z^2 = 1$, $x = k$. Since $x = x$ with no restriction, the surface is a circular cylinder with radius 1 whose axis is the x-axis.

4. $\mathbf{r}(x,\theta) = \langle x, x\cos\theta, x\sin\theta\rangle$, so the corresponding parametric equations for the surface are $x = x$, $y = x\cos\theta$, $z = x\sin\theta$. For any point (x,y,z) on the surface, we have $y^2 + z^2 = x^2\cos^2\theta + x^2\sin^2\theta = x^2$. With $x = x$ and no restrictions on the parameters, the surface is $x^2 = y^2 + z^2$, which we recognize as a circular cone whose axis is the x-axis.

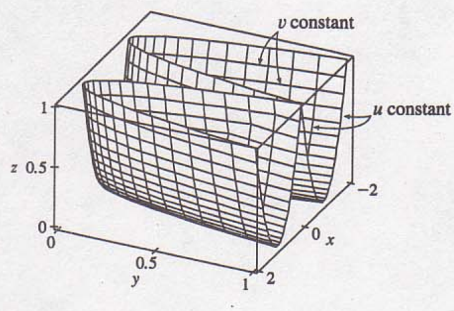
5. $\mathbf{r}(u,v) = \left\langle u^2+1, v^3+1, u+v\right\rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u^2 + 1$, $y = v^3 + 1$, $z = u + v$, $-1 \le u \le 1$, $-1 \le v \le 1$. If we keep u constant at u_0, $x = u_0^2 + 1$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^3 + 1$, a constant, so these grid curves are the curves parallel to the xz-plane.



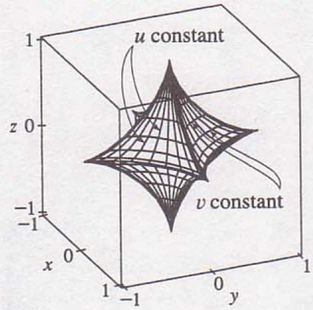
6. $\mathbf{r}(u,v) = \left\langle u+v, u^2, v^2\right\rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u + v$, $y = u^2$, $z = v^2$, $-1 \le u \le 1$, $-1 \le v \le 1$. If $u = u_0$ is constant, $y = u_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xz-plane. If $v = v_0$ is constant, $z = v_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.



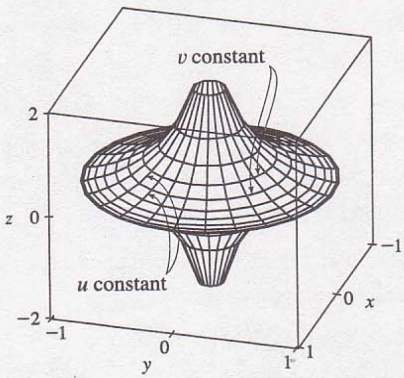
7. $\mathbf{r}(u,v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$.

The surface has parametric equations $x = \cos^3 u \cos^3 v$, $y = \sin^3 u \cos^3 v$, $z = \sin^3 v$, $0 \le u \le \pi$, $0 \le v \le 2\pi$. Note that if $v = v_0$ is constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curves must be the curves parallel to the xy-plane. The vertically oriented grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos^3 u_0 \cos^3 v$, $y = \sin^3 u_0 \cos^3 v$, $z = \sin^3 v$. These curves lie in vertical planes that contain the z-axis.



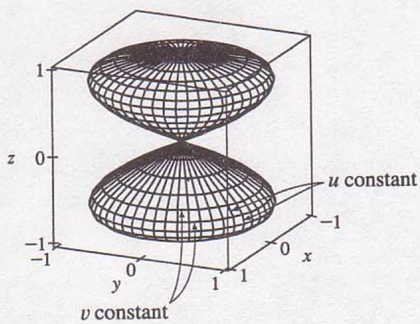
8. $\mathbf{r}(u,v) = \langle \cos u \sin v, \sin u \sin v, \cos v + \ln\tan(v/2) \rangle$.

The surface has parametric equations $x = \cos u \sin v$, $y = \sin u \sin v$, $z = \cos v + \ln\tan(v/2)$, $0 \le u \le 2\pi$, $0.1 \le v \le 6.2$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin v_0$, $y = \sin u \sin v_0$, $z = \cos v_0 + \ln\tan(v_0/2)$ which represent a circle of radius $\sin v_0$ in the plane $z = \cos v_0 + \ln\tan(v_0/2)$. So the circular grid curves we see lying horizontally are the grid curves with v constant. The vertically oriented grid curves correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin v$, $y = \sin u_0 \sin v$, $z = \cos v + \ln\tan(v/2)$. These curves lie in vertical planes that contain the z-axis.



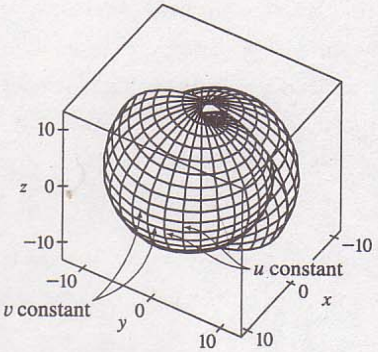
9. $x = \cos u \sin 2v$, $y = \sin u \sin 2v$, $z = \sin v$.

The complete graph of the surface is given by the parametric domain $0 \le u \le \pi$, $0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin 2v_0$, $y = \sin u \sin 2v_0$, $z = \sin v_0$ which represent a circle of radius $\sin 2v_0$ in the plane $z = \sin v_0$. So the circular grid curves we see lying horizontally are the grid curves which have v constant. The vertical grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin 2v$ and $y = \sin u_0 \sin 2v$ with $z = \sin v$ which has a "figure-eight" shape.



10. $x = u \sin u \cos v$, $y = u \cos u \cos v$, $z = u \sin v$.

We graph the portion of the surface with parametric domain $0 \le u \le 4\pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = u \sin u \cos v_0$, $y = u \cos u \cos v_0$, $z = u \sin v_0$. The equations for x and y show that the projections onto the xy-plane give a spiral shape, so the corresponding grid curves are the almost-horizontal spiral curves we see. The vertical grid curves, which look approximately circular, correspond to $u = u_0$ being held constant, giving $x = u_0 \sin u_0 \cos v$, $y = u_0 \cos u_0 \cos v$, $z = u_0 \sin v$.



11. $\mathbf{r}(u, v) = \cos v\, \mathbf{i} + \sin v\, \mathbf{j} + u\, \mathbf{k}$. The parametric equations for the surface are $x = \cos v$, $y = \sin v$, $z = u$. Then $x^2 + y^2 = \cos^2 v + \sin^2 v = 1$ and $z = u$ with no restriction on u, so we have a circular cylinder, graph IV. The grid curves with u constant are the horizontal circles we see in the plane $z = u$. If v is constant, both x and y are constant with z free to vary, so the corresponding grid curves are the lines on the cylinder parallel to the z-axis.

12. $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + u\, \mathbf{k}$. The parametric equations for the surface are $x = u \cos v$, $y = u \sin v$, $z = u$. Then $x^2 + y^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 = z^2$, which represents the equation of a cone with axis the z-axis, graph V. The grid curves with u constant are the horizontal circles we see, corresponding to the equations $x^2 + y^2 = u^2$ in the plane $z = u$. If v is constant, x, y, z are each scalar multiples of u, corresponding to the straight line grid curves through the origin.

13. $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$. The parametric equations for the surface are $x = u \cos v$, $y = u \sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $x = u^3$, $y = u \sin v$, $z = u \cos v$. Then $y^2 + z^2 = u^2 \sin v^2 + u^2 \cos v^2 = u^2$, so if u is held constant, each grid curve is a circle of radius u in the plane $x = u^3$. The graph then must be graph III. If v is held constant, so $v = v_0$, we have $y = u \sin v_0$ and $z = u \cos v_0$. Then $y = (\tan v_0)\, z$, so the grid curves we see running lengthwise along the surface in the planes $y = kz$ correspond to keeping v constant.

15. $x = (u - \sin u) \cos v$, $y = (1 - \cos u) \sin v$, $z = u$. If u is held constant, x and y give an equation of an ellipse in the plane $z = u$, thus the grid curves are horizontally oriented ellipses. Note that when $u = 0$, the "ellipse" is the single point $(0, 0, 0)$, and when $u = \pi$, we have $y = 0$ while x ranges from $-\pi$ to π, a line segment parallel to the x-axis in the plane $z = \pi$. This is the upper "seam" we see in graph II. When v is held constant, $z = u$ is free to vary, so the corresponding grid curves are the curves we see running up and down along the surface.

16. $x = (1 - u)(3 + \cos v) \cos 4\pi u$, $y = (1 - u)(3 + \cos v) \sin 4\pi u$, $z = 3u + (1 - u) \sin v$. These equations correspond to graph VI: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3, 0, 0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2} \cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2} \sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. From Example 3, parametric equations for the plane through the point $(1, 2, -3)$ that contains the vectors $\mathbf{a} = \langle 1, 1, -1 \rangle$ and $\mathbf{b} = \langle 1, -1, 1 \rangle$ are $x = 1 + u(1) + v(1) = 1 + u + v$, $y = 2 + u(1) + v(-1) = 2 + u - v$, $z = -3 + u(-1) + v(1) = -3 - u + v$.

18. Letting x and y be the parameters, parametric equations are $x = x$, $y = y$, $z = -\sqrt{1 + x^2 + y^2}$ (since the surface lies below the rectangle) where $-1 \le x \le 1$ and $-3 \le y \le 3$.
Alternate Solution: Using cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = -\sqrt{1 + r^2}$ where $-1 \le r\cos\theta \le 1$ and $-3 \le r\sin\theta \le 3$.

19. $x = x$, $y = 6 - 3x^2 - 2z^2$, $z = z$ where $3x^2 + 2z^2 \le 6$ since $y \ge 0$. Then the associated vector equation is $\mathbf{r}(x, y) = x\,\mathbf{i} + (6 - 3x^2 - 2z^2)\,\mathbf{j} + z\,\mathbf{k}$.

20. $x = 4 - y^2 - 2z^2$, $y = y$, $z = z$ where $y^2 + 2z^2 \le 4$ since $x \ge 0$. Then the associated vector equation is $\mathbf{r}(x, y) = (4 - y^2 - 2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

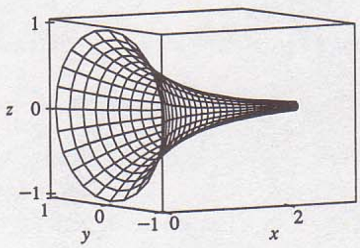
21. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = 2$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $2 \le x^2 + y^2 \le 4$.
Alternate Solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$.

22. In cylindrical coordinates, parametric equations are $x = \sin\theta$, $y = y$, $z = \cos\theta$, $0 \le \theta \le 2\pi$, $-1 \le y \le 3$.

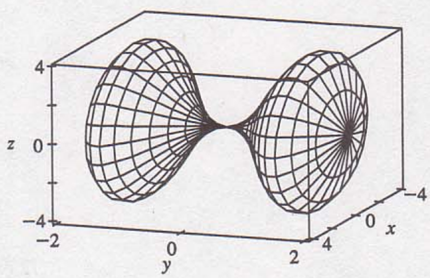
23. The surface is a disc with radius 4 and center $(0, 0, 5)$. Thus, $x = r\cos\theta$, $y = r\sin\theta$, $z = 5$ where $0 \le r \le 4$, $0 \le \theta \le 2\pi$ is a parametric representation of the surface.
Alternate Solution: In rectangular coordinates we could represent the surface as $x = x$, $y = y$, $z = 5$ where $0 \le x^2 + y^2 \le 16$.

24. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \le x^2 + y^2 \le 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ, we have $x = s\cos\theta$, $y = s\sin\theta$, $z = 3 + s\cos\theta$ where $0 \le s \le 1$ and $0 \le \theta \le 2\pi$.

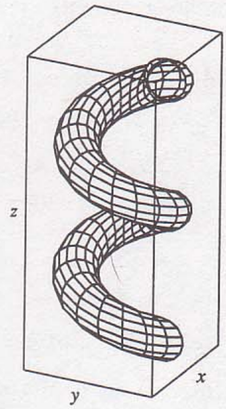
25. Using Equations 3, we have the parametrization $x = x$, $y = e^{-x}\cos\theta$, $z = e^{-x}\sin\theta$, $0 \le x \le 3$, $0 \le \theta \le 2\pi$.



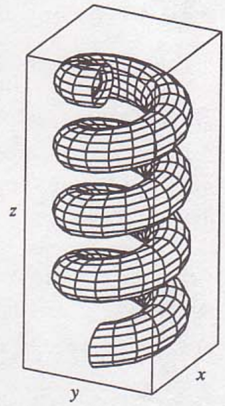
26. Letting θ be the angle of rotation about the y-axis, we have the parametrization $x = (4y^2 - y^4)\cos\theta$, $y = y$, $z = (4y^2 - y^4)\sin\theta$, $-2 \le y \le 2$, $0 \le \theta \le 2\pi$.



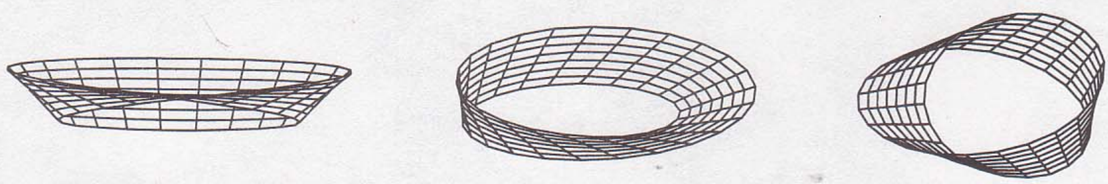
27. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \le u \le \pi$ while the original surface requires $0 \le u \le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.

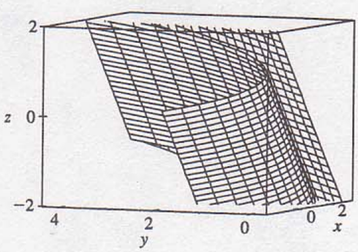


28. First we graph the surface as viewed from the front, then from two additional viewpoints.

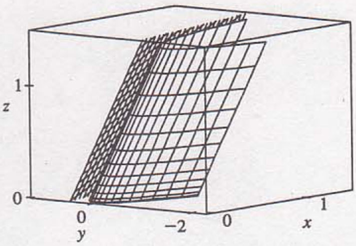


The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 17.7 [ET 16.7].)

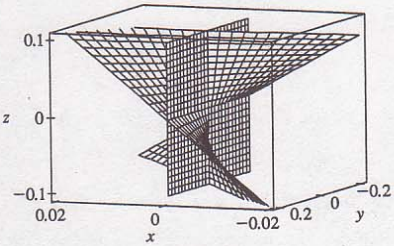
29. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u - v)\,\mathbf{k}$.
$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2, 3, 0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.



30. $\mathbf{r}(u,v) = \langle u^2, u - v^2, v^2 \rangle$.
$\mathbf{r}_u = \langle 2u, 1, 0 \rangle$ and $\mathbf{r}_v = \langle 0, -2v, 2v \rangle$, so
$\mathbf{r}_u \times \mathbf{r}_v = \langle 2v, -4uv, -4uv \rangle$. The point $(1, 0, 1)$ corresponds to
$u = 1$, $v = \pm 1$. So a normal vector to the surface at $(1, 0, 1)$ is
$\pm \langle 2, -4, -4 \rangle$ and an equation of the tangent plane is
$2x - 4y - 4z = -2$ or $x - 2y - 2z + 1 = 0$.



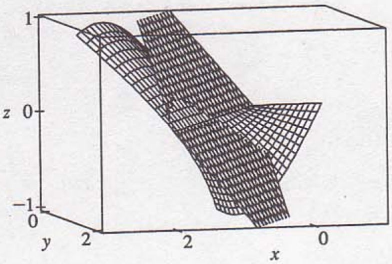
31. $\mathbf{r}(u,v) = uv\,\mathbf{i} + ue^v\,\mathbf{j} + ve^u\,\mathbf{k}$.
$\mathbf{r}_u = \langle v, e^v, ve^u \rangle$, $\mathbf{r}_v = \langle u, ue^v, e^u \rangle$, and
$\mathbf{r}_u \times \mathbf{r}_v = e^{u+v}(1 - uv)\,\mathbf{i} + e^u(uv - v)\,\mathbf{j} + e^v(uv - u)\,\mathbf{k}$. The point $(0, 0, 0)$ corresponds to $u = 0$, $v = 0$. Thus a normal vector to the surface at $(0, 0, 0)$ is $\mathbf{i}$, and an equation of the tangent plane is $x = 0$.



32. $\mathbf{r}(u,v) = (u + v)\,\mathbf{i} + u\cos v\,\mathbf{j} + v\sin u\,\mathbf{k}$.
$\mathbf{r}_u = \langle 1, \cos v, v\cos u \rangle$, $\mathbf{r}_v = \langle 1, -u\sin v, \sin u \rangle$, and

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v \sin u + uv \cos u \sin v,$$
$$v\cos u - \sin u, -u\sin v - \cos v \rangle$$

The point $(1, 1, 0)$ corresponds to $u = 1$, $v = 0$. Thus a normal vector to the surface at $(1, 1, 0)$ is $\langle \sin 1, -\sin 1, -1 \rangle$, and an equation of the tangent plane is $(\sin 1)\,x - (\sin 1)\,y - z = 0$.



33. Here $z = f(x,y) = 4 - x - 2y$ and D is the disk $x^2 + y^2 \le 4$. Thus, by Formula 9,
$A(S) = \iint_D \sqrt{1 + (-1)^2 + (-2)^2}\, dA = \sqrt{6} \iint_D dA = \sqrt{6}A(D) = 4\sqrt{6}\pi$.

34. $z = f(x,y) = x + y^2$ with $0 \le x \le y$, $0 \le y \le 1$. Thus, by Formula 9,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 1 + 4y^2}\, dA = \int_0^1 \int_0^y \sqrt{2 + 4y^2}\, dx\, dy = \int_0^1 \left[x\sqrt{2 + 4y^2} \right]_{x=0}^{x=y} dy \\ &= \int_0^1 y\sqrt{2 + 4y^2}\, dy = 2\left(\tfrac{1}{24}\right)\left(2 + 4y^2\right)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(6\sqrt{6} - 2\sqrt{2}\right) \\ &= \tfrac{3}{\sqrt{6}} - \tfrac{1}{3\sqrt{2}} \end{aligned}$$

35. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 \sqrt{1 + 4r^2}\, r\, dr \\ &= [\theta]_0^{2\pi} \left[\tfrac{1}{12}\left(1 + 4r^2\right)^{3/2} \right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

36. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \le y^2 + z^2 \le 9$. Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\mathbf{i} + \mathbf{j}) \times (2z\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$.

Note: In general, if $x = f(y, z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} - \dfrac{\partial f}{\partial z}\mathbf{k}$, and

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\, dA.$$

Then

$$\begin{aligned} A(S) &= \iint_{0 \le y^2 + z^2 \le 9} \sqrt{1 + 4y^2 + 4z^2}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr = 2\pi \left[\tfrac{1}{12}\left(1 + 4r^2\right)^{3/2}\right]_0^3 = \tfrac{\pi}{6}\left(37\sqrt{37} - 1\right) \end{aligned}$$

37. A parametric representation of the surface is $x = x$, $y = 4x + z^2$, $z = z$ with $0 \le x \le 1$, $0 \le z \le 1$. Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 4\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 4\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$.

Note: In general, if $y = f(x, z)$ then $\mathbf{r}_z \times \mathbf{r}_x = -\dfrac{\partial f}{\partial x}\mathbf{i} + \mathbf{j} - \dfrac{\partial f}{\partial x}\mathbf{k}$ and

$A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\, dA$. Then

$$\begin{aligned} A(S) &= \int_0^1 \int_0^1 \sqrt{17 + 4z^2}\, dx\, dz = \int_0^1 \sqrt{17 + 4z^2}\, dz \\ &= \tfrac{1}{2}\left(z\sqrt{17 + 4z^2} + \tfrac{17}{2} \ln\left|2z + \sqrt{4z^2 + 17}\right|\right)\Big]_0^1 = \tfrac{\sqrt{21}}{2} + \tfrac{17}{4}\left[\ln\left(2 + \sqrt{21}\right) - \ln\sqrt{17}\right] \end{aligned}$$

38. Let S_1 be that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$ by symmetry. On S_1, $z = \sqrt{a^2 - x^2}$ so $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2}} = \dfrac{a}{\sqrt{a^2 - x^2}}$. Hence

$A(S_1) = \displaystyle\iint_{0 \le x^2 + y^2 \le a^2} \frac{a}{\sqrt{a^2 - x^2}}\, dA = \int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2}}\, dy\, dx = \int_{-a}^{a} 2a\, dx = 4a^2$. Thus $A(S) = 8a^2$.

Alternate Solution: If $A(S_2)$ is the surface area in the first octant, then $A(S) = 8A(S_2)$. A parametric representation of the surface in the first octant is $x = a\sin\theta$, $y = y$, $z = a\cos\theta$ (θ being the angle in the xz-plane measured from the positive z-axis), where $0 \le \theta \le \frac{\pi}{2}$ and $0 \le y \le a\cos\theta$. The restrictions on y follow from: $x^2 + y^2 \le a^2$ or $a^2\sin^2\theta + y^2 \le a^2$ so $y^2 \le a^2\left(1 - \sin^2\theta\right)$; thus in the first octant $0 \le y \le a\cos\theta$. Then $\mathbf{r}_y \times \mathbf{r}_\theta = \langle -a\sin\theta, 0, -a\cos\theta \rangle$ and $A(S_2) = \int_0^{\pi/2} \int_0^{a\cos\theta} a\, dy\, d\theta = \int_0^{\pi/2} a^2 \cos\theta\, d\theta = a^2$. Hence $A(S) = 8a^2$.

39. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, $z = a\cos\phi$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$. For D, $0 \le \phi \le \frac{\pi}{2}$ and for each fixed ϕ, $\left(x - \frac{1}{2}a\right)^2 + y^2 \le \left(\frac{1}{2}a\right)^2$ or $\left[a\sin\phi\cos\theta - \frac{1}{2}a\right]^2 + a^2\sin^2\phi\sin^2\theta \le (a/2)^2$ implies $a^2\sin^2\phi - a^2\sin\phi\cos\theta \le 0$ or $\sin\phi\,(\sin\phi - \cos\theta) \le 0$. But $0 \le \phi \le \frac{\pi}{2}$, so $\cos\theta \ge \sin\phi$ or $\sin\left(\frac{\pi}{2} + \theta\right) \ge \sin\phi$ or $\phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi$. Hence $D = \left\{(\phi, \theta) \mid 0 \le \phi \le \frac{\pi}{2}, \phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi\right\}$. Then

$$\begin{aligned} A(S_1) &= \int_0^{\pi/2} \int_{\phi - (\pi/2)}^{(\pi/2) - \phi} a^2 \sin\phi\, d\theta\, d\phi = a^2 \int_0^{\pi/2} (\pi - 2\phi)\sin\phi\, d\phi \\ &= a^2\left[(-\pi\cos\phi) - 2(-\phi\cos\phi + \sin\phi)\right]_0^{\pi/2} = a^2(\pi - 2) \end{aligned}$$

Thus $A(S) = 2a^2(\pi - 2)$.

Alternate Solution: Working on S_1 we could parametrize the portion of the sphere by $x = x$, $y = y$, $z = \sqrt{a^2 - x^2 - y^2}$. Then $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$ and

$$A(S_1) = \iint_{0 \le (x - (a/2))^2 + y^2 \le (a/2)^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -a\left(a^2 - r^2\right)^{1/2}\Big]_{r=0}^{r=a\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} a^2\left[1 - \left(1 - \cos^2\theta\right)^{1/2}\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} a^2(1 - |\sin\theta|)\, d\theta = 2a^2 \int_0^{\pi/2} (1 - \sin\theta)\, d\theta = 2a^2\left(\tfrac{\pi}{2} - 1\right)$$

Thus $A(S) = 4a^2\left(\frac{\pi}{2} - 1\right) = 2a^2(\pi - 2)$.

Notes:

(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and then multiplying by 4. However, if you set up S_1 as above and arrived at $A(S_1) = a^2\pi$, you now see your error.

40. $\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$. Then

$$A(S) = \int_0^\pi \int_0^1 \sqrt{1 + u^2}\, du\, dv = \int_0^\pi dv \int_0^1 \sqrt{1 + u^2}\, du$$

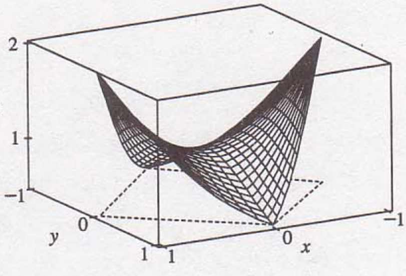
$$= \pi\left[\tfrac{u}{2}\sqrt{u^2 + 1} + \tfrac{1}{2}\ln\left|u + \sqrt{u^2 + 1}\right|\right]_0^1 = \tfrac{\pi}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

41. $\mathbf{r}_u = \langle v, 1, 1\rangle$, $\mathbf{r}_v = \langle u, 1, -1\rangle$ and $\mathbf{r}_u \times \mathbf{r}_v = \langle -2, u + v, v - u\rangle$. Then

$$A(S) = \iint_{u^2 + v^2 \le 1} \sqrt{4 + 2u^2 + 2v^2}\, dA = \int_0^{2\pi}\int_0^1 r\sqrt{4 + 2r^2}\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{4 + 2r^2}\, dr$$

$$= 2\pi\left[\tfrac{1}{6}\left(4 + 2r^2\right)^{3/2}\right]_0^1 = \tfrac{\pi}{3}\left(6\sqrt{6} - 8\right) = \pi\left(2\sqrt{6} - \tfrac{8}{3}\right)$$

42. Let $f(x, y) = \dfrac{1 + x^2}{1 + y^2}$. Then $f_x = \dfrac{2x}{1 + y^2}$, $f_y = (1 + x^2)\left[-\dfrac{2y}{(1 + y^2)^2}\right] = -\dfrac{2y(1 + x^2)}{(1 + y^2)^2}$. We use a CAS to estimate $\int_{-1}^1 \int_{-(1 - |x|)}^{1 - |x|} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx \approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1 - |x|) \le y \le 1 - |x|$ as the y-range in our plot command.



43. (a) The midpoints of the four squares are $\left(\frac{1}{4},\frac{1}{4}\right)$, $\left(\frac{1}{4},\frac{3}{4}\right)$, $\left(\frac{3}{4},\frac{1}{4}\right)$, and $\left(\frac{3}{4},\frac{3}{4}\right)$; the derivatives of the function $f(x,y)=x^2+y^2$ are $f_x(x,y)=2x$ and $f_y(x,y)=2y$, so the Midpoint Rule gives

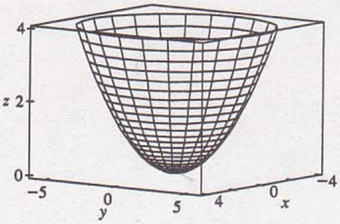
$$\begin{aligned} A(S) &= \int_0^1\int_0^1 \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\,dy\,dx \\ &\approx \tfrac{1}{4}\left(\sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2+\left[2\left(\tfrac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2+\left[2\left(\tfrac{3}{4}\right)\right]^2+1}\right. \\ &\qquad \left.+\sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2+\left[2\left(\tfrac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2+\left[2\left(\tfrac{3}{4}\right)\right]^2+1}\right) \\ &= \tfrac{1}{4}\left(\sqrt{\tfrac{3}{2}}+2\sqrt{\tfrac{7}{2}}+\sqrt{\tfrac{11}{2}}\right)\approx 1.8279 \end{aligned}$$

(b) A CAS estimates the integral to be $A(S)=\int_0^1\int_0^1\sqrt{f_x^2+f_y^2+1}\,dy\,dx=\int_0^1\int_0^1\sqrt{4x^2+4y^2+1}\,dy\,dx\approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

44. (a) $\mathbf{r}_u=a\cos v\,\mathbf{i}+b\sin v\,\mathbf{j}+2u\,\mathbf{k}$, $\mathbf{r}_v=-au\sin v\,\mathbf{i}+bu\cos v\,\mathbf{j}+0\,\mathbf{k}$, and $\mathbf{r}_u\times\mathbf{r}_v=-2bu^2\cos v\,\mathbf{i}-2au^2\sin v\,\mathbf{j}+abu\,\mathbf{k}$.
$A(S)=\int_0^{2\pi}\int_0^2|\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv=\int_0^{2\pi}\int_0^2\sqrt{4b^2u^4\cos^2 v+4a^2u^4\sin^2 v+a^2b^2u^2}\,du\,dv$

(b) $x^2=a^2u^2\cos^2 v$, $y^2=b^2u^2\sin^2 v$, $z=u^2 \;\Rightarrow\; x^2/a^2+y^2/b^2=u^2=z$ which is an elliptic paraboloid. To find D, notice that $0\le u\le 2 \;\Rightarrow\; 0\le z\le 4 \;\Rightarrow\; 0\le x^2/a^2+y^2/b^2\le 4$. Therefore, using Formula 9, we have $A(S)=\int_{-2a}^{2a}\int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}}\sqrt{1+(2x/a^2)^2+(2y/b^2)^2}\,dy\,dx$.

(c)



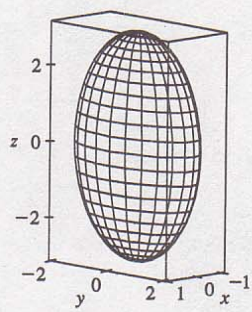
(d) We substitute $a=2$, $b=3$ in the integral in part (a) to get $A(S)=\int_0^{2\pi}\int_0^2 2u\sqrt{9u^2\cos^2 v+4u^2\sin^2 v+9}\,du\,dv$. We use a CAS to estimate the integral accurate to four decimal places. To speed up the calculation, we can set `Digits:=7;` (in Maple) or use the approximation command `N` (in Mathematica). We find that $A(S)\approx 115.6596$.

45. (a) $x=a\sin u\cos v$, $y=b\sin u\sin v$, $z=c\cos u \;\Rightarrow$

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}&=(\sin u\cos v)^2+(\sin u\sin v)^2+(\cos u)^2\\ &=\sin^2 u+\cos^2 u=1\end{aligned}$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.

(b)



(c) From the parametric equations (with $a=1$, $b=2$, and $c=3$), we calculate $\mathbf{r}_u=\cos u\cos v\,\mathbf{i}+2\cos u\sin v\,\mathbf{j}-3\sin u\,\mathbf{k}$ and

$\mathbf{r}_v = -\sin u \sin v\,\mathbf{i} + 2\sin u\cos v\,\mathbf{j}$. So $\mathbf{r}_u \times \mathbf{r}_v = 6\sin^2 u\cos v\,\mathbf{i} + 3\sin^2 u\sin v\,\mathbf{j} + 2\sin u\cos u\,\mathbf{k}$, and the surface area is given by

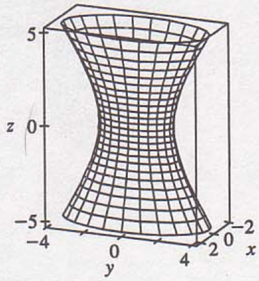
$$\begin{aligned} A(S) &= \textstyle\int_0^{2\pi}\int_0^{\pi} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \\ &= \textstyle\int_0^{2\pi}\int_0^{\pi} \sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\,du\,dv \end{aligned}$$

46. (a) $x = a\cosh u\cos v$, $y = b\cosh u\sin v$, $z = c\sinh u \quad\Rightarrow$

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} &= \cosh^2 u\cos^2 v + \cosh^2 u\sin^2 v - \sinh^2 u \\ &= \cosh^2 u - \sinh^2 u = 1 \end{aligned}$$

and the parametric equations represent a hyperboloid of one sheet.

(b)



(c) $\mathbf{r}_u = \sinh u\cos v\,\mathbf{i} + 2\sinh u\sin v\,\mathbf{j} + 3\cosh u\,\mathbf{k}$ and

$\mathbf{r}_v = -\cosh u\sin v\,\mathbf{i} + 2\cosh u\cos v\,\mathbf{j}$, so

$\mathbf{r}_u \times \mathbf{r}_v = -6\cosh^2 u\cos v\,\mathbf{i} - 3\cosh^2 u\sin v\,\mathbf{j} + 2\cosh u\sinh u\,\mathbf{k}$. We integrate between $u = \sinh^{-1}(-1) = -\ln(1+\sqrt{2})$ and $u = \sinh^{-1} 1 = \ln(1+\sqrt{2})$, since then z varies between -3 and 3, as desired. So the surface area is

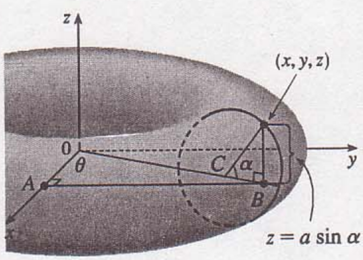
$$\begin{aligned} A(S) &= \textstyle\int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \\ &= \textstyle\int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} \sqrt{36\cosh^4 u\cos^2 v + 9\cosh^4 u\sin^2 v + 4\cosh^2 u\sinh^2 u}\,du\,dv \end{aligned}$$

47. $\mathbf{r}(u,v) = \langle \cos^3 u\cos^3 v, \sin^3 u\cos^3 v, \sin^3 v\rangle$, so $\mathbf{r}_u = \langle -3\cos^2 u\sin u\cos^3 v, 3\sin^2 u\cos u\cos^3 v, 0\rangle$,
$\mathbf{r}_v = \langle -3\cos^3 u\cos^2 v\sin v, -3\sin^3 u\cos^2 v\sin v, 3\sin^2 v\cos v\rangle$, and
$\mathbf{r}_u \times \mathbf{r}_v = \langle 9\cos u\sin^2 u\cos^4 v\sin^2 v, 9\cos^2 u\sin u\cos^4 v\sin^2 v, 9\cos^2 u\sin^2 u\cos^5 v\sin v\rangle$. Then

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= 9\sqrt{\cos^2 u\sin^4 u\cos^8 v\sin^4 v + \cos^4 u\sin^2 u\cos^8 v\sin^4 v + \cos^4 u\sin^4 u\cos^{10} v\sin^2 v} \\ &= 9\sqrt{\cos^2 u\sin^2 u\cos^8 v\sin^2 v\,(\sin^2 v + \cos^2 u\sin^2 u\cos^2 v)} \\ &= 9\cos^4 v\,|\cos u\sin u\sin v|\,\sqrt{\sin^2 v + \cos^2 u\sin^2 u\cos^2 v} \end{aligned}$$

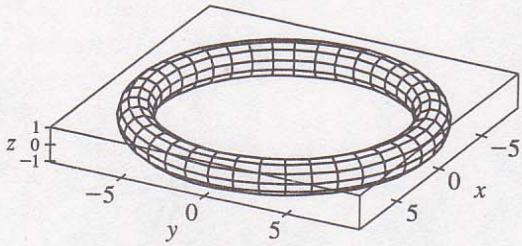
Using a CAS, we have
$A(S) = \int_0^{\pi}\int_0^{2\pi} 9\cos^4 v\,|\cos u\sin u\sin v|\,\sqrt{\sin^2 v + \cos^2 u\sin^2 u\cos^2 v}\,dv\,du \approx 4.4506$.

48. (a)

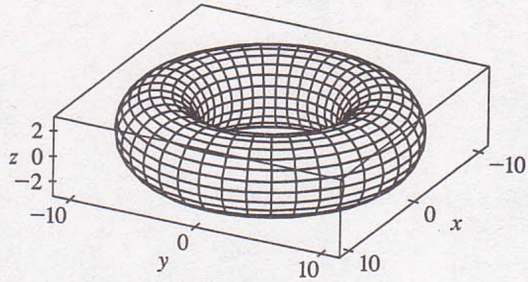


Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But $|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that $y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so $x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x = b\cos\theta + a\cos\alpha\cos\theta$, $y = b\sin\theta + a\cos\alpha\sin\theta$, $z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.

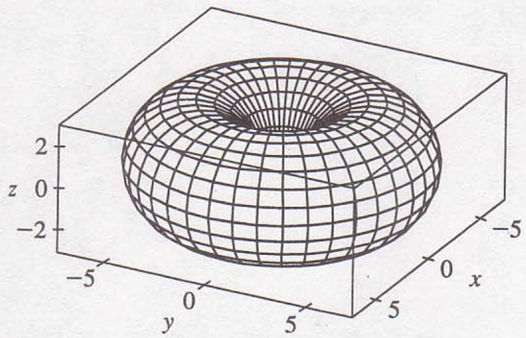
(b)



$a = 1, b = 8$



$a = 3, b = 8$



$a = 3, b = 4$

(c) $x = b\cos\theta + a\cos\alpha\cos\theta$, $y = b\sin\theta + a\cos\alpha\sin\theta$, $z = a\sin\alpha$, so
$\mathbf{r}_\alpha = \langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha\rangle$, $\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\cos\theta - a^2\cos\alpha\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\cos\theta - a^2\sin\alpha\cos^2\theta\right)\mathbf{j} \\ &\quad + \left(-ab\cos^2\alpha\sin\theta - a^2\cos^2\alpha\sin\theta\cos\theta - ab\sin^2\alpha\sin\theta - a^2\sin^2\alpha\sin\theta\cos\theta\right)\mathbf{k} \\ &= -a\,(b + a\cos\alpha)\left[(\cos\theta\cos\alpha)\,\mathbf{i} + (\sin\theta\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}\right]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a\,(b + a\cos\alpha)\sqrt{\cos^2\theta\cos^2\alpha + \sin^2\theta\cos^2\alpha + \sin^2\alpha} = a\,(b + a\cos\alpha)$.
Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence
$A(S) = \int_0^{2\pi}\int_0^{2\pi} a\,(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab$.

17.7 Surface Integrals ET 16.7

1. Each face of the cube has surface area $2^2 = 4$, and the points P_{ij}^* are the points where the cube intersects the coordinate axes. Here, $f(x, y, z) = \sqrt{x^2 + 2y^2 + 3z^2}$, so by Definition 1,

$$\begin{aligned}\iint_S f(x, y, z)\,dS &\approx [f(1, 0, 0)](4) + [f(-1, 0, 0)](4) + [f(0, 1, 0)](4) + [f(0, -1, 0)](4) \\ &\quad + [f(0, 0, 1)](4) + [f(0, 0, -1)](4) \\ &= 4\left(1 + 1 + 2\sqrt{2} + 2\sqrt{3}\right) = 8\left(1 + \sqrt{2} + \sqrt{3}\right) \approx 33.170\end{aligned}$$

2. Each quarter-cylinder has surface area $\frac{1}{4}\left[2\pi\,(1)\,(2)\right] = \pi$, and the top and bottom disks have surface area $\pi\,(1)^2 = \pi$. We can take $(0,0,1)$ as a sample point in the top disk, $(0,0,-1)$ in the bottom disk, and $(\pm1,0,0)$, $(0,\pm1,0)$ in the four quarter-cylinders. Then $\iint_S f\,(x,y,z)\;dS$ can be approximated by the Riemann sum $f\,(1,0,0)\,(\pi) + f\,(-1,0,0)\,(\pi) + f\,(0,1,0)\,(\pi) + f\,(0,-1,0)\,(\pi) + f\,(0,0,1)\,(\pi) + f\,(0,0,-1)\,(\pi) =$ $(2+2+3+3+4+4)\,\pi = 18\pi \approx 56.5$.

3. We can use the xz- and yz-planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S = \frac{1}{8}\,(4)\,\pi\left(\sqrt{50}\right)^2 = 25\pi$. Then $(\pm3,\pm4,5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\begin{aligned}\iint_H f\,(x,y,z)\;dS &\approx f\,(3,4,5)\,\Delta S + f\,(3,-4,5)\,\Delta S + f\,(-3,4,5)\,\Delta S + f\,(-3,-4,5)\,\Delta S\\ &= (7+8+9+12)\,(25\pi) = 900\pi \approx 2827\end{aligned}$$

4. On the surface, $f\,(x,y,z) = g\left(\sqrt{x^2+y^2+z^2}\right) = g\,(2) = -5$. So since the area of a sphere is $4\pi r^2$, $\iint_S f\,(x,y,z)\,dS = \iint_S g\,(2)\,dS = -5\iint_S dS = -5\left[4\pi\,(2)^2\right] = -80\pi$.

5. $z = 1+2x+3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 2,

$$\begin{aligned}\iint_S x^2yz\,dS &= \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA\\ &= \int_0^3\int_0^2 x^2y\,(1+2x+3y)\sqrt{4+9+1}\,dy\,dx\\ &= \sqrt{14}\int_0^3\int_0^2 \left(x^2y+2x^3y+3x^2y^2\right)dy\,dx\\ &= \sqrt{14}\int_0^3\left[\tfrac{1}{2}x^2y^2+x^3y^2+x^2y^3\right]_{y=0}^{y=2}dx\\ &= \sqrt{14}\int_0^3\left(10x^2+4x^3\right)dx = \sqrt{14}\left[\tfrac{10}{3}x^3+x^4\right]_0^3 = 171\sqrt{14}\end{aligned}$$

6. S is the region in the plane $2x+y+z=2$ or $z = 2-2x-y$ over $D = \{(x,y)\mid 0\le x\le1, 0\le y\le 2-2x\}$. Thus

$$\begin{aligned}\iint_S xy\,dS &= \iint_D xy\sqrt{(-2)^2+(-1)^2+1}\,dA\\ &= \sqrt{6}\int_0^1\int_0^{2-2x} xy\,dy\,dx = \sqrt{6}\int_0^1\left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=2-2x}dx\\ &= \tfrac{\sqrt{6}}{2}\int_0^1\left(4x-8x^2+4x^3\right)dx = \tfrac{\sqrt{6}}{2}\left(2-\tfrac{8}{3}+1\right) = \tfrac{\sqrt{6}}{6}\end{aligned}$$

7. S is the part of the plane $z = 1-x-y$ over the region $D = \{(x,y)\mid 0\le x\le1, 0\le y\le 1-x\}$. Thus

$$\begin{aligned}\iint_S yz\,dS &= \iint_D y\,(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\,dA\\ &= \sqrt{3}\int_0^1\int_0^{1-x}\left(y-xy-y^2\right)dy\,dx = \sqrt{3}\int_0^1\left[\tfrac{1}{2}y^2-\tfrac{1}{2}xy^2-\tfrac{1}{3}y^3\right]_{y=0}^{y=1-x}dx\\ &= \sqrt{3}\int_0^1\tfrac{1}{6}\,(1-x)^3\,dx = -\tfrac{\sqrt{3}}{24}\,(1-x)^4\Big]_0^1 = \tfrac{\sqrt{3}}{24}\end{aligned}$$

8. $z = \frac{2}{3}\left(x^{3/2}+y^{3/2}\right)$ and

$$\begin{aligned}\iint_S y\,dS &= \iint_D y\sqrt{\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+1}\,dA = \int_0^1\int_0^1 y\sqrt{x+y+1}\,dx\,dy\\ &= \int_0^1 y\left[\tfrac{2}{3}\,(x+y+1)^{3/2}\right]_{x=0}^{x=1}dy = \int_0^1 \tfrac{2}{3}y\left[(y+2)^{3/2}-(y+1)^{3/2}\right]dy\end{aligned}$$

Substituting $u = y + 2$ in the first term and $t = y + 1$ in the second, we have

$$\begin{aligned}\iint_S y\,dS &= \tfrac{2}{3}\int_2^3 (u-2)\,u^{3/2}\,du - \tfrac{2}{3}\int_1^2 (t-1)\,t^{3/2}\,dt \\ &= \tfrac{2}{3}\left[\tfrac{2}{7}u^{7/2} - \tfrac{4}{5}u^{5/2}\right]_2^3 - \tfrac{2}{3}\left[\tfrac{2}{7}t^{7/2} - \tfrac{2}{5}t^{5/2}\right]_1^2 \\ &= \tfrac{2}{3}\left[\tfrac{2}{7}\left(3^{7/2} - 2^{7/2}\right) - \tfrac{4}{5}\left(3^{5/2} - 2^{5/2}\right) - \tfrac{2}{7}\left(2^{7/2} - 1\right) + \tfrac{2}{5}\left(2^{5/2} - 1\right)\right] \\ &= \tfrac{2}{3}\left(\tfrac{18}{35}\sqrt{3} + \tfrac{8}{35}\sqrt{2} - \tfrac{4}{35}\right) = \tfrac{4}{105}\left(9\sqrt{3} + 4\sqrt{2} - 2\right)\end{aligned}$$

9. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + (x^2 + 4z)\,\mathbf{j} + z\,\mathbf{k}$, $0 \le x \le 2$, $0 \le z \le 2$. Then $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\,\mathbf{j}) \times (4\mathbf{j} + \mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2 + 17}$. Thus

$$\begin{aligned}\iint_S x\,dS &= \int_0^2\int_0^2 x\sqrt{4x^2+17}\,dx\,dz = \int_0^2 dz\,\int_0^2 x\sqrt{4x^2+17}\,dx \\ &= 2\left[\frac{1}{12}\left(4x^2+17\right)^{3/2}\right]_0^2 = \frac{33\sqrt{33} - 17\sqrt{17}}{6}\end{aligned}$$

10. $\mathbf{r}(y,z) = (4 - y^2 - z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $0 \le y^2 + z^2 \le 4$, so $\mathbf{r}_y \times \mathbf{r}_z = (-2y\mathbf{i} + \mathbf{j}) \times (-2z\mathbf{i} + \mathbf{k}) = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{4y^2 + 4z^2 + 1}$. Then

$$\begin{aligned}\iint_S \left(y^2 + z^2\right) dS &= \iint_{y^2+z^2\le 4} \left(y^2+z^2\right)\sqrt{4y^2+4z^2+1}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{4r^2+1}\,r\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta\,\int_0^2 r^3\sqrt{4r^2+1}\,dr\end{aligned}$$

Substituting $u = 4r^2 + 1$, so $du = 8r\,dr$ and $r = \frac{1}{4}(u-1)$, gives

$$\begin{aligned}\iint_S \left(y^2+z^2\right) dS &= 2\pi\int_1^{17} \tfrac{1}{8}\tfrac{1}{4}(u-1)\sqrt{u}\,du = \tfrac{\pi}{16}\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{17} \\ &= \tfrac{\pi}{16}\left[\tfrac{2}{5}\left(289\sqrt{17} - 1\right) - \tfrac{2}{3}\left(17\sqrt{17} - 1\right)\right] \\ &= \tfrac{\pi}{16}\left(\tfrac{1564}{15}\sqrt{17} + \tfrac{4}{15}\right) = \tfrac{\pi}{60}\left(391\sqrt{17} + 1\right)\end{aligned}$$

11. S is the part of the plane $z = y + 3$ over the disk $D = \{(x,y) \mid x^2 + y^2 \le 1\}$. Thus

$$\begin{aligned}\iint_S yz\,dS &= \iint_D y\,(y+3)\,\sqrt{(0)^2 + (1)^2 + 1}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^1 r\sin\theta\,(r\sin\theta + 3)\,r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\left[\tfrac{1}{4}r^4\sin^2\theta + r^3\sin\theta\right]_{r=0}^{r=1} d\theta = \sqrt{2}\int_0^{2\pi}\left(\tfrac{1}{4}\sin^2\theta + \sin\theta\right) d\theta \\ &= \sqrt{2}\left[\tfrac{1}{4}\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) - \cos\theta\right]_0^{2\pi} = \tfrac{\pi}{2\sqrt{2}}\end{aligned}$$

12. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 2$; and the back, S_3, in the plane $y = 0$. On S_1: using cylindrical coordinates, $\mathbf{r}(\theta, y) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + \cos\theta\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le y \le 2 - \sin\theta$, $|\mathbf{r}_\theta \times \mathbf{r}_y| = 1$ and $\iint_{S_1} xy\,dS = \int_0^{2\pi}\int_0^{2-\sin\theta} (\sin\theta)\,y\,dy\,d\theta = \int_0^{2\pi}\left[2\sin\theta - 2\sin^2\theta + \frac{1}{3}\sin^3\theta\right] d\theta = -2\pi$.
On S_2: $\mathbf{r}(x,z) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = |-\mathbf{i} - \mathbf{j}| = \sqrt{2}$, where $x^2 + z^2 \le 1$ and

$$\begin{aligned}\iint_{S_2} xy\,dS &= \iint_{x^2+z^2\le 1} x\,(2-x)\,\sqrt{2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{2}\left(2r\sin\theta - r^2\sin^2\theta\right) r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\left[\tfrac{2}{3}\sin\theta - \tfrac{1}{4}\sin^2\theta\right] d\theta = -\tfrac{\sqrt{2}}{4}\pi\end{aligned}$$

On S_3: $y = 0$ so $\iint_{S_3} xy\,dS = 0$. Hence $\iint_S xy\,dS = -2\pi - \frac{\sqrt{2}}{4}\pi = -\frac{1}{4}\left(8 + \sqrt{2}\right)\pi$.

13. Using spherical coordinates and Example 17.6.10 [ET 16.6.10] we have
$\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$. Then
$\iint_S (x^2z + y^2z)\,dS = \int_0^{2\pi}\int_0^{\pi/2} (4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta = 16\pi\sin^4\phi\big]_0^{\pi/2} = 16\pi.$

14. Using spherical coordinates, $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$, and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$ (see Example 17.6.10 [ET 16.6.10]). Then
$\iint_S xyz\,dS = \int_0^{2\pi}\int_0^{\pi/4} (\sin^3\phi\cos\phi\cos\theta\sin\theta)\,d\phi\,d\theta = 0$ since $\int_0^{2\pi}\cos\theta\sin\theta\,d\theta = 0$.

15. Using cylindrical coordinates, we have $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, and $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$.
$\iint_S (x^2y + z^2)\,dS = \int_0^{2\pi}\int_0^2 (27\cos^2\theta\sin\theta + z^2)\,3\,dz\,d\theta = \int_0^{2\pi}(162\cos^2\theta\sin\theta + 8)\,d\theta = 16\pi$

16. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.
On S_1: $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$,
$\iint_{S_1} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^2 (9 + z^2)\,3\,dz\,d\theta = 2\pi(54 + 8) = 124\pi.$
On S_2: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,
$\iint_{S_2} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 + 4)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4} + 18\right) = \frac{153}{2}\pi.$
On S_3: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,
$\iint_{S_3} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 + 0)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81}{2}\pi.$
Hence $\iint_S (x^2 + y^2 + z^2)\,dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi.$

17. $\mathbf{r}(u,v) = uv\,\mathbf{i} + (u+v)\,\mathbf{j} + (u-v)\,\mathbf{k}$, $u^2 + v^2 \le 1$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4 + 2u^2 + 2v^2}$ (see Exercise 17.6.41 [ET 16.6.41]). Then

$$\begin{aligned}\iint_S yx\,dS &= \iint_{u^2+v^2\le 1} (u^2 - v^2)\sqrt{4 + 2u^2 + 2v^2}\,dA = \int_0^{2\pi}\int_0^1 r^2(\cos^2\theta - \sin^2\theta)\sqrt{4 + 2r^2}\,r\,dr\,d\theta \\ &= \left[\int_0^{2\pi}(\cos^2\theta - \sin^2\theta)\,d\theta\right]\left[\int_0^1 r^3\sqrt{4 + 2r^2}\,dr\right] = 0\end{aligned}$$

since the first integral is 0.

18. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; \mathbf{r}_u \times \mathbf{r}_v = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\,\mathbf{k} \;\Rightarrow$
$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1 + u^2}$, so $\iint_S \sqrt{1 + x^2 + y^2}\,dS = \int_0^\pi\int_0^1 \sqrt{1 + u^2}\sqrt{1 + u^2}\,du\,dv = \frac{4}{3}\pi.$

19. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1] \times [0,1]$, so by Equation 8

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D [-xy(-2x) - yz(-2y) + zx]\,dA \\ &= \int_0^1\int_0^1 \left[2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2)\right]dy\,dx \\ &= \int_0^1 \left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right)dx = \tfrac{713}{180}\end{aligned}$$

20. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + 4x^2\,\mathbf{j} + yz\,\mathbf{k}$, $z = g(x,y) = xe^y$, and D is the square $[0,1] \times [0,1]$, so by Equation 8

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \left[-xy(e^y) - 4x^2(xe^y) + yz\right]dA = \int_0^1\int_0^1 (-xye^y - 4x^3e^y + xye^y)\,dy\,dx \\ &= \int_0^1 \left[-4x^3e^y\right]_{y=0}^{y=1}dx = (e-1)\int_0^1(-4x^3)\,dx = 1 - e\end{aligned}$$

21. $\mathbf{F}(x,y,z) = xze^y\,\mathbf{i} - xze^y\,\mathbf{j} + z\,\mathbf{k}$, $z = g(x,y) = 1 - x - y$, and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$.
Since S has downward orientation, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \left[-xze^y(-1) - (-xze^y)(-1) + z\right]dA = -\int_0^1\int_0^{1-x}(1 - x - y)\,dy\,dx \\ &= -\int_0^1 \left(\tfrac{1}{2}x^2 - x + \tfrac{1}{2}\right)dx = -\tfrac{1}{6}\end{aligned}$$

22. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z^4\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the disk $\{(x,y) \mid x^2+y^2 \le 1\}$. Since S has downward orientation, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \left[-x\left(\frac{x}{\sqrt{x^2+y^2}}\right) - y\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^4\right] dA \\ &= -\iint_D \left[\frac{-x^2-y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^4\right] dA = -\int_0^{2\pi}\int_0^1 \left(\frac{-r^2}{r} + r^4\right) r\,dr\,d\theta \\ &= -\int_0^{2\pi} d\theta \int_0^1 \left(r^5 - r^2\right) dr = -2\pi\left(\tfrac{1}{6} - \tfrac{1}{3}\right) = \frac{\pi}{3}\end{aligned}$$

23. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = 3\sin\phi\cos\theta\,\mathbf{i} + 3\sin\phi\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$ and
$\mathbf{r}_\phi \times \mathbf{r}_\theta = 9\sin^2\phi\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\cos\phi\,\mathbf{k}$. Then
$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 27\sin^3\phi\cos^2\theta + 27\sin^3\phi\sin^2\theta + 27\sin\phi\cos^2\phi = 27\sin\phi$ and
$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} 27\sin\phi\,d\phi\,d\theta = (2\pi)(54) = 108\pi$.

24. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = -4\sin\phi\sin\theta\,\mathbf{i} + 4\sin\phi\cos\theta\,\mathbf{j} + 12\cos\phi\,\mathbf{k}$ and
$\mathbf{r}_\phi \times \mathbf{r}_\theta = 16\sin^2\phi\cos\theta\,\mathbf{i} + 16\sin^2\phi\sin\theta\,\mathbf{j} + 16\sin\phi\cos\phi\,\mathbf{k}$. Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) &= -64\sin^3\phi\sin\theta\cos\theta + 64\sin^3\phi\sin\theta\cos\theta + 192\sin\phi\cos^2\phi \\ &= 192\sin\phi\cos^2\phi\end{aligned}$$

and $\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/2} 192\sin\phi\cos^2\phi\,d\phi\,d\theta = 2\pi\left[-64\cos^3\phi\right]_0^{\pi/2} = 128\pi$.

25. Let S_1 be the paraboloid $y = x^2+z^2$, $0 \le y \le 1$ and S_2 the disk $x^2+z^2 \le 1$, $y = 1$. Since S is a closed surface, we use the outward orientation. On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+z^2\le 1}\left[-(x^2+z^2) - 2z^2\right] dA = -\int_0^{2\pi}\int_0^1 \left(r^2 + 2r^2\cos^2\theta\right) r\,dr\,d\theta \\ &= -\int_0^{2\pi} \tfrac{1}{4}\left(1 + 2\cos^2\theta\right) d\theta = -\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) = -\pi\end{aligned}$$

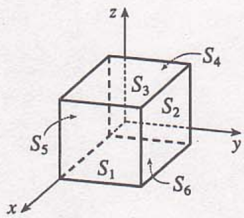
On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} (1)\,dA = \pi$. Hence
$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\pi + \pi = 0$.

26. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 2$; and the back, S_3, in the plane $y = 0$.
On S_1: $\mathbf{F}(\mathbf{r}(\theta,y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \quad\Rightarrow$

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{2-\sin\theta} \left(\sin^2\theta + 5\cos\theta\right) dy\,d\theta \\ &= \int_0^{2\pi} \left(2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\cos\theta\right) d\theta = 2\pi\end{aligned}$$

On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{i} + \mathbf{j}$.
$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} [x + (2-x)]\,dA = 2\pi$.
On S_3: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + 5\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ so $\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = 0$. Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\pi$.

27. Here S consists of the six faces of the cube as labeled in the figure. On S_1:



$\mathbf{F} = \mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = 4$;

S_2: $\mathbf{F} = x\mathbf{i} + 2\mathbf{j} + 3z\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 2\,dx\,dz = 8$;

S_3: $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + 3\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 3\,dx\,dy = 12$;

S_4: $\mathbf{F} = -\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 4$;

S_5: $\mathbf{F} = x\mathbf{i} - 2\mathbf{j} + 3z\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = 8$;

S_6: $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 3\,dx\,dy = 12$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^{6} \iint_{S_i} \mathbf{F} \cdot d\mathbf{S} = 48$.

28. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad \mathbf{r}_u \times \mathbf{r}_v = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(u,v)) = u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + v^2\mathbf{k}$. Then

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^\pi \int_0^1 \left(u\sin^2 v - u\cos^2 v + uv^2\right) du\,dv = \int_0^\pi \int_0^1 \left(-u\cos 2v + uv^2\right) du\,dv \\ &= \int_0^\pi \left[-\tfrac{1}{2}\cos 2v + \tfrac{1}{2}v^2\right] dv = \tfrac{1}{6}\pi^3.\end{aligned}$$

29. $z = xy \quad\Rightarrow\quad \partial z/\partial x = y$, $\partial z/\partial y = x$, so by Formula 2, a CAS gives

$$\iint_S xyz\,dS = \int_0^1\int_0^1 xy\,(xy)\,\sqrt{y^2 + x^2 + 1}\,dx\,dy \approx 0.1642.$$

30. As in Exercise 29, we use a CAS to calculate

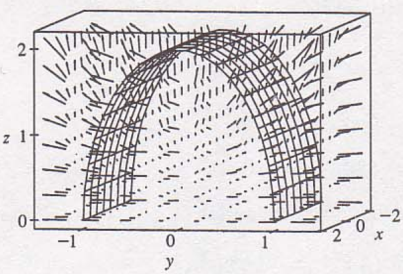
$$\begin{aligned}\iint_S x^2yz\,dS &= \int_0^1\int_0^1 x^2y\,(xy)\,\sqrt{y^2 + x^2 + 1}\,dx\,dy \\ &= \tfrac{1}{60}\sqrt{3} - \tfrac{1}{12}\ln\left(1 + \sqrt{3}\right) - \tfrac{1}{192}\ln\left(\sqrt{2} + 1\right) + \tfrac{317}{2880}\sqrt{2} + \tfrac{1}{24}\ln 2.\end{aligned}$$

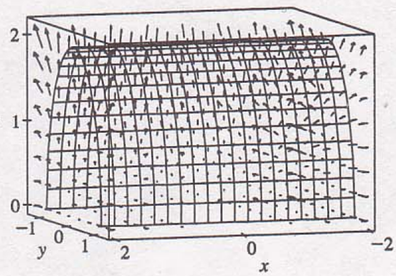
31. We use Formula 2 with $z = 3 - 2x^2 - y^2 \quad\Rightarrow\quad \partial z/\partial x = -4x$, $\partial z/\partial y = -2y$. The boundaries of the region $3 - 2x^2 - y^2 \geq 0$ are $-\sqrt{\frac{3}{2}} \leq x \leq \sqrt{\frac{3}{2}}$ and $-\sqrt{3 - 2x^2} \leq y \leq \sqrt{3 - 2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation takes a very long time) to calculate

$$\iint_S x^2y^2z^2\,dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}} \int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2y^2\left(3 - 2x^2 - y^2\right)^2 \sqrt{16x^2 + 4y^2 + 1}\,dy\,dx \approx 3.4895.$$

32. The flux of $\mathbf{F}$ across S is given by $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$. Now on S, $z = g(x,y) = 2\sqrt{1 - y^2}$, so $\partial g/\partial x = 0$ and $\partial g/\partial y = -2y\left(1 - y^2\right)^{-1/2}$. Therefore, by (8),

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_{-2}^{2}\int_{-1}^{1} \left(-x^2y\left[-2y\left(1 - y^2\right)^{-1/2}\right] + \left[2\sqrt{1 - y^2}\right]^2 e^{x/5}\right) dy\,dx \\ &= \tfrac{1}{3}\left(16\pi + 80e^{2/5} - 80e^{-2/5}\right)\end{aligned}$$





33. If S is given by $y = h(x, z)$, then S is also the level surface $f(x, y, z) = y - h(x, z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \dfrac{-h_x\,\mathbf{i} + \mathbf{j} - h_z\,\mathbf{k}}{\sqrt{h_x^2 + 1 + h_z^2}}$, and $-\mathbf{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (8), using Formula 2 to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \frac{\dfrac{\partial h}{\partial x}\mathbf{i} - \mathbf{j} + \dfrac{\partial h}{\partial z}\mathbf{k}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + 1 + \left(\dfrac{\partial h}{\partial z}\right)^2}} \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2}\, dA$$

where D is the projection of $f(x, y, z)$ onto the xz-plane. Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P\frac{\partial h}{\partial x} - Q + R\frac{\partial h}{\partial z}\right) dA.$$

34. If S is given by $x = k(y, z)$, then S is also the level surface $f(x, y, z) = x - k(y, z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \dfrac{\mathbf{i} - k_y\,\mathbf{j} - k_z\,\mathbf{k}}{\sqrt{1 + k_y^2 + k_z^2}}$, and since the x-component is positive this is the unit normal that points forward. Now we proceed as in the derivation of (8), using Formula 2 for

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \frac{\mathbf{i} - \dfrac{\partial k}{\partial y}\mathbf{j} - \dfrac{\partial k}{\partial z}\mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial k}{\partial y}\right)^2 + \left(\dfrac{\partial k}{\partial z}\right)^2}} \sqrt{1 + \left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2}\, dA\end{aligned}$$

where D is the projection of $f(x, y, z)$ onto the yz-plane. Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P - Q\frac{\partial k}{\partial y} - R\frac{\partial k}{\partial z}\right) dA.$$

35. $m = \iint_S K\, dS = K \cdot 4\pi\left(\frac{1}{2}a^2\right) = 2\pi a^2 K$; by symmetry $M_{xz} = M_{yz} = 0$, and $M_{xy} = \iint_S zK\, dS = K\int_0^{2\pi}\int_0^{\pi/2} (a\cos\phi)\left(a^2\sin\phi\right) d\phi\, d\theta = 2\pi K a^3\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \pi K a^3$. Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{2}a\right)$.

36. S is given by $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2 + y^2}{x^2 + y^2}} = \sqrt{2}$ so

$$\begin{aligned} m &= \iint_S \left(10 - \sqrt{x^2 + y^2}\right) dS = \iint_{1 \le x^2 + y^2 \le 16} \left(10 - \sqrt{x^2 + y^2}\right)\sqrt{2}\, dA \\ &= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10 - r)\, r\, dr\, d\theta = 2\pi\sqrt{2}\left[5r^2 - \tfrac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\pi \end{aligned}$$

37. (a) $I_z = \iint_S \left(x^2 + y^2\right)\rho(x, y, z)\, dS$

(b) $I_z = \iint_S \left(x^2 + y^2\right)\left(10 - \sqrt{x^2 + y^2}\right) dS = \iint_{1 \le x^2 + y^2 \le 16} \left(x^2 + y^2\right)\left(10 - \sqrt{x^2 + y^2}\right)\sqrt{2}\, dA$

$= \int_0^{2\pi}\int_1^4 \sqrt{2}\left(10r^3 - r^4\right) dr\, d\theta = 2\sqrt{2}\pi\left(\frac{4329}{10}\right) = \frac{4329}{5}\sqrt{2}\pi$

38. S is given by $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$.

(a) $m = \iint_S k\, dS = k\iint_{0 \le x^2 + y^2 \le a^2} \sqrt{2}\, dS = \sqrt{2}a^2 k\pi$; by symmetry $M_{xz} = M_{yz} = 0$, and $M_{xy} = \iint_S zk\, dS = k\int_0^{2\pi}\int_0^a \sqrt{2}r^2\, dr\, d\theta = \frac{2}{3}\sqrt{2}a^3 k\pi$. Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2}{3}a\right)$.

(b) $I_z = \iint_S \left(x^2 + y^2\right) k\, dS = \int_0^{2\pi}\int_0^a \sqrt{2}kr^3\, dr\, d\theta = 2\pi\sqrt{2}k\left(\frac{1}{4}a^4\right) = \frac{\sqrt{2}}{2}\pi k a^4$.

39. $\rho(x,y,z) = 1200$, $\mathbf{V} = y\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{F} = \rho\mathbf{V} = (1200)\,(y\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k})$. S is given by $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \left[9 - \frac{1}{4}\left(x^2+y^2\right)\right]\mathbf{k}$, $0 \le x^2+y^2 \le 36$ and $\mathbf{r}_x \times \mathbf{r}_y = \frac{1}{2}x\,\mathbf{i} + \frac{1}{2}y\,\mathbf{j} + \mathbf{k}$. Thus the rate of flow is given by

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{0\le x^2+y^2\le 36} (1200)\left(\tfrac{1}{2}xy + \tfrac{1}{2}y + \left[9 - \tfrac{1}{4}\left(x^2+y^2\right)\right]\right) dA \\ &= 1200\int_0^6\int_0^{2\pi}\left[\tfrac{1}{2}r^2\sin\theta\cos\theta + \tfrac{1}{2}r\sin\theta + 9 - \tfrac{1}{4}r^2\right] r\,d\theta\,dr \\ &= 1200\int_0^6 2\pi\left(9r - \tfrac{1}{4}r^3\right)dr = (1200)\,(2\pi)\,(81) = 194{,}400\pi\end{aligned}$$

40. $\rho(x,y,z) = 1500$, $\mathbf{F} = \rho\mathbf{V} = (1500)\,(-y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k})$. S is given by $\mathbf{r}(\phi,\theta) = 5\sin\phi\cos\theta\,\mathbf{i} + 5\sin\phi\sin\theta\,\mathbf{j} + 5\cos\phi\,\mathbf{k}$, $0\le\phi\le\pi$, $0\le\theta\le2\pi$, and $\mathbf{r}_\phi\times\mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\sin\phi\cos\phi\,\mathbf{k}$. Thus the rate of outward flow is

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= 1500\int_0^{2\pi}\int_0^{\pi}\left(-125\sin^3\phi\sin\theta\cos\theta + 125\sin^3\phi\sin\theta\cos\theta + 250\sin\phi\cos^2\phi\right)d\phi\,d\theta \\ &= (3000\pi)\,(250)\left(-\tfrac{1}{3}\cos^3\phi\right)\Big]_0^{\pi} = 500{,}000\pi.\end{aligned}$$

41. S consists of the hemisphere S_1 given by $z = \sqrt{a^2-x^2-y^2}$ and the disk S_2 given by $0\le x^2+y^2\le a^2$, $z=0$. On S_1: $\mathbf{E} = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + 2a\cos\phi\,\mathbf{k}$, $\mathbf{T}_\phi\times\mathbf{T}_\theta = a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$. Thus

$$\begin{aligned}\iint_{S_1}\mathbf{E}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2}\left(a^3\sin^3\phi + 2a^3\sin\phi\cos^2\phi\right)d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2}\left(a^3\sin\phi + a^3\sin\phi\cos^2\phi\right)d\phi\,d\theta = (2\pi)\,a^3\left(1+\tfrac{1}{3}\right) = \tfrac{8}{3}\pi a^3\end{aligned}$$

On S_2: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\mathbf{r}_y\times\mathbf{r}_x = -\mathbf{k}$ so $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S} = 0$. Hence the total charge is $q = \epsilon_0\iint_S\mathbf{E}\cdot d\mathbf{S} = \frac{8}{3}\pi a^3\epsilon_0$.

42. Referring to the figure in Exercise 27, on

S_1: $\mathbf{E} = \mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_y\times\mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1}\mathbf{E}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = 4$;

S_2: $\mathbf{E} = x\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dx\,dz = 4$;

S_3: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3}\mathbf{E}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dx\,dy = 4$;

S_4: $\mathbf{E} = -\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4}\mathbf{E}\cdot d\mathbf{S} = 4$.

Similarly $\iint_{S_5}\mathbf{E}\cdot d\mathbf{S} = \iint_{S_6}\mathbf{E}\cdot d\mathbf{S} = 4$. Hence $q = \epsilon_0\iint_S\mathbf{E}\cdot d\mathbf{S} = \epsilon_0\sum\limits_{i=1}^{6}\iint_{S_i}\mathbf{E}\cdot d\mathbf{S} = 24\epsilon_0$.

43. $K\nabla u = 6.5\,(4y\,\mathbf{j} + 4z\,\mathbf{k})$. S is given by $\mathbf{r}(x,\theta) = x\,\mathbf{i} + \sqrt{6}\cos\theta\,\mathbf{j} + \sqrt{6}\sin\theta\,\mathbf{k}$ and since we want the inward heat flow, we use $\mathbf{r}_x\times\mathbf{r}_\theta = -\sqrt{6}\cos\theta\,\mathbf{j} - \sqrt{6}\sin\theta\,\mathbf{k}$. Then the rate of heat flow inward is given by $\iint_S(-K\,\nabla u)\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^4 -(6.5)\,(-24)\,dx\,d\theta = (2\pi)\,(156)\,(4) = 1248\pi$.

44. $u(x,y,z) = c/\sqrt{x^2+y^2+z^2}$,

$$\mathbf{F} = -K\nabla u = -K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{i} - \frac{cy}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{j} - \frac{cz}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{k}\right]$$
$$= \frac{cK}{(x^2+y^2+z^2)^{3/2}}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

and the outward unit normal is $\mathbf{n} = \frac{1}{a}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$.

Thus $\mathbf{F}\cdot\mathbf{n} = \dfrac{cK}{a\,(x^2+y^2+z^2)^{3/2}}\,(x^2+y^2+z^2)$, but on S, $x^2+y^2+z^2 = a^2$ so $\mathbf{F}\cdot\mathbf{n} = \dfrac{cK}{a^2}$. Hence the rate of heat flow across S is $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{cK}{a^2}\iint_S dS = \frac{cK}{a^2}\left(4\pi a^2\right) = 4\pi Kc$.

17.8 Stokes' Theorem ET 16.8

1. Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2+y^2 = 4$, $z = 0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_P \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ (where C is the boundary curve).

2. The plane $z = 5$ intersects the paraboloid $z = 9 - x^2 - y^2$ in the circle $x^2+y^2 = 4$, $z = 5$. This boundary curve C is oriented in the counterclockwise direction, so the vector equation is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 10\sin t\,\mathbf{i} + 10\cos t\,\mathbf{j} + 4\cos t\sin t\,\mathbf{k}$, and by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}\left(-20\sin^2 t + 20\cos^2 t\right)dt$$
$$= 20\int_0^{2\pi}\cos 2t\,dt = 0$$

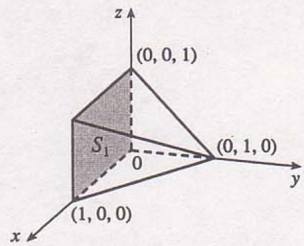
3. The boundary curve C is the circle $x^2+y^2 = 4$, $z = 0$ oriented in the counterclockwise direction. The vector equation is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$ and $\mathbf{F}(\mathbf{r}(t)) = (2\cos t)^2 e^{(2\sin t)(0)}\,\mathbf{i} + (2\sin t)^2 e^{(2\cos t)(0)}\,\mathbf{j} + (0)^2 e^{(2\cos t)(2\sin t)}\,\mathbf{k} = 4\cos^2 t\,\mathbf{i} + 4\sin^2 t\,\mathbf{j}$. Then, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}\left(-8\cos^2 t\sin t + 8\sin^2 t\cos t\right)dt$$
$$= 8\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = 0$$

4. C is the circle $y^2+z^2 = 4$, $x = \sqrt{5}$ with vector equation $\mathbf{r}(t) = \sqrt{5}\,\mathbf{i} + 2\cos t\,\mathbf{j} + 2\sin t\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t)) = \left[\sqrt{5} + \tan^{-1}(4\cos t\sin t)\right]\mathbf{i} + 8\cos^2 t\sin t\,\mathbf{j} + 2\sin t\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -16\cos^2 t\sin^2 t + 4\sin t\cos t = -2 + 2\cos 2t + 2\sin 2t$. Thus $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = 2\int_0^{2\pi}(-1 + \cos 2t + \sin 2t)\,dt = -4\pi$.

5. C is the square in the plane $z = -1$. By (3), $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl}\mathbf{F} = x^2 z\,\mathbf{i} + (xy - 2xyz)\,\mathbf{j} + (y - xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = y - xz = x + y$ on S_2, where $z = -1$. Thus $\iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1}(x+y)\,dx\,dy = 0$ so $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$.

6. Here S consists of the 4 sides of the pyramid but not the base in the xz-plane. Call the base S_1. Then $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the boundary of the base. To avoid calculating four line integrals, apply Stokes' Theorem again. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$. But $\operatorname{curl}\mathbf{F} = (2xy - e^z)\,\mathbf{i} - y^2\,\mathbf{j} - x\,\mathbf{k}$ and $\mathbf{n} = \mathbf{j}$, so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = -y^2 = 0$ on S_1, $\iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$ and $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$.



7. $\operatorname{curl}\mathbf{F} = -2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $x + y + z = 1$ over $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 17.7.8 [ET 16.7.8], we have $z = g(x, y) = 1 - x - y$, $P = -2z$, $Q = -2x$, $R = -2y$, and

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D [-(-2z)(-1) - (-2x)(-1) + (-2y)]\,dA \\ &= \int_0^1 \int_0^{1-x} (-2)\,dy\,dx = -2\int_0^1 (1 - x)\,dx = -1\end{aligned}$$

8. $\operatorname{curl}\mathbf{F} = e^x\,\mathbf{k}$ and S is the portion of the plane $2x + y + 2z = 2$ over $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x\}$. We orient S upward and use Equation 17.7.8 [ET 16.7.8] with $z = g(x, y) = 1 - x - \frac{1}{2}y$:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D (0 + 0 + e^x)\,dA = \int_0^1 \int_0^{2-2x} e^x\,dy\,dx \\ &= \int_0^1 (2 - 2x)\,e^x\,dx = \left[(2 - 2x)\,e^x + 2e^x\right]_0^1 \quad \text{(by integrating by parts)} \\ &= 2e - 4\end{aligned}$$

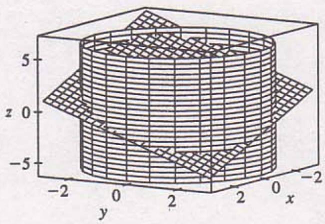
9. The curve of intersection is an ellipse in the plane $z = x + 4$ with unit normal $\mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$ and $\operatorname{curl}\mathbf{F} = 5\,\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k}$ so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = -\frac{1}{\sqrt{2}}$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = -\iint_S \frac{1}{\sqrt{2}}\,dS = -\frac{1}{\sqrt{2}}$ (surface area of planar ellipse) $= -\frac{1}{\sqrt{2}}\pi\,(2)\,(2\sqrt{2}) = -4\pi$. (Recall that the area of an ellipse with semiaxes a and b is πab.)

10. S is the part of the surface $z = 1 - x^2 - y^2$ in the first octant. $\operatorname{curl}\mathbf{F} = 2y\,\mathbf{i} - 2x\,\mathbf{j}$. Using Equation 17.7.8 [ET 16.7.8] with $g(x, y) = 1 - x^2 - y^2$, $P = 2y$, $Q = -2x$, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D [-2y(-2x) + (2x)(-2y)]\,dA = \iint_D 0\,dA = 0$.

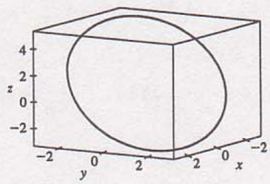
11. (a) The curve of intersection is an ellipse in the plane $x + y + z = 1$ with unit normal $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\operatorname{curl}\mathbf{F} = x^2\,\mathbf{j} + y^2\,\mathbf{k}$ and $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{3}}(x^2 + y^2)$. Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \tfrac{1}{\sqrt{3}}(x^2 + y^2)\,dS = \iint_{x^2+y^2\le 9} (x^2 + y^2)\,dx\,dy \\ &= \int_0^{2\pi}\int_0^3 r^3\,dr\,d\theta = 2\pi\left(\tfrac{81}{4}\right) = \tfrac{81\pi}{2}\end{aligned}$$

(b)



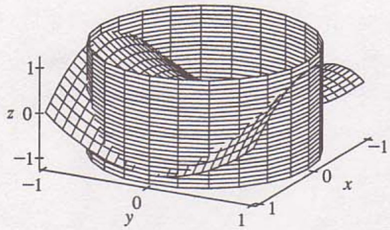
(c) One possible parametrization is $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3\cos t - 3\sin t$, $0 \le t \le 2\pi$.



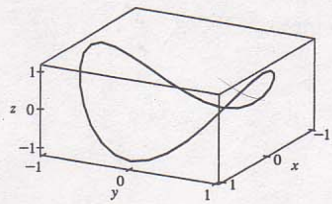
12. (a) S is the part of the surface $z = y^2 - x^2$ that lies above the unit disk D.
$\operatorname{curl}\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (x^2 - x^2)\,\mathbf{k} = x\,\mathbf{i} - y\,\mathbf{j}$. Using Equation 17.7.8 [ET 16.7.8] with $g(x,y) = y^2 - x^2$, $P = x$, $Q = -y$, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D [-x(-2x) - (-y)(2y)]\,dA = 2\iint_D (x^2+y^2)\,dA\\ &= 2\int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta = 2(2\pi)\left[\tfrac14 r^4\right]_0^1 = \pi\end{aligned}$$

(b)



(c) One possible set of parametric equations is $x = \cos t$, $y = \sin t$, $z = \sin^2 t - \cos^2 t$, $0 \le t \le 2\pi$.



13. The boundary curve C is the circle $x^2 + y^2 = 9$, $z = 0$ oriented in the counterclockwise direction as viewed from $(0,0,1)$. Then $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{F}(\mathbf{r}(t)) = 9\sin t\,\mathbf{i} - 18\cos t\,\mathbf{k}$ and $\mathbf{F}\cdot\mathbf{r}'(t) = -27\sin^2 t$. Thus $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-27\sin^2 t)\,dt = -27\pi$. Now $\operatorname{curl}\mathbf{F} = -4\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}$, so

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+y^2\le 9}(-8x + 12y - 3)\,dA = \int_0^{2\pi}\int_0^3(-8r\cos\theta + 12r\sin\theta - 3)\,r\,dr\,d\theta\\ &= \int_0^3(-3r)(2\pi)\,dr = -27\pi\end{aligned}$$

14. C: $x^2 + y^2 = a^2$, $z = 0$, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(a^2\sin t\cos t)(-a\sin t)\,dt = -\frac13 a^3\sin^3 t\big]_0^{2\pi} = 0$. Then $\operatorname{curl}\mathbf{F} = -y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \dfrac{x}{(a^2-x^2-y^2)^{1/2}}\,\mathbf{i} + \dfrac{y}{(a^2-x^2-y^2)^{1/2}}\,\mathbf{j} + \mathbf{k}$. Hence

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+y^2\le a^2}\left[-\frac{yx}{(a^2-x^2-y^2)^{1/2}} - y - x\right]dA\\ &= -\int_0^a\int_0^{2\pi}\left[\frac{r^2\cos\theta\sin\theta}{\sqrt{a^2-r^2}} + r\sin\theta + r\cos\theta\right]r\,d\theta\,dr = 0\end{aligned}$$

since $\int_0^{2\pi}\sin\theta\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\cos\theta\sin\theta\,d\theta = 0$. Notice that for this reason, it's much easier to integrate with respect to θ first.

15. The x-, y-, and z-intercepts of the plane are all 1, so C consists of the three line segments C_1: $\mathbf{r}_1(t) = (1-t)\,\mathbf{i} + t\,\mathbf{j}$, $0 \le t \le 1$, C_2: $\mathbf{r}_2(t) = (1-t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 1$, and C_3: $\mathbf{r}_3(t) = t\,\mathbf{i} + (1-t)\,\mathbf{k}$, $0 \le t \le 1$. Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1[t\,\mathbf{i} + (1-t)\,\mathbf{k}]\cdot(-\mathbf{i}+\mathbf{j})\,dt + \int_0^1[(1-t)\,\mathbf{i} + t\,\mathbf{j}]\cdot(-\mathbf{j}+\mathbf{k})\,dt + \int_0^1[(1-t)\,\mathbf{j} + t\,\mathbf{k}]\cdot(\mathbf{i}-\mathbf{k})\,dt\\ &= \int_0^1(-3t)\,dt = -\tfrac32\end{aligned}$$

Now $\operatorname{curl}\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^{1-x}(-3)\,dy\,dx = -\frac32$.

16. The components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$, and both the curve C and the surface S meet the requirements of Stokes' Theorem. If there is a vector field $\mathbf{G}$ where $\mathbf{F} = \operatorname{curl} \mathbf{G}$, then Stokes' Theorem says $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{G} \cdot d\mathbf{S}$ depends only on the values of $\mathbf{G}$ on C, and hence is independent of the choice of S. By Theorem 17.5.11 [ET 16.5.11], $\operatorname{div} \operatorname{curl} \mathbf{G} = 0$, so $\operatorname{div} \mathbf{F} = 0 \;\Leftrightarrow$
$(3ax^2 - 3z^2) + (x^2 + 3by^2) + (3cz^2) = 0 \;\Leftrightarrow\; (3a+1)\,x^2 + 3by^2 + (3c-3)\,z^2 = 0 \;\Leftrightarrow$
$a = -\frac{1}{3}, b = 0, c = 1$.

17. $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^x + z^2 & y^y + x^2 & z^z + y^2 \end{vmatrix} = 2y\,\mathbf{i} + 2z\,\mathbf{j} + 2x\,\mathbf{k}$ and $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

To parametrize the surface, let $x = 2\cos\theta \sin\phi$, $y = 2\sin\theta\sin\phi$, $z = 2\cos\phi$, so that
$\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$, and
$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\,\mathbf{i} + 4\sin^2\phi\sin\theta\,\mathbf{j} + 4\sin\phi\cos\phi\,\mathbf{k}$. Then
$\operatorname{curl} \mathbf{F}\,(\mathbf{r}(\phi,\theta)) = 4\sin\phi\sin\theta\,\mathbf{i} + 4\cos\phi\,\mathbf{j} + 4\sin\phi\cos\theta\,\mathbf{k}$, and
$\operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\sin\theta\cos\theta + 16\cos\phi\sin^2\phi\sin\theta + 16\sin^2\phi\cos\phi\cos\theta$. Therefore

$$\begin{aligned}
\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA \\
&= 16\left[\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta\right]\left[\int_0^{\pi/2} \sin^3\phi\,d\phi\right] + 16\left[\int_0^{\pi/2} \sin\theta\,d\theta\right]\left[\int_0^{\pi/2} \sin^2\phi\cos\phi\,d\phi\right] \\
&\qquad + 16\left[\int_0^{\pi/2} \cos\theta\,d\theta\right]\left[\int_0^{\pi/2} \sin^2\phi\cos\phi\,d\phi\right] \\
&= 8\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2} + 16\,(1)\left[\tfrac{1}{3}\sin^3\phi\right]_0^{\pi/2} + 16\,(1)\left[\tfrac{1}{3}\sin^3\phi\right]_0^{\pi/2} \\
&= 8\left[0 + 1 + 0 - \tfrac{1}{3}\right] + 16\left(\tfrac{1}{3}\right) + 16\left(\tfrac{1}{3}\right) = \tfrac{16}{3} + \tfrac{16}{3} + \tfrac{16}{3} = 16
\end{aligned}$$

18. $\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz = \int_C \mathbf{F} \cdot d\mathbf{r}$, where
$\mathbf{F}(x,y,z) = (y + \sin x)\,\mathbf{i} + (z^2 + \cos y)\,\mathbf{j} + x^3\,\mathbf{k} \;\Rightarrow\; \operatorname{curl} \mathbf{F} = -2z\,\mathbf{i} - 3x^2\,\mathbf{j} - \mathbf{k}$. Since
$\sin 2t = 2\sin t\cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto the xy-plane is the unit disk D ($x^2 + y^2 \le 1$). C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 17.7.8 [ET 16.7.8] with $g(x,y) = 2xy$,
$P = -2\,(2xy) = -4xy$, $Q = -3x^2$, $R = -1$, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= -\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -\iint_D \left[-(-4xy)(2y) - (-3x^2)(2x) - 1\right] dA \\
&= -\iint_D \left(8xy^2 + 6x^3 - 1\right) dA = -\int_0^{2\pi}\int_0^1 \left(8r^3\cos\theta\sin^2\theta + 6r^3\cos^3\theta - 1\right) r\,dr\,d\theta \\
&= -\int_0^{2\pi} \left(\tfrac{8}{5}\cos\theta\sin^2\theta + \tfrac{6}{5}\cos^3\theta - \tfrac{1}{2}\right) r\,dr\,d\theta \\
&= -\left[\tfrac{8}{15}\sin^3\theta + \tfrac{6}{5}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right) - \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi
\end{aligned}$$

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S. Then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{H_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} + \iint_{H_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ so $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ as desired.

20. (a) By Exercise 17.5.26 [ET 16.5.26], $\operatorname{curl}(f\nabla g) = f\operatorname{curl}(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ since $\operatorname{curl}(\nabla g) = \mathbf{0}$. Hence by Stokes' Theorem $\int_C (f\nabla g)\cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g)\cdot d\mathbf{S}$.

(b) As in (a), $\operatorname{curl}(f\nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f\nabla f)\cdot d\mathbf{r} = \iint_S [\operatorname{curl}(f\nabla f)]\cdot d\mathbf{S} = 0$.

(c) As in (a),

$$\begin{aligned}\operatorname{curl}(f\nabla g + g\nabla f) &= \operatorname{curl}(f\nabla g) + \operatorname{curl}(g\nabla f) \qquad \text{(by Exercise 17.5.24 [ET 16.5.24])}\\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} \qquad [\text{since } \mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})]\end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g + g\nabla f)\cdot d\mathbf{S} = 0$.

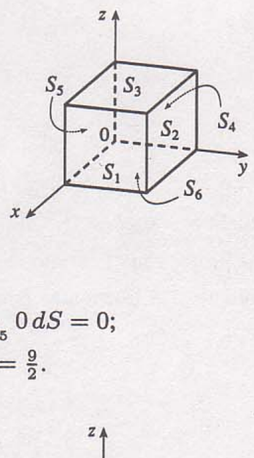
17.9 The Divergence Theorem ET 16.9

1. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

2. (a) The vectors that end near P_1 are shorter than the vectors that start near P_1, so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2, so the net flow is inward and P_2 is a sink.

(b) $\mathbf{F}(x,y) = \langle x, y^2\rangle \Rightarrow \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 1 + 2y$. The y-value at P_1 is positive, so $\operatorname{div}\mathbf{F} = 1 + 2y$ is positive, thus P_1 is a source. At P_2, $y < -1$, so $\operatorname{div}\mathbf{F} = 1 + 2y$ is negative, and P_2 is a sink.

3. $\operatorname{div}\mathbf{F} = 3 + x + 2x = 3 + 3x$, so



$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (3x+3)\,dx\,dy\,dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, on S_1: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and

$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3$;

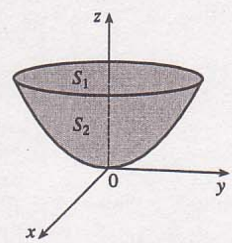
S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$ and $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} x\,dS = \frac{1}{2}$;

S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_3} 2x\,dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4} \mathbf{F}\cdot d\mathbf{S} = 0$; S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_5} 0\,dS = 0$;

S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_6} 0\,dS = 0$. Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{9}{2}$.

4. $\operatorname{div}\mathbf{F} = 8z$, so



$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 8zr\,dz\,dr\,d\theta = 2\pi\int_0^1 (4r - 4r^5)\,dr = \frac{8}{3}\pi$.

On S_1: $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3\pi$.

S_2: $\mathbf{F} = (x^3 + xy^2)\,\mathbf{i} + (y^3 + yx^2)\,\mathbf{j} + 3(x^2+y^2)^2\,\mathbf{k}$,

$-(\mathbf{r}_x \times \mathbf{r}_y) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}$ and

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+y^2\le 1} (-x^4 - y^4 - 2x^2y^2)\,dA\\ &= -\int_0^{2\pi}\int_0^1 r^5\,dr\,d\theta = -\frac{\pi}{3}\end{aligned}$$

Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = 3\pi - \frac{\pi}{3} = \frac{8}{3}\pi$.

5. $\operatorname{div}\mathbf{F} = x + y + z$, so

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^1 (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r^2\cos\theta + r^2\sin\theta + \tfrac{1}{2}r\right)dr\,d\theta$$
$$= \int_0^{2\pi}\left(\tfrac{1}{3}\cos\theta + \tfrac{1}{3}\sin\theta + \tfrac{1}{4}\right)d\theta = \tfrac{1}{4}(2\pi) = \tfrac{\pi}{2}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge.
On S_1, $z = 1$, $\mathbf{n} = \mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$, so
$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} x\,dS = \int_0^{2\pi}\int_0^1 (r\cos\theta)\,r\,dr\,d\theta = [\sin\theta]_0^{2\pi}\left[\tfrac{1}{3}r^3\right]_0^1 = 0$. On S_2, $z = 0$, $\mathbf{n} = -\mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i}$ so $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} 0\,dS = 0$. S_3 is given by $\mathbf{r}(\theta, z) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 1$. Then $\mathbf{r}_\theta \times \mathbf{r}_z = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and

$$\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_\theta\times\mathbf{r}_z)\,dA = \int_0^{2\pi}\int_0^1\left(\cos^2\theta\sin\theta + z\sin^2\theta\right)dz\,d\theta$$
$$= \int_0^{2\pi}\left(\cos^2\theta\sin\theta + \tfrac{1}{2}\sin^2\theta\right)d\theta = \left[-\tfrac{1}{3}\cos^3\theta + \tfrac{1}{4}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \tfrac{\pi}{2}$$

Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = 0 + 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

6. $\operatorname{div}\mathbf{F} = 1 + 1 + 1 = 3$, so $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 3\,dV = 3\,(\text{volume of ball}) = 3\left(\frac{4}{3}\pi\right) = 4\pi$. To find $\iint_S \mathbf{F}\cdot d\mathbf{S}$ we use spherical coordinates. S is the unit sphere, represented by $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$ (see Example 17.6.10 [ET 16.6.10]) and $\mathbf{F}(\mathbf{r}(\phi,\theta)) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$. Thus

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,dA = \int_0^{2\pi}\int_0^{\pi}\left(\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi\right)d\phi\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi = (2\pi)(2) = 4\pi$$

7. $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(3y^2z^3) + \dfrac{\partial}{\partial y}(9x^2yz^2) + \dfrac{\partial}{\partial z}(4xy^2) = 9x^2z^2$, so by the Divergence Theorem,
$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 9x^2z^2\,dV = \int_{-1}^1\int_{-1}^1\int_{-1}^1 9x^2z^2\,dx\,dy\,dz = 8$.

8. $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(x^2y) + \dfrac{\partial}{\partial y}(-x^2z) + \dfrac{\partial}{\partial z}(z^2y) = 2xy + 2zy$, so by the Divergence Theorem,
$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (2xy + 2yz)\,dV = \int_0^1\int_0^2\int_0^3 (2xy + 2yz)\,dx\,dy\,dz = 24$.

9. $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(-xz) + \dfrac{\partial}{\partial y}(-yz) + \dfrac{\partial}{\partial z}(z^2) = -z - z + 2z = 0$,
so $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 0\,dV = 0$.

10. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (5y)\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 5y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[5(1-x)y - 5y^2\right]dy\,dx$
$$= \int_0^1\left[\tfrac{5}{2}(1-x)^3 - \tfrac{5}{3}(1-x)^3\right]dx = \tfrac{5}{24}$$

11. $\operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (3y^2 + 3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 \left(3r^2\cos^2\theta + 3r^2\sin^2\theta\right)r\,dx\,dr\,d\theta$$
$$= 3\int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_{-1}^2 dx = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2}$$

12. $\operatorname{div}\mathbf{F} = 3x^2y - 2x^2y - x^2y = 0$, so $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

13. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^4\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\frac{3}{5}\sin\phi\,d\phi = \frac{12}{5}\pi$

14. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r^3\,dz\,dr\,d\theta = 2\pi\int_0^2 \left(12r^3 - 3r^5\right)dr = 32\pi$

15. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 2y\,dV = \iint_{x^2+y^2\le 9}\int_{y-3}^{0} 2y\,dz\,dA = \int_0^{2\pi}\int_0^3\int_{-3+r\sin\theta}^{0} \left(2r^2\sin\theta\right) dz\,dr\,d\theta$

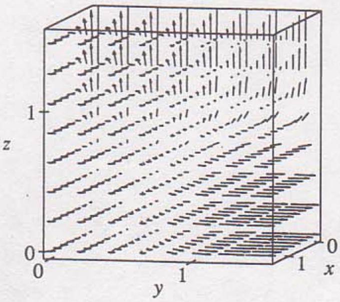
$$= \int_0^{2\pi}\int_0^3 \left(6r^2\sin\theta - 2r^3\sin^2\theta\right) dr\,d\theta = \int_0^{2\pi}\left[54\sin\theta - \tfrac{81}{2}\sin^2\theta\right] d\theta = -\tfrac{81}{2}\pi$$

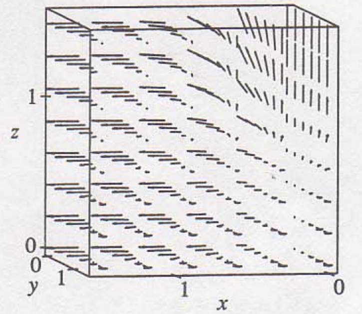
16. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3\left(x^2+y^2+1\right) dV = \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 3\left(\rho^2\sin^2\phi+1\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$= 2\pi\int_0^{\pi/2}\left[\tfrac{93}{5}\sin^3\phi + 7\sin\phi\right] d\phi = 2\pi\left[\tfrac{93}{5}\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) - 7\cos\phi\right]_0^{\pi/2} = \tfrac{194}{5}\pi$$

17. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \sqrt{3-x^2}\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_0^{2-x^4-y^4}\sqrt{3-x^2}\,dz\,dy\,dx = \frac{341}{60}\sqrt{2} + \frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

18.





By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of the cube is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\left[\cos x\cos^2 y + 3\sin^2 y\cos y\cos^4 z + 5\sin^4 z\cos z\cos^6 x\right] dz\,dy\,dx = \tfrac{19}{64}\pi^2$$

19. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{k}) = -x^2z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D\left(-y^2\right) dA = -\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\, r\,dr\,d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since

$\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}\left(z^2x\right) + \dfrac{\partial}{\partial y}\left(\tfrac{1}{3}y^3 + \tan z\right) + \dfrac{\partial}{\partial z}\left(x^2z + y^2\right) = z^2 + y^2 + x^2$, we use spherical coordinates to get

$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi$. Finally

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi$.

20. As in the hint to Exercise 19, we create a closed surface $S_2 = S\cup S_1$, where S is the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2 + y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F}\cdot(-\mathbf{k}) = -z = -1$ on S_1. So

$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}(-1)\,dS = -A\left(S_1\right) = -\pi$. Let E be the region bounded by S_2. Then

$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV = \int_0^1\int_0^{2\pi}\int_1^{2-r^2} r\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi}\left(r - r^3\right) d\theta\,dr$

$$= (2\pi)\tfrac{1}{4} = \tfrac{\pi}{2}.$$

Thus the flux of $\mathbf{F}$ across S is $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

21. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2+y^2+z^2\right)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right) = \dfrac{\left(x^2+y^2+z^2\right) - 3x^2}{\left(x^2+y^2+z^2\right)^{5/2}}$ with similar

expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3\left(x^2+y^2+z^2\right) - 3\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{5/2}} = 0$, except at $(0, 0, 0)$ where it is undefined.

22. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \left(2x+2y+z^2\right) dS$, so $\mathbf{F}\cdot\mathbf{n} = 2x+2y+z^2$. But for S, $\mathbf{n} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i}+2\,\mathbf{j}+z\,\mathbf{k}$ and $\operatorname{div}\mathbf{F} = 1$. If $B = \left\{(x,y,z) \mid x^2+y^2+z^2 \le 1\right\}$, then $\iint_S \left(2x+2y+z^2\right) dS = \iiint_B dV = V(B) = \frac{4}{3}\pi\,(1)^3 = \frac{4}{3}\pi$.

23. $\iint_S \mathbf{a}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{a}\,dV = 0$ since $\operatorname{div}\mathbf{a} = 0$.

24. $\frac{1}{3}\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\iiint_E \operatorname{div}\mathbf{F}\,dV = \frac{1}{3}\iiint_E 3\,dV = V(E)$

25. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

26. $\iint_S D_{\mathbf{n}} f\,dS = \iint_S (\nabla f\cdot\mathbf{n})\,dS = \iiint_E \operatorname{div}(\nabla f)\,dV = \iiint_E \nabla^2 f\,dV$

27. $\iint_S (f\nabla g)\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}(f\nabla g)\,dV = \iiint_E \left(f\nabla^2 g + \nabla g\cdot\nabla f\right) dV$ by Exercise 17.5.25 [ET 16.5.25].

28. $\iint_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_E \left[\left(f\nabla^2 g + \nabla g\cdot\nabla f\right) - \left(g\nabla^2 f + \nabla g\cdot\nabla f\right)\right] dV$ (by Exercise 27). But $\nabla g\cdot\nabla f = \nabla f\cdot\nabla g$, so that $\iint_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_E \left(f\nabla^2 g - g\nabla^2 f\right) dV$.

29. If $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1\,\mathbf{i} + fc_2\,\mathbf{j} + fc_3\,\mathbf{k}$. Then $\operatorname{div}\mathbf{F} = \operatorname{div} f\mathbf{c} = \dfrac{\partial f}{\partial x}c_1 + \dfrac{\partial f}{\partial y}c_2 + \dfrac{\partial f}{\partial z}c_3 = \nabla f\cdot\mathbf{c}$ and the divergence theorem says $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV \;\Rightarrow\; \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{c}\,dV$. In particular, if $\mathbf{c} = \mathbf{i}$ then $\iint_S f\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{i}\,dV \;\Rightarrow\; \displaystyle\iint_S fn_1\,dS = \iiint_E \frac{\partial f}{\partial x}\,dV$ (where $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$). Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\displaystyle\iint_S fn_2\,dS = \iiint_E \frac{\partial f}{\partial y}\,dV$, and $\mathbf{c} = \mathbf{k}$ gives $\displaystyle\iint_S fn_3\,dS = \iiint_E \frac{\partial f}{\partial z}\,dV$. Then

$$\begin{aligned}
\textstyle\iint_S f\mathbf{n}\,dS &= \left(\textstyle\iint_S fn_1\,dS\right)\mathbf{i} + \left(\textstyle\iint_S fn_2\,dS\right)\mathbf{j} + \left(\textstyle\iint_S fn_3\,dS\right)\mathbf{k} \\
&= \left(\iiint_E \frac{\partial f}{\partial x}\,dV\right)\mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y}\,dV\right)\mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z}\,dV\right)\mathbf{k} \\
&= \iiint_E \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) dV = \iiint_E \nabla f\,dV
\end{aligned}$$

as desired.

30. By Exercise 29, $\iint_S p\mathbf{n}\,dS = \iiint_E \nabla p\,dV$, so

$$\begin{aligned}
\mathbf{F} &= -\textstyle\iint_S p\mathbf{n}\,dS = -\iiint_E \nabla p\,dV = -\iiint_E \nabla(\rho g z)\,dV = -\iiint_E (\rho g\,\mathbf{k})\,dV \\
&= -\rho g\left(\textstyle\iiint_E dV\right)\mathbf{k} = -\rho g V(E)\,\mathbf{k}
\end{aligned}$$

But the weight of the displaced liquid is volume × density × $g = \rho g V(E)$, thus $\mathbf{F} = -W\mathbf{k}$ as desired.

17 Review

ET 16

CONCEPT CHECK

1. See Definitions 1 and 2 in Section 17.1 [ET 16.1]. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f.

(b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

3. (a) See Definition 17.2.2 [ET 16.2.2].

(b) We normally evaluate the line integral using Formula 17.2.3 [ET 16.2.3].

(c) The mass is $m = \int_C \rho(x, y)\, ds$, and the center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{1}{m}\int_C x\rho(x, y)\, ds$, $\overline{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds$.

(d) See (5) and (6) in Section 17.2 [ET 16.2] for plane curves; we have similar definitions when C is a space curve (see the equation preceding (10) in Section 17.2 [ET 16.2]).

(e) For plane curves, see Equations 17.2.7 [ET 16.2.7]. We have similar results for space curves (see the equation preceding (10) in Section 17.2 [ET 16.2]).

4. (a) See Definition 17.2.13 [ET 16.2.13].

(b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

(c) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

5. See Theorem 17.3.2 [ET 16.3.2].

6. (a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

(b) See Theorem 17.3.4 [ET 16.3.4].

7. See the statement of Green's Theorem on page 1102 [ET 1068].

8. See Equations 17.4.5 [ET 16.4.5].

9. (a) $\operatorname{curl} \mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}$

(b) $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$

(c) For $\operatorname{curl} \mathbf{F}$, see the discussion accompanying Figure 1 on page 1112 [ET 1078] as well as Figure 6 and the accompanying discussion on page 1142 [ET 1108]. For $\operatorname{div} \mathbf{F}$, see the discussion following Example 5 on page 1113 [ET 1079] as well as the discussion preceding (8) on page 1149 [ET 1115].

10. See Theorem 17.3.6 [ET 16.3.6]; see Theorem 17.5.4 [ET 16.5.4].

11. (a) See (1) and (2) and the accompanying discussion in Section 17.6 [ET 16.6]; See Figure 4 and the accompanying discussion on page 1118 [ET 1084].

(b) See Definition 17.6.6 [ET 16.6.6].

(c) See Equation 17.6.9 [ET 16.6.9].

12. (a) See (1) in Section 17.7 [ET 16.7].

(b) We normally evaluate the surface integral using Formula 17.7.3 [ET 16.7.3].

(c) See Formula 17.7.2 [ET 16.7.2].

(d) The mass is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\, dS$, $\overline{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\, dS$, $\overline{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\, dS$.

13. (a) See Figures 7 and 8 and the accompanying discussion in Section 17.7 [ET 16.7]. A Möbius strip is a nonorientable surface; see Figures 5 and 6 and the accompanying discussion on page 1131 [ET 1097].

(b) See Definition 17.7.7 [ET 16.7.7].

(c) See Formula 17.7.9 [ET 16.7.9].

(d) See Formula 17.7.8 [ET 16.7.8].

14. See the statement of Stokes' Theorem on page 1139 [ET 1105].

15. See the statement of the Divergence Theorem on page 1145 [ET 1111].

16. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

TRUE-FALSE QUIZ

1. False; div $\mathbf{F}$ is a scalar field.

2. True. (See Definition 17.5.1 [ET 16.5.1].)

3. True, by Theorem 17.5.3 [ET 16.5.3] and the fact that div $\mathbf{0} = 0$.

4. True, by Theorem 17.3.2 [ET 16.3.2].

5. False. See Exercise 17.3.33 [ET 16.3.33]. (But the assertion is true if D is simply-connected; see Theorem 17.3.6 [ET 16.3.6].)

6. False. See the discussion accompanying Figure 8 on page 1086 [ET 1052].

7. True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

8. False by Theorem 17.5.11 [ET 16.5.11], because if it were true, then div curl $\mathbf{F} = 3 \neq 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

(b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and div $\mathbf{F}\,(P)$ is positive.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \leq x \leq 1$ so

$\int_C x\, ds = \int_0^1 x\sqrt{1 + (2x)^2}\, dx = \frac{1}{12}\left(1 + 4x^2\right)^{3/2}\Big]_0^1 = \frac{1}{12}\left(5\sqrt{5} - 1\right)$.

3. $\int_C x^3 z\, ds = \int_0^{\pi/2} (2\sin t)^3 (2\cos t)\sqrt{(2\cos t)^2 + (1)^2 + (-2\sin t)^2}\, dt = \int_0^{\pi/2} \left(16\sin^3 t\cos t\right)\sqrt{5}\, dt$

$= 4\sqrt{5}\sin^4 t\Big]_0^{\pi/2} = 4\sqrt{5}$

4. $\int_C xy\, dx + y\, dy = \int_0^{\pi/2} (x\sin x + \sin x\cos x)\, dx = -x\cos x + \sin x - \frac{1}{4}\cos 2x\Big]_0^{\pi/2} = \frac{3}{2}$

5. $x = \cos t \Rightarrow dx = -\sin t\, dt$, $y = \sin t \Rightarrow dy = \cos t\, dt$, $0 \leq t \leq 2\pi$ and

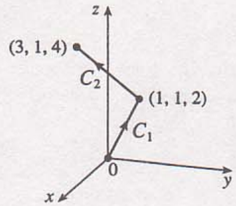
$\int_C x^3 y\, dx - x\, dy = \int_0^{2\pi} \left(-\cos^3 t\sin^2 t - \cos^2 t\right) dt = \int_0^{2\pi} \left(-\cos^3 t\sin^2 t - \cos^2 t\right) dt = -\pi$

Or: Since C is a simple closed curve, apply Green's Theorem giving

$\iint_{x^2+y^2\leq 1} \left(-1 - x^3\right) dA = \int_0^1 \int_0^{2\pi} \left(-r - r^4\cos^3\theta\right) d\theta = -\pi$.

6. $\int_C x\sin y\,dx + xyz\,dz = \int_0^1 \left(t\sin t^2 + 3t^8\right) dt = -\frac{1}{2}\cos t^2 + \frac{1}{3}t^9\Big]_0^1 = \frac{5}{6} - \frac{1}{2}\cos 1$

7.



C_1: $x = t$, $y = t$, $z = 2t$, $0 \le t \le 1$;

C_2: $x = 1 + 2t$, $y = 1$, $z = 2 + 2t$, $0 \le t \le 1$.

Then $\int_C y\,dx + z\,dy + x\,dz = \int_0^1 5t\,dt + \int_0^1 (4 + 4t)\,dt = \frac{17}{2}$.

8. $\mathbf{F}(\mathbf{r}(t)) = -t^7\,\mathbf{i} + e^{-t^3}\,\mathbf{j}$, $\mathbf{F}\cdot\mathbf{r}'(t) = -2t^8 - 3t^2e^{-t^3}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(-2t^8 - 3t^2e^{-t^3}\right) dt = -\frac{2}{9}t^9 + e^{-t^3}\Big]_0^1 = e^{-1} - \frac{11}{9}$.

9. $\mathbf{F}(\mathbf{r}(t)) = \left(2t + t^2\right)\mathbf{i} + t^4\,\mathbf{j} + 4t^4\,\mathbf{k}$, $\mathbf{F}\cdot\mathbf{r}'(t) = 4t + 2t^2 + 2t^5 + 16t^7$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(4t + 2t^2 + 2t^5 + 16t^7\right) dt = 5$.

10. (a) C: $x = 3 - 3t$, $y = \frac{\pi}{2}t$, $z = 3t$, $0 \le t \le 1$. Then

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[3t\,\mathbf{i} + (3 - 3t)\,\mathbf{j} + \tfrac{\pi}{2}t\,\mathbf{k}\right]\cdot\left[-3\,\mathbf{i} + \tfrac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right] dt = \int_0^1 \left[-9t + \tfrac{3\pi}{2}\right] dt$$
$$= \tfrac{1}{2}(3\pi - 9)$$

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k})\cdot(-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt$

$$= \int_0^{\pi/2} \left(-9\sin^2 t + 3\cos t + 3t\cos t\right) dt$$
$$= \left[-\tfrac{9}{2}(t - \sin t\cos t) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2} = -\tfrac{9\pi}{4} + 3 + \tfrac{3\pi}{2} - 3 = -\tfrac{3\pi}{4}$$

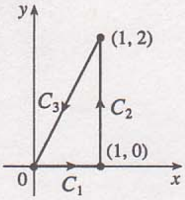
11. $\dfrac{\partial(\sin y)}{\partial y} = \cos y = \dfrac{\partial(x\cos y + \sin y)}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Hence there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = \sin y$ implies $f(x, y) = x\sin y + g(y)$ and $f_y(x, y) = x\cos y + g'(y)$. But $f_y(x, y) = x\cos y + \sin y$, so $g'(y) = \sin y$ and $f(x, y) = x\sin y - \cos y + K$ is a potential function for $\mathbf{F}$.

12. $\operatorname{curl}\mathbf{F} = (2y - 2y)\,\mathbf{i} + (2z - 2z)\,\mathbf{j} + \left(6xy^2 - 6xy^2\right)\mathbf{k} = \mathbf{0}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, so $\mathbf{F}$ is conservative, by Theorem 17.5.4 [ET 16.5.4]. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = 2xy^3 + z^2$ implies $f(x, y, z) = x^2y^3 + xz^2 + g(y, z)$ and $f_y(x, y, z) = 3x^2y^2 + g_y(y, z)$. But $f_y(x, y, z) = 3x^2y^2 + 2yz$ so $g(y, z) = y^2z + h(z)$. Then $f(x, y, z) = x^2y^3 + xz^2 + y^2z + h(z)$ implies $f_z(x, y, z) = 2xz + y^2 + h'(z)$. But $f_z(x, y, z) = y^2 + 2xz$ so $h'(z) = 0$. So a potential function for $\mathbf{F}$ is $f(x, y, z) = x^2y^3 + xz^2 + y^2z + K$.

13. Since $\dfrac{\partial\left(2x + y^2 + 3x^2y\right)}{\partial y} = 2y + 3x^2 = \dfrac{\partial\left(2xy + x^3 + 3y^2\right)}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x, y) = x^2 + xy^2 + x^3y + y^3 + K$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(\pi, 0) - f(0, 0) = \pi^2$.

14. Let $f(x, y, z) = x^2yz + xy^2z + K$. Then $\nabla f = (2xyz + y^2z)\,\mathbf{i} + (x^2z + 2xyz)\,\mathbf{j} + (2xy + y^2x)\,\mathbf{k} = \mathbf{F}$ so $\mathbf{F}$ is conservative. Now $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{r}(1) = 2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 3, 4) - f(1, 1, 1) = 118$.

15.



C_1: $0 \le x \le 1$, $y = 0$; C_2: $x = 1$, $0 \le y \le 2$; C_3: $x = x$, $y = 2x$, $x = 1$ to $x = 0$. Then $\int_C xy\,dx + x^2\,dy = \int_0^1 0\,dx + \int_0^2 (0 + 1)\,dy + \int_1^0 (2x^2 + 2x^2)\,dx = \frac{2}{3}$. Using Green's Theorem, we have

$$\begin{aligned}\int_C xy\,dx + x^2\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(x^2) - \tfrac{\partial}{\partial y}(xy)\right] dA = \iint_D (2x - x)\,dA \\ &= \int_0^1 \int_0^{2x} x\,dy\,dx = \tfrac{2}{3}\end{aligned}$$

16.
$$\begin{aligned}\int_C (1 + \tan x)\,dx + (x^2 + e^y)\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(x^2 + e^y) - \tfrac{\partial}{\partial y}(1 + \tan x)\right] dA \\ &= \iint_D (2x - 0)\,dA = \int_0^1 \int_0^{\sqrt{x}} 2x\,dy\,dx = \tfrac{4}{5}\end{aligned}$$

17.
$$\begin{aligned}\int_C x^2y\,dx - xy^2\,dy &= \iint_{x^2 + y^2 \le 4} \left[\tfrac{\partial}{\partial x}(-xy^2) - \tfrac{\partial}{\partial y}(x^2y)\right] dA \\ &= \iint_{x^2 + y^2 \le 4} (-y^2 - x^2)\,dA = -\int_0^{2\pi} \int_0^2 r^3\,dr\,d\theta = -8\pi\end{aligned}$$

18. $\operatorname{curl} \mathbf{F} = (-2z \sin y - 0)\,\mathbf{i} + (x^2 - 0)\,\mathbf{j} + (2 \sin y - 0)\,\mathbf{k} = -2z \sin y\,\mathbf{i} + x^2\,\mathbf{j} + 2 \sin y\,\mathbf{k}$,
$\operatorname{div} \mathbf{F} = 2xz + 2x \cos y + 2 \cos y$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl} \mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}&\mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} \\ &\quad= \left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right] \\ &\qquad - \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j} + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

and

$$\begin{aligned}(\mathbf{G} \cdot \nabla)\,\mathbf{F} - (\mathbf{F} \cdot \nabla)\,\mathbf{G} = \Bigg[&\left(P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\frac{\partial Q_1}{\partial x} + Q_2\frac{\partial Q_1}{\partial y} + R_2\frac{\partial Q_1}{\partial z}\right)\mathbf{j} \\ &+ \left(P_2\frac{\partial R_1}{\partial x} + Q_2\frac{\partial R_1}{\partial y} + R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}\Bigg] \\ - \Bigg[&\left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\frac{\partial Q_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j} \\ &+ \left(P_1\frac{\partial R_2}{\partial x} + Q_1\frac{\partial R_2}{\partial y} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}\Bigg]\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} \\
&= \left[\left(P_1\frac{\partial Q_2}{\partial y} + Q_2\frac{\partial P_1}{\partial x}\right) - \left(P_2\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial P_2}{\partial y}\right)\right. \\
&\qquad\qquad \left. - \left(P_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial P_2}{\partial z}\right) + \left(P_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i} \\
&\quad + \left[\left(Q_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial Q_2}{\partial z}\right)\right. \\
&\qquad\qquad \left. - \left(P_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial P_1}{\partial x}\right) + \left(P_2\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j} \\
&\quad + \left[\left(P_2\frac{\partial R_1}{\partial x} + R_1\frac{\partial P_2}{\partial x}\right) - \left(P_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial P_1}{\partial x}\right)\right. \\
&\qquad\qquad \left. - \left(Q_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\frac{\partial R_1}{\partial y} + R_1\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k} \\
&= \left[\frac{\partial}{\partial y}(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}(P_2R_1 - P_1R_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}(P_1Q_2 - P_2Q_1)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}(Q_1R_2 - Q_2R_1)\right]\mathbf{k} \\
&= \operatorname{curl}(\mathbf{F}\times\mathbf{G})
\end{aligned}$$

21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x,y) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx + g(y)\,dy = \iint_D \left[\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right] dA = \iint_D 0\,dA = 0.$

22.
$$\begin{aligned}
\nabla^2(fg) &= \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\right) \quad \text{(Product Rule)} \\
&= \frac{\partial^2 f}{\partial x^2}g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}g + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} \\
&\qquad + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}g + 2\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2} \quad \text{(Product Rule)} \\
&= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle\cdot\left\langle\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z}\right\rangle \\
&= f\nabla^2 g + g\nabla^2 f + 2\nabla f\cdot\nabla g
\end{aligned}$$

Another Method: Using the rules in Exercises 15.6.35(b) [ET 14.6.35(b)] and 17.5.25 [ET 16.5.25], we have

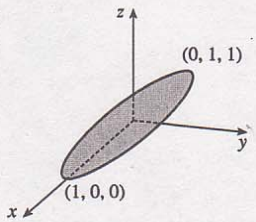
$$\begin{aligned}\nabla^2(fg) &= \nabla\cdot\nabla(fg) = \nabla\cdot(g\nabla f + f\nabla g) = \nabla g\cdot\nabla f + g\nabla\cdot\nabla f + \nabla f\cdot\nabla g + f\nabla\cdot\nabla g\\ &= g\nabla^2 f + f\nabla^2 g + 2\nabla f\cdot\nabla g\end{aligned}$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C f_y\,dx - f_x\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(-f_x) - \tfrac{\partial}{\partial y}(f_y)\right] dA = -\iint_D (f_{xx}+f_{yy})\,dA\\ &= -\iint_D 0\,dA = 0\end{aligned}$$

Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

24. (a) $x^2+y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2+y^2=1$. But also $y=z$, so C lies on the plane $y=z$. Thus C is the intersection of the plane $y=z$ and the cylinder $x^2+y^2=1$.



(b) Apply Stokes' Theorem, $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$:

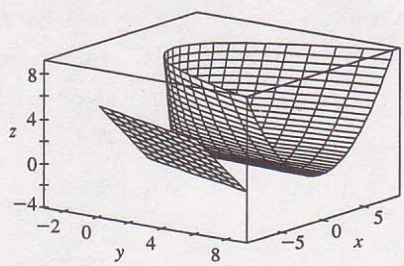
$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z\end{vmatrix}\\ &= \left\langle -2y\csc^2 z - \left(-2y\csc^2 z\right), 0, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}\end{aligned}$$

Therefore $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0$.

25. $z = f(x,y) = x^2 + 2y$ with $0\le x\le 1$, $0\le y\le 2x$. Thus
$A(S) = \iint_D \sqrt{1+4x^2+4}\,dA = \int_0^1\int_0^{2x}\sqrt{5+4x^2}\,dy\,dx = \int_0^1 2x\sqrt{5+4x^2}\,dx$
$= \frac{1}{6}\left(5+4x^2\right)^{3/2}\Big]_0^1 = \frac{1}{6}\left(27-5\sqrt{5}\right)$.

26. (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u\times\mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4,-2,1)$ corresponds to $u=1$, $v=2$ (or $u=-1$, $v=-2$ but $\mathbf{r}_u\times\mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4,-2,1)$ is $2\mathbf{i}+8\mathbf{j}+8\mathbf{k}$ and the equation of the tangent plane is $2x+8y+8z=0$ or $x+4y+4z=0$.

(b)



(c) By Definition 17.6.6 [ET 16.6.6], the area of S is given by
$A(S) = \int_0^3\int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\,dv\,du = 2\int_0^3\int_{-3}^3 \sqrt{u^4+4u^2v^2+v^4}\,dv\,du.$

(d) By Equation 17.7.9 [ET 16.7.9], the surface integral is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^3\int_{-3}^3 \left\langle \frac{(u^2)^2}{1+(v^2)^2}, \frac{(v^2)^2}{1+(-uv)^2}, \frac{(-uv)^2}{1+(u^2)^2} \right\rangle \cdot \langle 2u^2, 4uv, 2v^2\rangle \, dv\, du$$
$$= \int_0^3\int_{-3}^3 \left(\frac{2u^6}{1+v^4} + \frac{4uv^5}{1+u^2v^2} + \frac{2u^2v^4}{1+u^4} \right) dv\, du \approx 1524.0190$$

27. $z = f(x,y) = x^2+y^2$ with $0 \le x^2+y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\, dS = \iint_{x^2+y^2\le 4} (x^2+y^2)\sqrt{4x^2+4y^2+1}\, dA = \int_0^{2\pi}\int_0^2 r^3\sqrt{1+4r^2}\, dr\, d\theta$$
$$= \tfrac{1}{60}\pi\left(391\sqrt{17}+1\right)$$

(Substitute $u = 1+4r^2$ and use tables.)

28. $z = f(x,y) = 4+x+y$ with $0 \le x^2+y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2z + y^2z)\, dS = \iint_{x^2+y^2\le 4} (x^2+y^2)(4+x+y)\sqrt{3}\, dA$$
$$= \int_0^2\int_0^{2\pi} \sqrt{3}r^3(4+r\cos\theta + r\sin\theta)\, d\theta\, dr = \int_0^2 8\pi\sqrt{3}r^3\, dr = 32\pi\sqrt{3}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (z-2)\, dV = \iiint_E z\, dV - 2\iiint_E dV = m\overline{z} - 2\left(\tfrac{4}{3}\pi 2^3\right) = -\tfrac{64}{3}\pi.$

Alternate Solution: $\mathbf{F}(\mathbf{r}(\phi,\theta)) = 4\sin\phi\cos\theta\cos\phi\,\mathbf{i} - 4\sin\phi\sin\theta\,\mathbf{j} + 6\sin\phi\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\,\mathbf{i} + 4\sin^2\phi\sin\theta\,\mathbf{j} + 4\sin\phi\cos\phi\,\mathbf{k}$, and

$\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 16\sin^3\phi\cos^2\theta\cos\phi - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta$. Then

$$\iint_S F\cdot dS = \int_0^{2\pi}\int_0^{\pi} \left(16\sin^3\phi\cos\phi\cos^2\theta - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta\right) d\phi\, d\theta$$
$$= \int_0^{2\pi} \tfrac{4}{3}\left(-16\sin^2\theta\right) d\theta = -\tfrac{64}{3}\pi$$

30. $z = f(x,y) = x^2+y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and

$\mathbf{F}(\mathbf{r}(x,y))\cdot(\mathbf{r}_x\times\mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+y^2\le 1} \left(-2x^3 - 2xy^2 + x^2 + y^2\right) dA$$
$$= \int_0^1\int_0^{2\pi} \left(-2r^3\cos^3\theta - 2r^3\cos\theta\sin^2\theta + r^2\right) r\, dr\, d\theta = \int_0^1 r^3(2\pi)\, dr = \tfrac{\pi}{2}$$

31. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ and

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\left(-\cos^2 t\sin t + \sin^2 t\cos t\right) dt = \tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\Big]_0^{2\pi} = 0.$

32. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so

$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t\sin t}\,\mathbf{k}$, and

$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -16\cos^2 t\sin^2 t + 4\sin t\cos t$. Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\left(-16\cos^2 t\sin^2 t + 4\sin t\cos t\right) dt$$
$$= \left[-16\left(-\tfrac{1}{4}\sin t\cos^3 t + \tfrac{1}{16}\sin 2t + \tfrac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi.$$

33. The surface is given by $x+y+z=1$ or $z = 1-x-y$, $0\le x\le 1$, $0 \le y \le 1-x$ and $\mathbf{r}_x\times\mathbf{r}_y = \mathbf{i}+\mathbf{j}+\mathbf{k}$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D (-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\, dA$$
$$= \iint_D (-1)\, dA = -(\text{area of } D) = -\tfrac{1}{2}$$

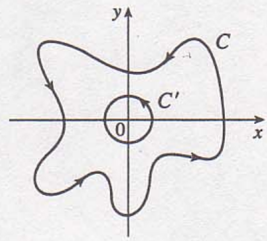
34. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3(x^2+y^2+z^2)\, dV = \int_0^{2\pi}\int_0^1\int_0^2 (3r^2+3z^2)\, r\, dz\, dr\, d\theta = 2\pi\int_0^1 (6r^3+8r)\, dr = 11\pi$

35. $\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_{x^2+y^2+z^2 \le 1} 3\, dV = 3\,(\text{volume of sphere}) = 4\pi$. Then
$\mathbf{F}(\mathbf{r}(\phi,\theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi = \sin\phi$ and
$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = (2\pi)(2) = 4\pi$.

36. Here we must use Equation 17.9.6 [ET 16.9.6] since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by
$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iiint_E \operatorname{div} \mathbf{F}\, dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, so
$\operatorname{div} \mathbf{F} = \nabla \cdot (|\mathbf{r}|^{-3}\mathbf{r}) = |\mathbf{r}|^{-3}(\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot (\nabla |\mathbf{r}|^{-3}) = |\mathbf{r}|^{-3}(3) + \mathbf{r} \cdot (-3|\mathbf{r}|^{-4})(\mathbf{r}|\mathbf{r}|^{-1}) = 0$. (Here we have used Exercises 17.5.30(a) [ET 16.5.30(a)] and 17.5.31(a) [ET 16.5.31(a)].) And $\mathbf{F} \cdot \mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$
on S_1. Thus $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS = \iint_{S_1} dS + \iiint_E 0\, dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi$.

37. Because $\operatorname{curl} \mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, and if $f(x,y,z) = x^3yz - 3xy + z^2$, then $\nabla f = \mathbf{F}$. Hence
$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0,3,0) - f(0,0,2) = 0 - 4 = -4$.

38. Let C' be the circle with center at the origin and radius a as in the figure. Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem



$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0, \text{ so}$$

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \left[\frac{2a^3\cos^3 t + 2a^3\cos t\sin^2 t - 2a\sin t}{a^2}(-a\sin t)\right. \\
&\qquad \left. + \frac{2a^3\sin^3 t + 2a^3\cos^2 t\sin t + 2a\cos t}{a^2}(a\cos t)\right] dt \\
&= \int_0^{2\pi} \frac{2a^2}{a^2}\, dt = 4\pi
\end{aligned}$$

39. By the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = 3\,(\text{volume of } E) = 3\,(8-1) = 21$.

40. The stated conditions allow us to use the Divergence Theorem. Hence
$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

Problems Plus

1. Let S_1 be the portion of $\Omega(S)$ between $S(a)$ and S, and let ∂S_1 be its boundary. Also let S_L be the lateral surface of S_1 [that is, the surface of S_1 except S and $S(a)$]. Applying the Divergence Theorem we have $\iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} \nabla\cdot\frac{\mathbf{r}}{r^3}\,dV$. But

$$\nabla\cdot\frac{\mathbf{r}}{r^3} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle$$
$$= \frac{(x^2+y^2+z^2-3x^2)+(x^2+y^2+z^2-3y^2)+(x^2+y^2+z^2-3z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

$\Rightarrow \iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} 0\,dV = 0$. On the other hand, notice that for the surfaces of ∂S_1 other than $S(a)$ and S, $\mathbf{r}\cdot\mathbf{n} = 0 \Rightarrow$

$$0 = \iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S_L} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$$
$$= \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$$

$\Rightarrow \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = -\iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$. Notice that on $S(a)$, $r = a \Rightarrow \mathbf{n} = -\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{a}$ and $\mathbf{r}\cdot\mathbf{r} = r^2 = a^2$, so that $-\iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{r}}{a^4}\,dS = \iint_{S(a)} \frac{a^2}{a^4}\,dS = \frac{1}{a^2}\iint_{S(a)} dS = \frac{\text{area of } S(a)}{a^2} = |\Omega(S)|$.

Therefore $|\Omega(S)| = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$.

2. By Green's Theorem

$$\int_C (y^3 - y)\,dx - 2x^3\,dy = \iint_D \left[\frac{\partial(-2x^3)}{\partial x} - \frac{\partial(y^3-y)}{\partial y}\right] dA = \iint_D (1 - 6x^2 - 3y^2)\,dA$$

Notice that for $6x^2 + 3y^2 > 1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2 + 3y^2 = 1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2 + 3y^2 = 1$.

3. The given line integral $\frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ can be expressed as $\int_C \mathbf{F}\cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x,y,z) = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \frac{1}{2}(bz - cy)\,\mathbf{i} + \frac{1}{2}(cx - az)\,\mathbf{j} + \frac{1}{2}(ay - bx)\,\mathbf{k}$. Then define S to be the planar interior of C, so S is an oriented, smooth surface. Stokes' Theorem says $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$. Now

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \left(\tfrac{1}{2}a + \tfrac{1}{2}a\right)\mathbf{i} + \left(\tfrac{1}{2}b + \tfrac{1}{2}b\right)\mathbf{j} + \left(\tfrac{1}{2}c + \tfrac{1}{2}c\right)\mathbf{k} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} = \mathbf{n}$$

so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{n} = |\mathbf{n}|^2 = 1$, hence $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S dS$ which is simply the surface area of S. Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ is the plane area enclosed by C.

4. (a) First we place the piston on coordinate axes so the top of the cylinder is at the origin and $x(t) \geq 0$ is the distance from the top of the cylinder to the piston at time t. Let C_1 be the curve traced out by the piston during one four-stroke cycle, so C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leq t \leq b$. (Thus, the curve lies on the positive x-axis and reverses direction several times.) The force on the piston is $AP(t)\,\mathbf{i}$, where A is the area of the top of the piston and $P(t)$ is the pressure in the cylinder at time t. As in Section 17.2 [ET 16.2], the work done on the piston is $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b AP(t)\,\mathbf{i} \cdot x'(t)\,\mathbf{i}\,dt = \int_a^b AP(t)\,x'(t)\,dt$. Here, the volume of the cylinder at time t is $V(t) = Ax(t) \quad \Rightarrow \quad V'(t) = Ax'(t) \quad \Rightarrow \quad \int_a^b AP(t)\,x'(t)\,dt = \int_a^b P(t)\,V'(t)\,dt$. Since the curve C in the PV-plane corresponds to the values of P and V at time t, $a \leq t \leq b$, we have

$W = \int_a^b AP(t)\,x'(t)\,dt = \int_a^b P(t)\,V'(t)\,dt = \int_C P\,dV$.

Another method: If we divide the time interval $[a, b]$ into n subintervals of equal length Δt, the amount of work done on the piston in the ith time interval is approximately $AP(t_i)\,[x(t_i) - x(t_{i-1})]$. Thus we estimate the total work done during one cycle to be $\sum\limits_{i=1}^{n} AP(t_i)\,[x(t_i) - x(t_{i-1})]$. If we allow $n \to \infty$, we have

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} AP(t_i)\,[x(t_i) - x(t_{i-1})] = \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)\,[Ax(t_i) - Ax(t_{i-1})] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)\,[V(t_i) - V(t_{i-1})] = \int_C P\,dV \end{aligned}$$

(b) Let C_L be the lower loop of the curve C and C_U the upper loop. Then $C = C_L \cup C_U$. C_L is positively oriented, so from Formula 17.4.5 [ET 16.4.5] we know the area of the lower loop in the PV-plane is given by $-\oint_{C_L} P\,dV$. C_U is negatively oriented, so the area of the upper loop is given by

$-\left(-\oint_{C_U} P\,dV\right) = \oint_{C_U} P\,dV$. From part (a),

$W = \int_C P\,dV = \int_{C_L \cup C_U} P\,dV = \oint_{C_L} P\,dV + \oint_{C_U} P\,dV = \oint_{C_U} P\,dV - \left(-\oint_{C_L} P\,dV\right)$, the difference of the areas enclosed by the two loops of C.

18 Second-Order Differential Equations ET 17

18.1 Second-Order Linear Equations ET 17.1

1. The auxiliary equation is $r^2 - 6r + 8 = 0 \quad \Rightarrow \quad (r-4)(r-2) = 0 \quad \Rightarrow \quad r = 4, r = 2$. Then by (8) the general solution is $y = c_1 e^{4x} + c_2 e^{2x}$.

2. The auxiliary equation is $r^2 - 4r + 8 = 0 \quad \Rightarrow \quad r = 2 \pm 2i$. Then by (11) the general solution is $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$.

3. The auxiliary equation is $r^2 + 8r + 41 = 0 \quad \Rightarrow \quad r = -4 \pm 5i$. Then by (11) the general solution is $y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$.

4. The auxiliary equation is $2r^2 - r - 1 = (2r+1)(r-1) = 0 \quad \Rightarrow \quad r = 1, r = -\frac{1}{2}$. Then the general solution is $y = c_1 e^{x} + c_2 e^{-x/2}$.

5. The auxiliary equation is $r^2 - 2r + 1 = (r-1)^2 = 0 \quad \Rightarrow \quad r = 1$. Then by (10), the general solution is $y = c_1 e^{x} + c_2 x e^{x}$.

6. The auxiliary equation is $3r^2 - 5r = r(3r-5) = 0 \quad \Rightarrow \quad r = 0, r = \frac{5}{3}$, so $y = c_1 + c_2 e^{5x/3}$.

7. The auxiliary equation is $4r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm\frac{1}{2}i$, so $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$.

8. The auxiliary equation is $16r^2 + 24r + 9 = (4r+3)^2 = 0 \quad \Rightarrow \quad r = -\frac{3}{4}$, so $y = c_1 e^{-3x/4} + c_2 x e^{-3x/4}$.

9. The auxiliary equation is $4r^2 + r = r(4r+1) = 0 \quad \Rightarrow \quad r = 0, r = -\frac{1}{4}$, so $y = c_1 + c_2 e^{-x/4}$.

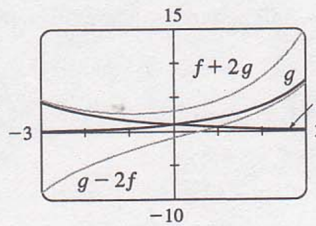
10. The auxiliary equation is $9r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm\frac{2}{3}i$, so $y = c_1 \cos\left(\frac{2}{3}x\right) + c_2 \sin\left(\frac{2}{3}x\right)$.

11. The auxiliary equation is $r^2 - 2r - 1 = 0 \quad \Rightarrow \quad r = 1 \pm \sqrt{2}$, so $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$.

12. The auxiliary equation is $r^2 - 6r + 4 = 0 \quad \Rightarrow \quad r = 3 \pm \sqrt{5}$, so $y = c_1 e^{(3+\sqrt{5})t} + c_2 e^{(3-\sqrt{5})t}$.

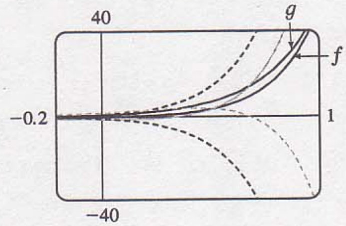
13. The auxiliary equation is $r^2 + r + 1 = 0 \quad \Rightarrow \quad r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so $y = e^{-t/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right]$.

14. $6r^2 - r - 2 = (2r+1)(3r-2) = 0$ so $y = c_1 e^{-x/2} + c_2 e^{2x/3}$. The solutions $(c_1, c_2) = (0, 1)$, $(1, 0)$, $(1, 2)$, $(-2, 1)$ are shown. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x \to \pm\infty$.

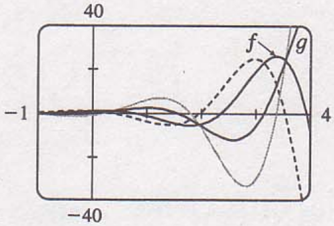


15. $r^2 - 8r + 16 = (r-4)^2 = 0$ so $y = c_1 e^{4x} + c_2 x e^{4x}$.

The graphs are all asymptotic to the x-axis as $x \to -\infty$, and as $x \to \infty$ the solutions tend to $\pm\infty$.



16. $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and the solution is $y = e^x (c_1 \cos 2x + c_2 \sin 2x)$. Graphs for $(c_1, c_2) = (1, 0), (0, 1), (1, -1), (-1, 2)$ are shown. The solutions are all asymptotic to the x-axis as $x \to -\infty$ and they all oscillate. The amplitudes of the oscillations become arbitrarily large as $x \to \infty$ and arbitrarily small as $x \to -\infty$.



17. $2r^2 + 5r + 3 = (2r + 3)(r + 1) = 0$, so $r = -\frac{3}{2}$, $r = -1$ and the general solution is $y = c_1 e^{-3x/2} + c_2 e^{-x}$. Then $y(0) = 3 \Rightarrow c_1 + c_2 = 3$ and $y'(0) = -4 \Rightarrow -\frac{3}{2}c_1 - c_2 = -4$, so $c_1 = 2$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = 2e^{-3x/2} + e^{-x}$.

18. $r^2 - 4 = (r + 2)(r - 2) = 0$ so the general solution is $y = c_1 e^{-2x} + c_2 e^{2x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y'(0) = -2c_1 + 2c_2$ so $c_1 = c_2 = \frac{1}{2}$ and the solution to the initial-value problem is $y = \frac{1}{2}\left(e^{-2x} + e^{2x}\right) = \cosh 2x$.

19. $r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$ and the general solution is $y = e^x (c_1 \cos x + c_2 \sin x)$. But $1 = y(0) = c_1$ and $2 = y'(0) = c_1 + c_2$ so the solution to the initial-value problem is $y = e^x (\cos x + \sin x)$.

20. $r^2 + 4r + 6 = 0 \Rightarrow r = -2 \pm \sqrt{2}i$ and the general solution is $y = e^{-2x}\left(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x\right)$. But $2 = y(0) = c_1$ and $4 = y'(0) = -2c_1 + \sqrt{2}c_2$, so the solution to the initial-value problem is $y = e^{-2x}\left(2 \cos \sqrt{2}x + 4\sqrt{2} \sin \sqrt{2}x\right)$.

21. $r^2 - 2r - 3 = (r - 3)(r + 1) = 0$ so the general solution is $y = c_1 e^{-x} + c_2 e^{3x}$. However the conditions are given at $x = 1$ so rewrite the general solution as $y = k_1 e^{-(x-1)} + k_2 e^{3(x-1)}$. Then $3 = y(1) = k_1 + k_2$ and $1 = y'(1) = -k_1 + 3k_2$ so $k_1 = 2$, $k_2 = 1$ and the solution to the initial-value problem is $y = 2e^{-(x-1)} + e^{3(x-1)}$.

22. $r^2 - 2r + 1 = (r - 1)^2 = 0$ so the general solution is $y = c_1 e^x + c_2 x e^x$. However the conditions are given at $x = 2$ so rewrite the general solution as $y = k_1 e^{x-2} + k_2 (x - 2) e^{x-2}$. Then $0 = y(2) = k_1$ and $1 = y'(2) = k_1 + k_2$ or $k_2 = 1$ and the solution to the initial-value problem is $y = (x - 2) e^{x-2}$.

23. $r^2 + 9 = 0 \Rightarrow r = \pm 3i$ and the general solution is $y = c_1 \cos 3x + c_2 \sin 3x$. But $0 = y\left(\frac{\pi}{3}\right) = -c_1$ and $1 = y'\left(\frac{\pi}{3}\right) = -3c_2$, so the solution to the initial-value problem is $y = -\frac{1}{3} \sin 3x$.

24. $r^2 + 4 = 0 \Rightarrow r = \pm 2i$ and the general solution is $y = c_1 \cos 2x + c_2 \sin 2x$. But $1 = y\left(\frac{\pi}{6}\right) = \frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2$ and $0 = y'\left(\frac{\pi}{6}\right) = -c_1\sqrt{3} + c_2$ so $c_1 = \frac{1}{2}$, $c_2 = \frac{\sqrt{3}}{2}$ and the solution to the initial-value problem is $y = \frac{1}{2}\left(\cos 2x + \sqrt{3} \sin 2x\right)$.

Alternate Solution: Rewrite the general solution as $y = k_1 \cos 2\left(x - \frac{\pi}{6}\right) + k_2 \sin 2\left(x - \frac{\pi}{6}\right)$. Then $1 = y\left(\frac{\pi}{6}\right) = k_1$ and $0 = y'\left(\frac{\pi}{6}\right) = 2k_2$ so the solution to the initial-value problem is $y = \cos 2\left(x - \frac{\pi}{6}\right)$. Verify that the answers agree.

25. $r^2 + 4r + 4 = (r + 2)^2 = 0$ so the general solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$. Then $0 = y(0) = c_1$, $3 = y(1) = c_2 e^{-2}$ so $c_2 = 3e^2$ and the solution of the boundary-value problem is $y = 3xe^{-2x+2}$.

26. $r^2 + 5r - 6 = (r + 6)(r - 1) = 0$ so the general solution is $y = c_1 e^x + c_2 e^{-6x}$. Then $0 = y(0) = c_1 + c_2$ and $1 = y(2) = c_1 e^2 + c_2 e^{-12}$ so $c_2 = \left(e^{-12} - e^2\right)^{-1}$, $c_1 = -\left(e^{-12} - e^{-2}\right)^{-1}$. The solution of the boundary-value problem is $y = \left(e^2 - e^{-12}\right)^{-1}\left(e^x - e^{-6x}\right)$.

27. $r^2+1=0 \Rightarrow r=\pm i$ and the general solution is $y=c_1\cos x+c_2\sin x$. But $1=y(0)=c_1$ and $0=y(\pi)=-c_1$ so there is no solution.

28. $r^2+9=0 \Rightarrow r=\pm 3i$ and the general solution is $y=c_1\cos 3x+c_2\sin 3x$. But $1=y(0)=c_1$ and $0=y\left(\frac{\pi}{2}\right)=-c_2$, so the solution to the boundary-value problem is $y=\cos 3x$.

29. $r^2-r-2=(r-2)(r+1)=0$ so the general solution is $y=c_1e^{-x}+c_2e^{2x}$. Then $1=y(-1)=c_1e+c_2e^{-2}$ and $0=y(1)=c_1e^{-1}+c_2e^2$ so $c_1=\dfrac{e^5}{e^6-1}$ and $c_2=\dfrac{e^2}{1-e^6}$ so the solution to the boundary-value problem is $y=\dfrac{e^5}{e^6-1}e^{-x}+\dfrac{e^2}{1-e^6}e^{2x}=\dfrac{1}{e^6-1}\left[e^{5-x}-e^{2(1+x)}\right]$.

30. $r^2+4r+3=(r+3)(r+1)=0$ so the general solution is $y=c_1e^{-x}+c_2e^{-3x}$. Then $0=y(1)=c_1e^{-1}+c_2e^{-3}$ and $2=y(3)=c_1e^{-3}+c_2e^{-9}$ so $c_1=\dfrac{2e^7}{e^4-1}$ and $c_2=\dfrac{2e^9}{1-e^4}$. Hence the solution to the boundary-value problem is $y=\dfrac{1}{e^4-1}\left(2e^{7-x}-2e^{9-3x}\right)$.

31. $r^2+4r+13=0 \Rightarrow r=-2\pm 3i$ and the general solution is $y=e^{-2x}(c_1\cos 3x+c_2\sin 3x)$. But $2=y(0)=c_1$ and $1=y\left(\frac{\pi}{2}\right)=e^{-\pi}(-c_2)$, so the solution to the boundary-value problem is $y=e^{-2x}(2\cos 3x-e^{\pi}\sin 3x)$.

32. $r^2+2r+5=0 \Rightarrow r=-1\pm 2i$ and the general solution is $y=e^{-x}(c_1\cos 2x+c_2\sin 2x)$. But $1=y(0)=c_1$ and $2=y(\pi)=e^{-\pi}(c_1)$ so there is no solution to the boundary-value problem.

33. (a) *Case 1* $(\lambda=0)$*:* $y''+\lambda y=0 \Rightarrow y''=0$ which has an auxiliary equation $r^2=0 \Rightarrow r=0 \Rightarrow$ $y=c_1+c_2x$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1$ and $0=y(L)=c_2L \Rightarrow c_1=c_2=0$. Thus, $y=0$.

Case 2 $(\lambda<0)$*:* $y''+\lambda y=0$ has auxiliary equation $r^2=-\lambda \Rightarrow r=\pm\sqrt{-\lambda}$ (distinct and real since $\lambda<0$) $\Rightarrow y=c_1e^{\sqrt{-\lambda}x}+c_2e^{-\sqrt{-\lambda}x}$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1+c_2$ $(\bigstar)$ and $0=y(L)=c_1e^{\sqrt{-\lambda}L}+c_2e^{-\sqrt{-\lambda}L}$ $(\dagger)$.

Multiplying $(\bigstar)$ by $e^{\sqrt{-\lambda}L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda}L}-e^{-\sqrt{-\lambda}L}\right)=0 \Rightarrow c_2=0$ and thus $c_1=0$ from $(\bigstar)$. Thus, $y=0$ for the cases $\lambda=0$ and $\lambda<0$.

(b) $y''+\lambda y=0$ has an auxiliary equation $r^2+\lambda=0 \Rightarrow r=\pm i\sqrt{\lambda} \Rightarrow y=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1$ and $0=y(L)=c_2\sin\sqrt{\lambda}L$ since $c_1=0$. Since we cannot have a trivial solution, $c_2\neq 0$ and thus $\sin\sqrt{\lambda}L=0 \Rightarrow \sqrt{\lambda}L=n\pi$ where n is an integer $\Rightarrow \lambda=n^2\pi^2/L^2$ and $y=c_2\sin(n\pi x/L)$ where n is an integer.

34. The auxiliary equation is $ar^2+br+c=0$. If $b^2-4ac>0$, then any solution is of the form $y(x)=c_1e^{r_1x}+c_2e^{r_2x}$ where $r_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$. But a, b, and c are all positive so both r_1 and r_2 are negative and $\lim_{x\to\infty}y(x)=0$. If $b^2-4ac=0$, then any solution is of the form $y(x)=c_1e^{rx}+c_2xe^{rx}$ where $r=-b/(2a)<0$ since a, b are positive. Hence $\lim_{x\to\infty}y(x)=0$. Finally if $b^2-4ac<0$, then any solution is of the form $y(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ where $\alpha=-b/(2a)<0$ since a and b are positive. Thus $\lim_{x\to\infty}y(x)=0$.

18.2 Nonhomogeneous Linear Equations ET 17.2

1. The auxiliary equation is $r^2 + 3r + 2 = (r+2)(r+1) = 0$, so the complementary solution is $y_c(x) = c_1 e^{-2x} + c_2 e^{-x}$. We try the particular solution $y_p(x) = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substituting into the differential equation, we have $(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$ or $2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$. Comparing coefficients gives $2A = 1$, $6A + 2B = 0$, and $2A + 3B + 2C = 0$, so $A = \frac{1}{2}$, $B = -\frac{3}{2}$, and $C = \frac{7}{4}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$.

2. The auxiliary equation is $r^2 + 9 = 0$ with roots $r = \pm 3i$, so the complementary solution is $y_c(x) = c_1 \cos(3x) + c_2 \sin(3x)$. Try the particular solution $y_p(x) = Ae^{3x}$, so $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substitution into the differential equation gives $9Ae^{3x} + 9(Ae^{3x}) = e^{3x}$ or $18Ae^{3x} = e^{3x}$. Thus $A = \frac{1}{18}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{18}e^{3x}$.

3. The auxiliary equation is $r^2 - 2r = r(r-2) = 0$, so the complementary solution is $y_c(x) = c_1 + c_2 e^{2x}$. Try the particular solution $y_p(x) = A\cos 4x + B\sin 4x$, so $y_p' = -4A\sin 4x + 4B\cos 4x$ and $y_p'' = -16A\cos 4x - 16B\sin 4x$. Substitution into the differential equation gives $(-16A\cos 4x - 16B\sin 4x) - 2(-4A\sin 4x + 4B\cos 4x) = \sin 4x \quad \Rightarrow$ $(-16A - 8B)\cos 4x + (8A - 16B)\sin 4x = \sin 4x$. Then $-16A - 8B = 0$ and $8A - 16B = 1 \quad \Rightarrow \quad A = \frac{1}{40}$ and $B = -\frac{1}{20}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^{2x} + \frac{1}{40}\cos 4x - \frac{1}{20}\sin 4x$.

4. The auxiliary equation is $r^2 + 6r + 9 = (r+3)^2 = 0$, so the complementary solution is $y_c(x) = c_1 e^{-3x} + c_2 x e^{-3x}$. Try the particular solution $y_p(x) = Ax + B$, so $y_p' = A$ and $y_p'' = 0$. Substitution into the differential equation gives $0 + 6A + 9(Ax + B) = 1 + x$ or $(9A)x + (6A + 9B) = 1 + x$. Comparing coefficients, we have $9A = 1$ and $6A + 9B = 1$, so $A = \frac{1}{9}$ and $B = \frac{1}{27}$. Thus the general solution is $y(x) = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{1}{9}x + \frac{1}{27}$.

5. The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is $y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{-x}$, so $y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \quad \Rightarrow \quad 10Ae^{-x} = e^{-x} \quad \Rightarrow \quad A = \frac{1}{10}$. Thus the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}$.

6. $y_c(x) = e^{-x}(c_1 x + c_2)$. Try $y_p(x) = x^2(Ax + B)e^{-x}$ so that no term in y_p is a solution of the complementary equation. Then $y_p' = \left[-Ax^3 + (3A - B)x^2 + 2Bx\right]e^{-x}$, $y_p'' = \left[Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B\right]e^{-x}$ and substitution gives $\left[Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B\right] + 2\left[-Ax^3 + (3A - B)x^2 + 2Bx\right] + (Ax^3 + Bx^2) = x \quad \Rightarrow$ $6Ax + 2B = x$. So $y_p(x) = x^2\left(\frac{1}{6}x\right)e^{-x}$ and the general solution is $y(x) = e^{-x}(c_1 x + c_2) + \frac{1}{6}x^3 e^{-x}$.

7. The auxiliary equation is $r^2 + 1 = 0$ with roots $r = \pm i$, so the complementary solution is $y_c(x) = c_1 \cos x + c_2 \sin x$. For $y'' + y = e^x$ try $y_{p_1}(x) = Ae^x$. Then $y_{p_1}' = y_{p_1}'' = Ae^x$ and substitution gives $Ae^x + Ae^x = e^x \quad \Rightarrow \quad A = \frac{1}{2}$, so $y_{p_1}(x) = \frac{1}{2}e^x$. For $y'' + y = x^3$ try $y_{p_2}(x) = Ax^3 + Bx^2 + Cx + D$.

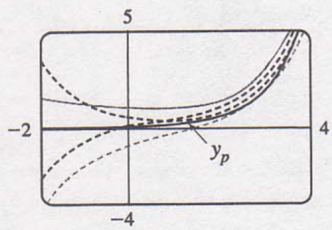
Then $y_{p2}' = 3Ax^2 + 2Bx + C$ and $y_{p2}'' = 6Ax + 2B$. Substituting, we have $6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$, so $A = 1$, $B = 0$, $6A + C = 0 \Rightarrow C = -6$, and $2B + D = 0 \Rightarrow D = 0$. Thus $y_{p2}(x) = x^3 - 6x$ and the general solution is $y(x) = y_c(x) + y_{p1}(x) + y_{p2}(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x + x^3 - 6x$. But $2 = y(0) = c_1 + \frac{1}{2} \Rightarrow c_1 = \frac{3}{2}$ and $0 = y'(0) = c_2 + \frac{1}{2} - 6 \Rightarrow c_2 = \frac{11}{2}$. Thus the solution to the initial-value problem is $y(x) = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x$.

8. The auxiliary equation is $r^2 - 4 = 0$ with roots $r = \pm 2$, so the complementary solution is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. Try $y_p(x) = e^x (A\cos x + B\sin x)$, so $y_p' = e^x(A\cos x + B\sin x + B\cos x - A\sin x)$ and $y_p'' = e^x(2B\cos x - 2A\sin x)$. Substitution gives $e^x(2B\cos x - 2A\sin x) - 4e^x(A\cos x + B\sin x) = e^x\cos x \Rightarrow$ $(2B - 4A)e^x\cos x + (-2A - 4B)e^x\sin x = e^x\cos x \Rightarrow A = -\frac{1}{5}$, $B = \frac{1}{10}$. Thus the general solution is $y(x) = c_1e^{2x} + c_2e^{-2x} + e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right)$. But $1 = y(0) = c_1 + c_2 - \frac{1}{5}$ and $2 = y'(0) = 2c_1 - 2c_2 - \frac{1}{10}$. Then $c_1 = \frac{9}{8}$, $c_2 = \frac{3}{40}$, and the solution to the initial-value problem is $y(x) = \frac{9}{8}e^{2x} + \frac{3}{40}e^{-2x} + e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right)$.

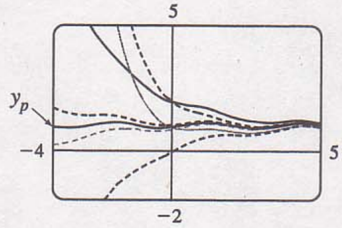
9. $y_c(x) = c_1e^x + c_2e^{-x}$. Try $y_p(x) = (Ax + B)e^{3x}$. Then $y_p' = e^{3x}(A + 3Ax + 3B)$ and $y_p'' = e^{3x}(3A + 3A + 9Ax + 9B)$. Substitution into the differential equation gives $e^{3x}[9Ax + 9B + 6A - (Ax + B)] = xe^{3x}$, so $A = \frac{1}{8}$, $B = -\frac{3}{32}$ and the general solution is $y(x) = c_1e^x + c_2e^{-x} + \left(\frac{1}{8}x - \frac{3}{32}\right)e^{3x}$. But $0 = y(0) = c_1 + c_2 - \frac{3}{32}$, $1 = y'(0) = c_1 - c_2 - \frac{9}{32} + \frac{1}{8}$ so the solution to the initial-value problem is $y(x) = \frac{5}{8}e^x - \frac{17}{32}e^{-x} + e^{3x}\left(\frac{1}{8}x - \frac{3}{32}\right)$.

10. $y_c(x) = c_1e^x + c_2e^{-2x}$. For $y'' + y' - 2y = x$ try $y_{p1}(x) = Ax + B$. Then $y_{p1}' = A$, $y_{p1}'' = 0$, and substitution gives $0 + A - 2(Ax + B) = x \Rightarrow A = -\frac{1}{2}$, $B = -\frac{1}{4}$, so $y_{p1}(x) = -\frac{1}{2}x - \frac{1}{4}$. For $y'' + y' - 2y = \sin 2x$ try $y_{p2}(x) = A\cos 2x + B\sin 2x$. Then $y_{p2}' = -2A\sin 2x + 2B\cos 2x$, $y_{p2}'' = -4A\cos 2x - 4B\sin 2x$, and substitution gives $(-4A\cos 2x - 4B\sin 2x) + (-2A\sin 2x + 2B\cos 2x) - 2(A\cos 2x + B\sin 2x) = \sin 2x \Rightarrow A = -\frac{1}{20}$, $B = -\frac{3}{20}$. Thus $y_{p2}(x) = -\frac{1}{20}\cos 2x + -\frac{3}{20}\sin 2x$ and the general solution is $y(x) = c_1e^x + c_2e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$. But $1 = y(0) = c_1 + c_2 - \frac{1}{4} - \frac{1}{20}$ and $0 = y'(0) = c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} \Rightarrow c_1 = \frac{17}{15}$ and $c_2 = \frac{1}{6}$. Thus the solution to the initial-value problem is $y(x) = \frac{17}{15}e^x + \frac{1}{6}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$.

11. $y_c(x) = c_1e^{-x/4} + c_2e^{-x}$. Try $y_p(x) = Ae^x$. Then $10Ae^x = e^x$, so $A = \frac{1}{10}$ and the general solution is $y(x) = c_1e^{-x/4} + c_2e^{-x} + \frac{1}{10}e^x$. The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as $x \to -\infty$), they all approach either ∞ or $-\infty$ as $x \to -\infty$. As $x \to \infty$, all solutions are asymptotic to $y_p = \frac{1}{10}e^x$.



12. The auxiliary equation is $(2r+1)(r+1)=0$, so $r=-1,-\frac{1}{2}$ and $y_c(x)=c_1e^{-x}+c_2e^{-x/2}$. For $2y''+3y'+y=1$, try $y_{p_1}(x)=A$; substituting gives $y_{p_1}(x)=1$. For $2y''+3y'+y=\cos 2x$ try $y_{p_2}=A\cos 2x+B\sin 2x \Rightarrow y'_{p_2}=-2A\sin 2x+2B\cos 2x$, $y''_{p_2}=-4A\cos 2x-4B\sin 2x$. Substituting into the differential equation gives $\cos 2x=(6B-7A)\cos 2x+(-7B-6A)\sin 2x$. Then solving the equations $6B-7A=1$ and $-7B-6A=0$ gives $A=-\frac{7}{85}$, $B=\frac{6}{85}$. Thus, $y_{p_2}(x)=-\frac{7}{85}\cos 2x+\frac{6}{85}\sin 2x$ and the general solution is $y(x)=c_1e^{-x}+c_2e^{-x/2}+1-\frac{7}{85}\cos 2x+\frac{6}{85}\sin 2x$.



The graph shows $y_p=y_{p_1}+y_{p_2}$ and several other solutions. Notice that all solutions are asymptotic to y_p as $x\to\infty$.

13. Since the roots of the auxiliary equation are complex, we need just try $y_p(x)=\left(Ax^4+Bx^3+Cx^2+Dx+E\right)e^{2x}$.

14. For $y''+6y'+2y=x^3$, try $y_{p_1}(x)=Ax^3+Bx^2+Cx+D$ and for $y''+6y'+2y=e^x\sin 2x$, try $y_{p_2}(x)=e^x(A\cos 2x+B\sin 2x)$.

15. Since $y_c(x)=e^x(c_1\cos x+c_2\sin x)$ we try $y_p(x)=xe^x(A\cos x+B\sin x)$.

16. Here $y_c(x)=c_1+c_2e^{-3x}$. For $y''+3y'=1$, try $y_{p_1}(x)=xA$, so that y_{p_1} is not a solution of the complementary equation. For $y''+3y'=xe^{-3x}$ try $y_{p_2}(x)=x(Ax+B)e^{-3x}$.

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u_1'=-\frac{Gy_2}{a(y_1y_2'-y_2y_1')} \quad \text{and} \quad u_2'=\frac{Gy_1}{a(y_1y_2'-y_2y_1')}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

17. (a) The complementary solution is $y_c(x)=c_1\cos 2x+c_2\sin 2x$. A particular solution is of the form $y_p(x)=Ax+B$. Thus, $4Ax+4B=x \Rightarrow A=\frac{1}{4}$ and $B=0 \Rightarrow y_p(x)=\frac{1}{4}x$. Thus, the general solution is $y=y_c+y_p=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}x$.

(b) In (a), $y_c(x)=c_1\cos 2x+c_2\sin 2x$, so set $y_1=\cos 2x$, $y_2=\sin 2x$. Then $y_1y_2'-y_2y_1'=2\cos^2 2x+2\sin^2 2x=2$ so $u_1'=-\frac{1}{2}x\sin 2x \Rightarrow$ $u_1(x)=-\frac{1}{2}\int x\sin 2x\,dx=-\frac{1}{4}\left(-x\cos 2x+\frac{1}{2}\sin 2x\right)$ (by parts) and $u_2'=\frac{1}{2}x\cos 2x$ $\Rightarrow u_2(x)=\frac{1}{2}\int x\cos 2x\,dx=\frac{1}{4}\left(x\sin 2x+\frac{1}{2}\cos 2x\right)$ (by parts). Hence $y_p(x)=-\frac{1}{4}\left(-x\cos 2x+\frac{1}{2}\sin 2x\right)\cos 2x+\frac{1}{4}\left(x\sin 2x+\frac{1}{2}\cos 2x\right)\sin 2x=\frac{1}{4}x$. Thus $y(x)=y_c(x)+y_p(x)=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}x$.

18. (a) Here $r^2-3r+2=0 \Rightarrow r=1$ or 2 and $y_c(x)=c_1e^{2x}+c_2e^x$. We try a particular solution of the form $y_p(x)=A\cos x+B\sin x \Rightarrow y_p'=-A\sin x+B\cos x$ and $y_p''=-A\cos x-B\sin x$. Then the equation $y''-3y'+2y=\sin x$ becomes $(A-3B)\cos x+(B+3A)\sin x=\sin x \Rightarrow A-3B=0$ and $B+3A=1 \Rightarrow A=\frac{3}{10}$ and $B=\frac{1}{10}$. Thus, $y_p(x)=\frac{3}{10}\cos x+\frac{1}{10}\sin x$. Therefore, the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{2x}+c_2e^x+\frac{3}{10}\cos x+\frac{1}{10}\sin x$.

(b) From (a) we know that $y_c(x) = c_1 e^{2x} + c_2 e^x$. Setting $y_1 = e^{2x}$, $y_2 = e^x$, we have $y_1 y_2' - y_2 y_1' = e^{3x} - 2e^{3x} = -e^{3x}$. Thus $u_1' = -\dfrac{\sin x e^x}{-e^{3x}} = \sin x e^{-2x}$ and $u_2' = \dfrac{\sin x e^{2x}}{-e^{3x}} = -\sin x e^{-x}$. Then $u_1(x) = \int e^{-2x} \sin x dx = \frac{1}{5} e^{-2x}(-2\sin x - \cos x)$ (by parts) and $u_2(x) = -\int e^{-x} \sin x dx = -\frac{1}{2} e^{-x}(-\sin x - \cos x)$. Thus $y_p(x) = \frac{1}{5}(-2\sin x - \cos x) + \frac{1}{2}(\sin x + \cos x) = \frac{1}{10}\sin x + \frac{3}{10}\cos x$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^x + \frac{1}{10}\sin x + \frac{3}{10}\cos x$.

19. (a) $r^2 - r = r(r-1) = 0 \Rightarrow r = 0, 1$, so the complementary solution is $y_c(x) = c_1 e^x + c_2 x e^x$. A particular solution is of the form $y_p(x) = Ae^{2x}$. Thus $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \Rightarrow Ae^{2x} = e^{2x} \Rightarrow A = 1$ $\Rightarrow y_p(x) = e^{2x}$. So a general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}$.

(b) From (a), $y_c(x) = c_1 e^x + c_2 x e^x$, so set $y_1 = e^x$, $y_2 = xe^x$. Then, $y_1 y_2' - y_2 y_1' = e^{2x}(1+x) - xe^{2x} = e^{2x}$ and so $u_1' = -xe^x \Rightarrow u_1(x) = -\int xe^x dx = -(x-1)e^x$ (by parts) and $u_2' = e^x \Rightarrow$ $u_2(x) = \int e^x \, dx = e^x$. Hence $y_p(x) = (1-x)e^{2x} + xe^{2x} = e^{2x}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}$.

20. (a) Here $r^2 - 2r + 1 = (r+1)^2 = 0 \Rightarrow r = -1$ and $y_c(x) = c_1 + c_2 e^x$ and so we try a particular solution of the form $y_p(x) = Axe^x$. Thus, after calculating the necessary derivatives, we get $y'' - y' = e^x \Rightarrow$ $Ae^x(2+x) - Ae^x(1+x) = e^x \Rightarrow A = 1$. Thus $y_p(x) = xe^x$ and the general solution is $y(x) = c_1 + c_2 e^x + xe^x$.

(b) From (a) we know that $y_c(x) = c_1 + c_2 e^x$, so setting $y_1 = 1$, $y_2 = e^x$, then $y_1 y_2' - y_2 y_1' = e^x - 0 = e^x$. Thus $u_1' = -e^{2x}/e^x = -e^x$ and $u_2' = e^x/e^x = 1$. Then $u_1(x) = -\int e^x dx = -e^x$ and $u_2(x) = x$. Thus $y_p(x) = -e^x + xe^x$ and the general solution is $y(x) = c_1 + c_2 e^x - e^x + xe^x = c_1 + c_3 e^x + xe^x$.

21. As in Example 6, $y_c(x) = c_1 \sin x + c_2 \cos x$, so set $y_1 = \sin x$, $y_2 = \cos x$. Then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$, so $u_1' = -\dfrac{\sec x \cos x}{-1} = 1 \Rightarrow u_1(x) = x$ and $u_2' = \dfrac{\sec x \sin x}{-1} = -\tan x \Rightarrow u_2(x) = -\int \tan x dx = \ln|\cos x| = \ln(\cos x)$ on $0 < x < \frac{\pi}{2}$. Hence $y_p(x) = x\sin x + \cos x \ln(\cos x)$ and the general solution is $y(x) = (c_1 + x)\sin x + [c_2 + \ln(\cos x)]\cos x$.

22. Setting $y_1 = \sin x$, $y_2 = \cos x$, then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$. Thus $u_1' = -\dfrac{\cot x \cos x}{-1} = \dfrac{\cos^2 x}{\sin x}$ and $u_2' = \dfrac{\cot x \sin x}{-1} = -\cos x$. Then $u_1(x) = \displaystyle\int \frac{\cos^2 x}{\sin x} dx = \int (\csc x - \sin x)\, dx = \ln(\csc x - \cot x) + \cos x$ and $u_2(x) = -\sin x$. Thus $y_p(x) = [\cos x + \ln(\csc x - \cot x)]\sin x + (-\sin x)(\cos x)$ and the general solution is $y(x) = c_1 \sin x + c_2 \cos x + \sin x \ln(\csc x - \cot x)$.

23. $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1 y_2' - y_2 y_1' = e^{3x}$. So $u_1' = \dfrac{-e^{2x}}{(1+e^{-x})e^{3x}} = -\dfrac{e^{-x}}{1+e^{-x}}$ and $u_1(x) = \displaystyle\int -\frac{e^{-x}}{1+e^{-x}} dx = \ln(1+e^{-x})$. $u_2' = \dfrac{e^x}{(1+e^{-x})e^{3x}} = \dfrac{e^x}{e^{3x}+e^{2x}}$ so $u_2(x) = \displaystyle\int \frac{e^x}{e^{3x}+e^{2x}} dx = \ln\left(\frac{e^x+1}{e^x}\right) - e^{-x} = \ln(1+e^{-x}) - e^{-x}$. Hence $y_p(x) = e^x \ln(1+e^{-x}) + e^{2x}\left[\ln(1+e^{-x}) - e^{-x}\right]$ and the general solution is $y(x) = [c_1 + \ln(1+e^{-x})]e^x + [c_2 - e^{-x} + \ln(1+e^{-x})]e^{2x}$.

24. $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and $y_1y_2' - y_2y_1' = -e^{-3x}$. So $u_1' = -\dfrac{(\sin e^x)\, e^{-2x}}{-e^{-3x}} = e^x \sin e^x$ and $u_2' = \dfrac{(\sin e^x)\, e^{-x}}{-e^{-3x}} = -e^{2x} \sin e^x$. Hence $u_1(x) = \int e^x \sin e^x dx = -\cos e^x$ and $u_2(x) = \int -e^{2x} \sin e^x dx = e^x \cos e^x - \sin e^x$. Then $y_p(x) = -e^{-x} \cos e^x - e^{-2x} [\sin e^x - e^x \cos e^x]$ and the general solution is $y(x) = (c_1 - \cos e^x)\, e^{-x} + [c_2 - \sin e^x + e^x \cos e^x]\, e^{-2x}$.

25. $y_1 = e^{-x}$, $y_2 = e^x$ and $y_1y_2' - y_2y_1' = 2$. So $u_1' = -\dfrac{e^x}{2x}$, $u_2' = \dfrac{e^{-x}}{2x}$ and $y_p(x) = -e^{-x} \displaystyle\int \frac{e^x}{2x} dx + e^x \int \frac{e^{-x}}{2x} dx$. Hence the general solution is

$$y(x) = \left(c_1 - \int \frac{e^x}{2x} dx\right) e^{-x} + \left(c_2 + \int \frac{e^{-x}}{2x} dx\right) e^x.$$

26. $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$ and $y_1y_2' - y_2y_1' = e^{-4x}$. Then $u_1' = \dfrac{-e^{-2x} x e^{-2x}}{x^3 e^{-4x}} = -\dfrac{1}{x^2}$ so $u_1(x) = x^{-1}$ and $u_2' = \dfrac{e^{-2x} e^{-2x}}{x^3 e^{-4x}} = \dfrac{1}{x^3}$ so $u_2(x) = -\dfrac{1}{2x^2}$. Thus $y_p(x) = \dfrac{e^{-2x}}{x} - \dfrac{xe^{-2x}}{2x^2} = \dfrac{e^{-2x}}{2x}$ and the general solution is $y(x) = e^{-2x} [c_1 + c_2 x + 1/(2x)]$.

18.3 Applications of Second-Order Differential Equations ET 17.3

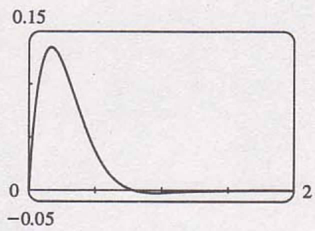
1. By Hooke's Law $k(0.6) = 20$ so $k = \frac{100}{3}$ is the spring constant and the differential equation is $3x'' + \frac{100}{3}x = 0$. The general solution is $x(t) = c_1 \cos\left(\frac{10}{3}t\right) + c_2 \sin\left(\frac{10}{3}t\right)$. But $0 = x(0) = c_1$ and $1.2 = x'(0) = \frac{10}{3}c_2$, so the position of the mass after t seconds is $x(t) = 0.36 \sin\left(\frac{10}{3}t\right)$.

2. $k(0.3) = 24.3$ or $k = 81$ is the spring constant and the resulting initial-value problem is $4x'' + 81x = 0$, $x(0) = -0.5$ (since compressed), $x'(0) = 0$. The general solution is $x(t) = c_1 \cos\left(\frac{9}{2}t\right) + c_2 \sin\left(\frac{9}{2}t\right)$. But $-0.2 = x(0) = c_1$ and $0 = x'(0) = \frac{9}{2}c_2$. Thus the position is given by $x(t) = -0.2 \cos(4.5t)$.

3. $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$. The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

4. (a) The differential equation is $3x'' + 30x' + 123x = 0$ with general solution $x(t) = e^{-5t}(c_1 \cos 4t + c_2 \sin 4t)$. Then $0 = x(0) = c_1$ and $2 = x'(0) = 4c_2$, so the position is given by $x(t) = \frac{1}{2}e^{-5t} \sin 4t$.

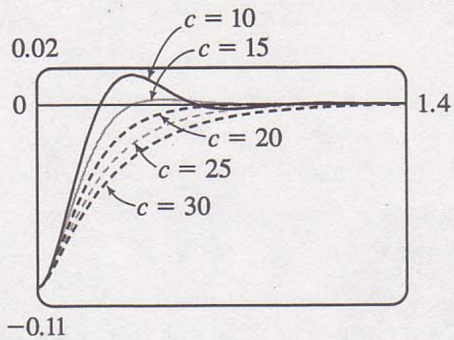
(b)



5. For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

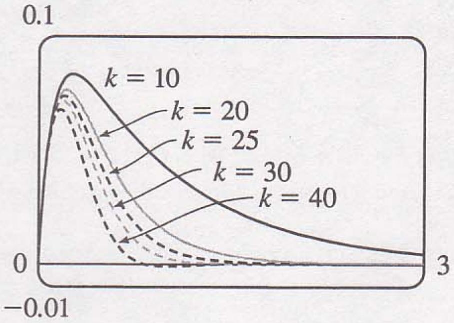
6. For critical damping we need $c^2 = 4mk$ or $c = 2\sqrt{mk} = 2\sqrt{3 \cdot 123} = 6\sqrt{41}$.

7. We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is $\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$. If $c = 10$, we have two complex roots $r = -5 \pm 5\sqrt{3}i$, so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1 \cos\left(5\sqrt{3}t\right) + c_2 \sin\left(5\sqrt{3}t\right)\right]$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = 5\sqrt{3}c_2 - 5c_1 \quad\Rightarrow\quad c_2 = -\frac{1}{10\sqrt{3}}$, so $x = e^{-5t}\left[-0.1\cos\left(5\sqrt{3}t\right) - \frac{1}{10\sqrt{3}}\sin\left(5\sqrt{3}t\right)\right]$. If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is $x = e^{-15t/2}\left[c_1\cos\left(\frac{5\sqrt{7}}{2}t\right) + c_2\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1 = x(0) = c_1$ and $0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}c_1 \quad\Rightarrow\quad c_2 = -\frac{3}{10\sqrt{7}}$. Thus $x = e^{-15t/2}\left[-0.1\cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{3}{10\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$. For $c = 20$, we have equal roots $r_1 = r_2 = -10$, so the oscillation is critically damped and the solution is $x = (c_1 + c_2t)\,e^{-10t}$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = -10c_1 + c_2 \quad\Rightarrow\quad c_2 = -1$, so $x = (-0.1 - t)\,e^{-10t}$. If $c = 25$ the auxiliary equation has roots $r_1 = -5$, $r_2 = -20$, so we have overdamping and the solution is $x = c_1e^{-5t} + c_2e^{-20t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -5c_1 - 20c_2 \quad\Rightarrow\quad c_1 = -\frac{2}{15}$ and $c_2 = \frac{1}{30}$, so $x = -\frac{2}{15}e^{-5t} + \frac{1}{30}e^{-20t}$. If $c = 30$ we have roots $r = -15 \pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x = c_1e^{\left(-15+5\sqrt{5}\right)t} + c_2e^{\left(-15-5\sqrt{5}\right)t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = \left(-15 + 5\sqrt{5}\right)c_1 + \left(-15 - 5\sqrt{5}\right)c_2 \quad\Rightarrow\quad c_1 = \frac{-5-3\sqrt{5}}{100}$ and $c_2 = \frac{-5+3\sqrt{5}}{100}$, so $x = \left(\frac{-5-3\sqrt{5}}{100}\right)e^{\left(-15+5\sqrt{5}\right)t} + \left(\frac{-5+3\sqrt{5}}{100}\right)e^{\left(-15-5\sqrt{5}\right)t}$.



8. We are given $m = 1$, $c = 10$, $x(0) = 0$ and $x'(0) = 1$. The differential equation is $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + kx = 0$ with auxiliary equation $r^2 + 10r + k = 0$. $k = 10$: the auxiliary equation has roots $r = -5 \pm \sqrt{15}$ so we have overdamping and the solution is $x = c_1e^{\left(-5+\sqrt{15}\right)t} + c_2e^{\left(-5-\sqrt{15}\right)t}$. Entering the initial conditions gives $c_1 = \frac{1}{2\sqrt{15}}$ and $c_2 = -\frac{1}{2\sqrt{15}}$, so $x = \frac{1}{2\sqrt{15}}e^{\left(-5+\sqrt{15}\right)t} - \frac{1}{2\sqrt{15}}e^{\left(-5-\sqrt{15}\right)t}$. $k = 20$: $r = -5 \pm \sqrt{5}$ and the solution is $x = c_1e^{\left(-5+\sqrt{5}\right)t} + c_2e^{\left(-5-\sqrt{5}\right)t}$ so again the motion is overdamped. The initial conditions give $c_1 = \frac{1}{2\sqrt{5}}$ and $c_2 = -\frac{1}{2\sqrt{5}}$, so $x = \frac{1}{2\sqrt{5}}e^{\left(-5+\sqrt{5}\right)t} - \frac{1}{2\sqrt{5}}e^{\left(-5-\sqrt{5}\right)t}$. $k = 25$: we have equal roots $r_1 = r_2 = -5$, so the motion is critically damped and the solution is $x = (c_1 + c_2t)\,e^{-5t}$. The initial conditions give $c_1 = 0$ and $c_2 = 1$, so $x = te^{-5t}$. $k = 30$: $r = -5 \pm \sqrt{5}i$ so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1\cos\left(\sqrt{5}t\right) + c_2\sin\left(\sqrt{5}t\right)\right]$. The initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{5}}$, so

$x = \frac{1}{\sqrt{5}} e^{-5t} \sin\left(\sqrt{5}t\right)$. $k = 40$: $r = -5 \pm \sqrt{15}i$ so we again have underdamping. The solution is $x = e^{-5t}\left[c_1 \cos\left(\sqrt{15}t\right) + c_2 \sin\left(\sqrt{15}t\right)\right]$, and the initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{15}}$. Thus $x = \frac{1}{\sqrt{15}} e^{-5t} \sin\left(\sqrt{15}t\right)$.



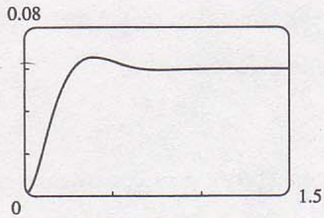
9. The differential equation is $mx'' + kx = F_0 \cos \omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$ with roots $\pm\sqrt{k/m}\, i = \pm \omega i$ so $x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$. Since $\omega_0 \neq \omega$, try $x_p(t) = A \cos \omega_0 t + B \sin \omega_0 t$. Then we need $(m)\left(-\omega_0^2\right)\left(A \cos \omega_0 t + B \sin \omega_0 t\right) + k\left(A \cos \omega_0 t + B \sin \omega_0 t\right) = F_0 \cos \omega_0 t$ or $A\left(k - m\omega_0^2\right) = F_0$ and $B\left(k - m\omega_0^2\right) = 0$. Hence $B = 0$ and $A = \dfrac{F_0}{k - m\omega_0^2} = \dfrac{F_0}{m\left(\omega^2 - \omega_0^2\right)}$ since $\omega^2 = \dfrac{k}{m}$. Thus the motion of the mass is given by $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \dfrac{F_0}{m\left(\omega^2 - \omega_0^2\right)} \cos \omega_0 t$.

10. As in Exercise 9, $x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t) = t\left(A \cos \omega t + B \sin \omega t\right)$. Then we need $m\left(2\omega B - \omega^2 At\right) \cos \omega t - m\left(2\omega A + \omega^2 Bt\right) \sin \omega t + kAt \cos \omega t + kBt \sin \omega t = F_0 \cos \omega t$ or $2m\omega B = F_0$ and $-2m\omega A = 0$ (noting $-m\omega^2 A + kA = 0$ and $-m\omega^2 B + kB = 0$ since $\omega^2 = k/m$). Hence the general solution is $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \left[F_0 t / (2m\omega)\right] \sin \omega t$.

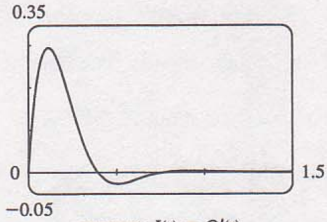
11. Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then $Q_c(t) = e^{-10t}\left(c_1 \cos 20t + c_2 \sin 20t\right)$ and try $Q_p(t) = A \Rightarrow 500A = 12$ or $A = \frac{3}{125}$. The general solution is $Q(t) = e^{-10t}\left(c_1 \cos 20t + c_2 \sin 20t\right) + \frac{3}{125}$. But $0 = Q(0) = c_1 + \frac{3}{125}$ and $Q'(t) = I(t) = e^{-10t}\left[\left(-10c_1 + 20c_2\right) \cos 20t + \left(-10c_2 - 20c_1\right) \sin 20t\right]$ but $0 = Q'(0) = -10c_1 + 20c_2$. Thus the charge is $Q(t) = -\frac{1}{250} e^{-10t}\left(6 \cos 20t + 3 \sin 20t\right) + \frac{3}{125}$ and the current is $I(t) = e^{-10t}\left(\frac{3}{5}\right) \sin 20t$.

12. (a) Here the initial-value problem for the charge is $2Q'' + 24Q' + 200Q = 12$ with $Q(0) = 0.001$ and $Q'(0) = 0$. Then $Q_c(t) = e^{-6t}\left(c_1 \cos 8t + c_2 \sin 8t\right)$ and try $Q_p(t) = A \Rightarrow A = \frac{3}{50}$ and the general solution is $Q(t) = e^{-6t}\left(c_1 \cos 8t + c_2 \sin 8t\right) + \frac{3}{50}$. But $0.001 = Q(0) = c + \frac{3}{50}$ so $c_1 = -0.059$. Also $Q'(t) = I(t) = e^{-6t}\left[\left(-6c_1 + 8c_2\right) \cos 8t + \left(-6c_2 - 8c_1\right) \sin 8t\right]$ and $0 = Q'(0) = -6c_1 + 8c_2$ so $c_2 = -0.04425$. Hence the charge is $Q(t) = -e^{-6t}\left(0.059 \cos 8t + 0.04425 \sin 8t\right) + \frac{3}{50}$ and the current is $I(t) = e^{-6t}\left(0.7375\right) \sin 8t$.

(b)



charge, $Q(t)$

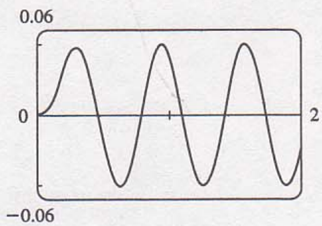


current, $I(t) = Q'(t)$

13. As in Exercise 11, $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ but $E(t) = 12 \sin 10t$ so try $Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives $(-100A + 200B + 500A) \cos 10t + (-100B - 200A + 500B) \sin 10t = 12 \sin 10t, \;\Rightarrow\; 400A + 200B = 0$ and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is $Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$. But $0 = Q(0) = c_1 - \frac{3}{250}$ so $c_1 = \frac{3}{250}$. Also $Q'(t) = \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t + e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t]$ and $0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2$ so $c_2 = -\frac{3}{500}$. Hence the charge is given by $Q(t) = e^{-10t}\left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t\right] - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$.

14. (a) As in Exercise 12, $Q_c(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t)$ but try $Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives $(-200A + 240B + 200A) \cos 10t + (-200B - 240A + 200B) \sin 10t = 12 \sin 10t$, so $B = 0$ and $A = -\frac{1}{20}$. Hence, the general solution is $Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{1}{20} \cos 10t$. But $0.001 = Q(0) = c_1 - \frac{1}{20}$, $Q'(t) = e^{-6t}[(-6c_1 + 8c_2) \cos 8t + (-6c_2 - 8c_1) \sin 8t] - \frac{1}{2} \sin 10t$ and $0 = Q'(0) = -6c_1 + 8c_2$, so $c_1 = 0.051$ and $c_2 = 0.03825$. Thus the charge is given by $Q(t) = e^{-6t}(0.051 \cos 8t + 0.03825 \sin 8t) - \frac{1}{20} \cos 10t$.

(b)



15. $x(t) = A \cos(\omega t + \delta) \;\Leftrightarrow\; x(t) = A[\cos \omega t \cos \delta - \sin \omega t \sin \delta] \;\Leftrightarrow\; x(t) = A\left(\frac{c_1}{A} \cos \omega t + \frac{c_2}{A} \sin \omega t\right)$ where $\cos \delta = c_1/A$ and $\sin \delta = -c_2/A \;\Leftrightarrow\; x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. (Note that $\cos^2 \delta + \sin^2 \delta = 1 \;\Rightarrow\; c_1^2 + c_2^2 = A^2$.)

16. (a) We approximate $\sin \theta$ by θ and, with $L = 1$ and $g = 9.8$, the differential equation becomes $\dfrac{d^2\theta}{dt^2} + 9.8\theta = 0$. The auxiliary equation is $r^2 + 9.8 = 0 \;\Rightarrow\; r = \pm\sqrt{9.8}i$, so the general solution is $\theta(t) = c_1 \cos\left(\sqrt{9.8}t\right) + c_2 \sin\left(\sqrt{9.8}t\right)$. Then $0.2 = \theta(0) = c_1$ and $1 = \theta'(0) = \sqrt{9.8}c_2 \;\Rightarrow\; c_2 = \frac{1}{\sqrt{9.8}}$, so the equation is $\theta(t) = 0.2 \cos\left(\sqrt{9.8}t\right) + \frac{1}{\sqrt{9.8}} \sin\left(\sqrt{9.8}t\right)$.

(b) $\theta'(t) = -0.2\sqrt{9.8}\sin\left(\sqrt{9.8}t\right) + \cos\left(\sqrt{9.8}t\right) = 0$ or $\tan\left(\sqrt{9.8}t\right) = \frac{5}{\sqrt{9.8}}$, so the critical numbers are $t = \frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right) + \frac{n}{\sqrt{9.8}}\pi$ (n any integer). The maximum angle from the vertical is $\theta\left(\frac{1}{\sqrt{9.8}}\tan^{-1}\frac{5}{\sqrt{9.8}}\right) \approx 0.377$ radians (or about $21.7\,°$).

(c) From part (b), the critical numbers of $\theta(t)$ are spaced $\frac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values is $2\left(\frac{\pi}{\sqrt{9.8}}\right)$. Thus the period of the pendulum is $\frac{2\pi}{\sqrt{9.8}} \approx 2.007$ seconds.

(d) $\theta(t) = 0 \quad\Rightarrow\quad 0.2\cos\left(\sqrt{9.8}t\right) + \frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}t\right) = 0 \quad\Rightarrow\quad \tan\left(\sqrt{9.8}t\right) = -0.2\sqrt{9.8} \quad\Rightarrow$ $t = \frac{1}{\sqrt{9.8}}\left[\tan^{-1}\left(-0.2\sqrt{9.8}\right) + \pi\right] \approx 0.825$ seconds.

(e) $\theta'(0.825) \approx -1.180$ rad/s.

18.4 Series Solutions ET 17.4

1. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$ and $y' - y = 0 \quad\Rightarrow\quad \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$. Replacing n by $n+1$ in the first sum gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=0}^{\infty} c_n x^n = 0$, so $\sum_{n=0}^{\infty}\left[(n+1)c_{n+1} - c_n\right]x^n = 0$. Equating coefficients gives $(n+1)c_{n+1} - c_n = 0$, so the recursion relation is $c_{n+1} = \dfrac{c_n}{n+1}$, $n = 0, 1, 2, \ldots$. Then $c_1 = c_0$, $c_2 = \dfrac{1}{2}c_1 = \dfrac{c_0}{2}$, $c_3 = \dfrac{1}{3}c_2 = \dfrac{1}{3}\cdot\dfrac{1}{2}c_0 = \dfrac{c_0}{3!}$, $c_4 = \dfrac{1}{4}c_3 = \dfrac{c_0}{4!}$, and in general, $c_n = \dfrac{c_0}{n!}$. Thus the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = c_0e^x.$$

2. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $\sum_{n=0}^{\infty} nc_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$. Replacing n by $n+1$ in the first sum and n by $n-1$ in the second gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$ or $c_1 + \sum_{n=1}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$. Thus $c_1 + \sum_{n=1}^{\infty}\left[(n+1)c_{n+1} - c_{n-1}\right]x^n = 0$. Equating coefficients gives $c_1 = 0$ and $(n+1)c_{n+1} - c_{n-1} = 0$. Thus the recursion relation is $c_{n+1} = \dfrac{c_{n-1}}{n+1}$, $n = 1, 2, \ldots$. But $c_1 = 0$, so $c_3 = 0$ and $c_5 = 0$ and in general $c_{2n+1} = 0$. Also $c_2 = \dfrac{c_0}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{c_0}{4\cdot 2} = \dfrac{c_0}{2^2\cdot 2!}$, $c_6 = \dfrac{c_4}{6} = \dfrac{c_0}{6\cdot 4\cdot 2} = \dfrac{c_0}{2^3\cdot 3!}$ and in general $c_{2n} = \dfrac{c_0}{2^n\cdot n!}$. Thus the solution is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n}x^{2n} = c_0\sum_{n=0}^{\infty}\frac{\left(x^2/2\right)^n}{n!} = c_0e^{x^2/2}$.

3. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$ and $-x^2y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2}x^n$. Hence the differential equation becomes $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=2}^{\infty} c_{n-2}x^n = 0$ or $c_1 + 2c_2x + \sum_{n=2}^{\infty}\left[(n+1)c_{n+1} - c_{n-2}\right]x^n = 0$. Equating coefficients gives $c_1 = c_2 = 0$ and $c_{n+1} = \dfrac{c_{n-2}}{n+1}$ for $n = 2, 3, \ldots$. But $c_1 = 0$, so $c_4 = 0$ and $c_7 = 0$ and in general $c_{3n+1} = 0$. Similarly $c_2 = 0$ so $c_{3n+2} = 0$. Finally $c_3 = \dfrac{c_0}{3}$, $c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{6\cdot 3} = \dfrac{c_0}{3^2\cdot 2!}$,

$c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{9\cdot 6\cdot 3} = \dfrac{c_0}{3^3\cdot 3!}, \ldots$, and $c_{3n} = \dfrac{c_0}{3^n\cdot n!}$. Thus the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}.$$

4. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \quad\Rightarrow\quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$. Then the differential equation becomes $(x-3)\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \quad\Rightarrow$

$\sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+1} - 3\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \quad\Rightarrow$

$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0 \quad\Rightarrow$

$\sum_{n=0}^{\infty}[(n+2)c_n - 3(n+1)c_{n+1}]x^n = 0$ (since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the recursion relation is $c_{n+1} = \dfrac{(n+2)c_n}{3(n+1)}$, $n = 0, 1, 2, \ldots$. Then $c_1 = \dfrac{2c_0}{3}$, $c_2 = \dfrac{3c_1}{3(2)} = \dfrac{3c_0}{3^2}$, $c_3 = \dfrac{4c_2}{3(3)} = \dfrac{4c_0}{3^3}$, $c_4 = \dfrac{5c_3}{3(4)} = \dfrac{5c_0}{3^4}$, and in general, $c_n = \dfrac{(n+1)c_0}{3^n}$. Thus the solution is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n$. $\left[\text{Note that } c_0 \displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n = \frac{9c_0}{(3-x)^2} \text{ for } |x| < 3.\right]$

5. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \quad\Rightarrow\quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. The differential equation becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + nc_n + c_n]x^n$ (since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives $(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$, thus the recursion relation is $c_{n+2} = \dfrac{-(n+1)c_n}{(n+2)(n+1)} = -\dfrac{c_n}{n+2}$, $n = 0, 1, 2, \ldots$. Then the even coefficients are given by $c_2 = -\dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = \dfrac{c_0}{2\cdot 4}$, $c_6 = -\dfrac{c_4}{6} = -\dfrac{c_0}{2\cdot 4\cdot 6}$, and in general, $c_{2n} = (-1)^n \dfrac{c_0}{2\cdot 4\cdot\cdots\cdot 2n} = \dfrac{(-1)^n c_0}{2^n n!}$. The odd coefficients are $c_3 = -\dfrac{c_1}{3}$, $c_5 = -\dfrac{c_3}{5} = \dfrac{c_1}{3\cdot 5}$, $c_7 = -\dfrac{c_5}{7} = -\dfrac{c_1}{3\cdot 5\cdot 7}$, and in general, $c_{2n+1} = (-1)^n \dfrac{c_1}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)} = \dfrac{(-2)^n n!\, c_1}{(2n+1)!}$. The solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}.$$

6. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$ then $y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. Hence the differential equation becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} - c_n]x^n = 0$. So the recursion relation is $c_{n+2} = \dfrac{c_n}{(n+2)(n+1)}$, $n = 0, 1, \ldots$. Given c_0 and c_1, $c_2 = \dfrac{c_0}{2\cdot 1}$, $c_4 = \dfrac{c_2}{4\cdot 3} = \dfrac{c_0}{4!}$, $c_6 = \dfrac{c_4}{6\cdot 5} = \dfrac{c_0}{6!}, \ldots$, $c_{2n} = \dfrac{c_0}{(2n)!}$ and $c_3 = \dfrac{c_1}{3\cdot 2}$, $c_5 = \dfrac{c_3}{5\cdot 4} = \dfrac{c_1}{5\cdot 4\cdot 3\cdot 2} = \dfrac{c_1}{5!}$, $c_7 = \dfrac{c_5}{7\cdot 6} = \dfrac{c_1}{7!}, \ldots$, $c_{2n+1} = \dfrac{c_1}{(2n+1)!}$. Thus the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} c_0 \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} c_1 \frac{x^{2n+1}}{(2n+1)!} = c_0 \cosh x + c_1 \sinh x$$

$$\left[\text{or } y(x) = c_0 \frac{e^x + e^{-x}}{2} + c_1 \frac{e^x - e^{-x}}{2} = \frac{c_0 + c_1}{2} e^x + \frac{c_0 - c_1}{2} e^{-x}\right].$$

7. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, $xy' = \sum_{n=0}^{\infty} n c_n x^n$ and $(x^2+1) y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + [n(n-1) + n - 1] c_n] x^n = 0$. The recursion relation is $c_{n+2} = -\dfrac{(n-1) c_n}{n+2}$, $n = 0, 1, 2, \ldots$. Given c_0 and c_1, $c_2 = \dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = -\dfrac{c_0}{2^2 \cdot 2!}$, $c_6 = -\dfrac{3c_4}{6} = (-1)^2 \dfrac{3c_0}{2^3 \cdot 3!}, \ldots$,
$c_{2n} = (-1)^{n-1} \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n-3)\, c_0}{2^n\, n!} = (-1)^{n-1} \dfrac{(2n-3)!\, c_0}{2^n 2^{n-2}\, n!\,(n-2)!} = (-1)^{n-1} \dfrac{(2n-3)!\, c_0}{2^{2n-2}\, n!\,(n-2)!}$ for $n = 2, 3, \ldots$. $c_3 = \dfrac{0 \cdot c_1}{3} = 0 \;\Rightarrow\; c_{2n+1} = 0$ for $n = 1, 2, \ldots$. Thus the solution is

$$y(x) = c_0 + c_1 x + c_0 \frac{x^2}{2} + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!}{2^{2n-2}\, n!\,(n-2)!} x^{2n}.$$

8. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$ and $-xy(x) = -\sum_{n=0}^{\infty} c_n x^{n+1} = -\sum_{n=1}^{\infty} c_{n-1} x^n$. The differential equation becomes $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0$ or $c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} - c_{n-1}] x^n = 0$. Equating coefficients gives $c_2 = 0$ and $c_{n+2} = \dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \ldots$. Since $c_2 = 0$, $c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Given c_0 and c_1, $c_3 = \dfrac{c_0}{3 \cdot 2}$, $c_6 = \dfrac{c_3}{6 \cdot 5} = \dfrac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \ldots$,
$c_{3n} = \dfrac{c_0}{3n(3n-1)(3n-3)(3n-4) \cdot \cdots \cdot 6 \cdot 5 \cdot 3 \cdot 2}$; also $c_4 = \dfrac{c_1}{4 \cdot 3}$, $c_7 = \dfrac{c_4}{7} = \dfrac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}$, $\ldots, c_{3n+1} = \dfrac{c_1}{(3n+1)3n(3n-2)(3n-3)\ldots 7 \cdot 6 \cdot 4 \cdot 3}$. The solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(3n-2)(3n-5) \cdot \cdots \cdot 7 \cdot 4}{(3n)!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(3n-1)(3n-4) \cdot \cdots \cdot 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}.$$

9. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$, $-xy'(x) = -\sum_{n=0}^{\infty} n c_n x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} - (n+1) c_n] x^n = 0$. Thus the recursion relation is $c_{n+2} = \dfrac{c_n}{n+2}$ for $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 1$ so $c_2 = \dfrac{1}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{1}{2 \cdot 4}$, $c_6 = \dfrac{c_4}{6} = \dfrac{1}{2 \cdot 4 \cdot 6}, \ldots$, $c_{2n} = \dfrac{1}{2^n\, n!}$. Also $c_1 = y'(0) = 0$ and by the recursion relation $c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Thus the solution to the initial-value problem is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\, n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}$.

10. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-2}^{\infty} (n+4)(n+3) c_{n+4} x^{n+2}$ $= 2c_2 + 6c_3 x + \sum_{n=0}^{\infty} (n+4)(n+3) c_{n+4} x^{n+2}$. Thus the differential equation becomes $2c_2 + 6c_3 x + \sum_{n=0}^{\infty} [(n+4)(n+3) c_{n+4} + c_n] x^{n+2} = 0$. So $c_2 = c_3 = 0$ and the recursion relation is $c_{n+4} = -\dfrac{c_n}{(n+4)(n+3)}$, $n = 0, 1, 2, \ldots$. But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion relation $c_{4n+1} = c_{4n+2} = c_{4n+3} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_0 = y(0) = 1$, so $c_4 = -\dfrac{1}{4 \cdot 3}$, $c_8 = -\dfrac{c_4}{8 \cdot 7} = \dfrac{(-1)^2}{8 \cdot 7 \cdot 4 \cdot 3}, \ldots, c_{4n} = \dfrac{(-1)^n}{4n(4n-1)(4n-4)(4n-5)\ldots 4 \cdot 3}$. Thus the solution to the initial-value problem is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = 1 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{4n(4n-1)\ldots 4 \cdot 3}$.

11. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} =$ $2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1}$ and the differential equation becomes $2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) c_{n+3} + (n+1) c_n] x^{n+1} = 0$. Then $c_2 = 0$ and the recursion relation is $c_{n+3} = -\dfrac{(n+1) c_n}{(n+3)(n+2)}$, $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0 = c_2$ and by the recursion relation $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_1 = y'(0) = 1$ so

$c_4 = -\dfrac{2}{4 \cdot 3}$, $c_7 = -\dfrac{5c_4}{7 \cdot 6} = (-1)^2 \dfrac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \dfrac{2^2 5^2}{7!}, \ldots, c_{3n+1} = (-1)^n \dfrac{2^2 5^2 \cdots (3n-1)^2}{(3n+1)!}$. Thus

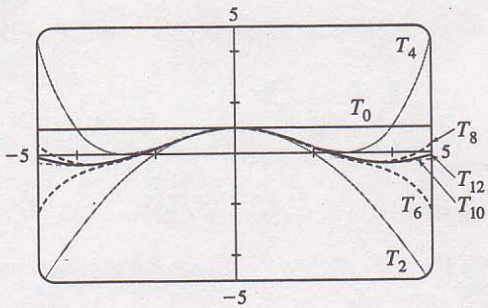
the solution is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=0}^{\infty} \left[(-1)^n \frac{2^2 5^2 \cdots (3n-1)^2 x^{3n+1}}{(3n+1)!} \right]$.

12. (a) Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^{n+2}$, $xy'(x) = \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=-1}^{\infty} (n+2) c_{n+2} x^{n+2} = c_1 x + \sum_{n=0}^{\infty} (n+2) c_{n+2} x^{n+2}$, and the differential equation becomes $c_1 x + \sum_{n=0}^{\infty} ([(n+2)(n+1) + (n+2)] c_{n+2} + c_n) x^{n+2} = 0$. Then $c_1 = 0$ and the recursion relation is $c_{n+2} = -\dfrac{c_n}{(n+2)^2}$, $n = 0, 1, 2, \ldots$. But $c_1 = y'(0) = 0$ so $c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_0 = y(0) = 1$, so $c_2 = -\dfrac{1}{2^2}$, $c_4 = -\dfrac{c_2}{4^2} = (-1)^2 \dfrac{1}{4^2 2^2} = (-1)^2 \dfrac{1}{2^4 (2!)^2}$,

$c_6 = -\dfrac{c_4}{6^2} = (-1)^3 \dfrac{1}{2^6 (3!)^2}, \ldots, c_{2n} = (-1)^n \dfrac{1}{2^{2n} (n!)^2}$. The solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} (n!)^2}.$$

(b) The Taylor polynomials T_0 to T_{12} are shown in the graph. Because T_{10} and T_{12} are close together throughout the interval $[-5, 5]$, it is reasonable to assume that T_{12} is a good approximation to the Bessel function on that interval.



18 Review ET 17

CONCEPT CHECK

1. (a) $ay'' + by' + cy = 0$ where a, b, and c are constants.

(b) $ar^2 + br + c = 0$

(c) If the auxiliary equation has two distinct real roots r_1 and r_2, the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If the roots are real and equal, the solution is $y = c_1 e^{rx} + c_2 x e^{rx}$ where r is the common root. If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

2. (a) An initial-value problem consists of finding a solution y of a second-order differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are constants.

(b) A boundary-value problem consists of finding a solution y of a second-order differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

3. (a) $ay'' + by' + cy = G(x)$ where a, b, and c are constants and G is a continuous function.

(b) The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution y_c of the complementary equation and y_p is any particular solution of the original differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

(c) See Examples 1–5 and the associated discussion in Section 18.2 [ET 17.2].

(d) See the discussion on pages 1170–1171 [ET 1136–1137].

4. Second-order linear differential equations can be used to describe the motion of a vibrating spring or to analyze an electric circuit; see the discussion in Section 18.3 [ET 17.3].

5. See Example 1 and the preceding discussion in Section 18.4 [ET 17.4].

TRUE-FALSE QUIZ

1. True. See Theorem 18.1.3 [ET 17.1.3].

2. False. The differential equation is not homogeneous.

3. True. $\cosh x$ and $\sinh x$ are linearly independent solutions of this linear homogeneous equation.

4. False. $y = Ae^x$ is a solution of the complementary equation, so we have to take $y_p(x) = Axe^x$.

EXERCISES

1. The auxiliary equation is $r^2 - 2r - 15 = 0 \quad \Rightarrow \quad (r-5)(r+3) = 0 \quad \Rightarrow \quad r = 5, r = -3$. Then the general solution is $y = c_1 e^{5x} + c_2 e^{-3x}$.

2. The auxiliary equation is $r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r = -2 \pm 3i$, so $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

3. The auxiliary equation is $r^2 + 3 = 0 \quad \Rightarrow \quad r = \pm\sqrt{3}\, i$. Then the general solution is $y = c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)$.

4. The auxiliary equation is $4r^2 + 4r + 1 = 0 \quad \Rightarrow \quad (2r+1)^2 = 0 \quad \Rightarrow \quad r = -\frac{1}{2}$, so the general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$.

5. $y_c(x) = e^{-x}(c_1 + c_2 x)$ and try $y_p(x) = A \cos 3x + B \sin 3x$. Then
$-9A \cos 3x - 9B \sin 3x - 6A \sin 3x + 6B \cos 3x + A \cos 3x + B \sin 3x = \sin 3x \quad \Rightarrow$
$6B - 8A = 0$ and $-6A - 8B = 1 \quad \Rightarrow \quad A = -\frac{3}{50}$, $B = -\frac{2}{25}$ and the general solution is
$y(x) = e^{-x}(c_1 + c_2 x) - \frac{3}{50} \cos 3x - \frac{2}{25} \sin 3x$.

6. $y_c(x) = c_1e^{-x} + c_2e^{3x}$ and try $y_p(x) = A\cos 4x + B\sin 4x$. Then $-16A\cos 4x - 16B\sin 4x + 8A\sin 4x - 8B\cos 4x - 3A\cos 4x - 3B\sin 4x = \cos 4x \quad\Rightarrow\quad -19A - 8B = 1$ and $-19B + 8A = 0 \quad\Rightarrow\quad A = -\frac{19}{425}$, $B = -\frac{8}{425}$, and the general solution is $y(x) = c_1e^{-x} + c_2e^{3x} - \frac{1}{425}(19\cos 4x + 8\sin 4x)$.

7. $y_c(x) = c_1\cos\left(\frac{3}{2}x\right) + c_2\sin\left(\frac{3}{2}x\right)$ and try $y_p(x) = Ax^2 + Bx + C$. Then $8A + 9Ax^2 + 9Bx + 9C = 2x^2 - 3$ $\Rightarrow\quad A = \frac{2}{9}$, $B = 0$, $C = -\frac{43}{81}$, and the general solution is $y(x) = c_1\cos\left(\frac{3}{2}x\right) + c_2\sin\left(\frac{3}{2}x\right) + \frac{2}{9}x^2 - \frac{43}{81}$.

8. $y_c(x) = e^{2x}(c_1\cos 4x + c_2\sin 4x)$ and try $y_p(x) = (Ax + B)e^x$. Then $e^x(Ax + B + 2A) + e^x(-4Ax - 4B - 4A) + (20Ax + 20B)e^x = xe^x \quad\Rightarrow\quad 17A = 1$ and $17B - 2A = 0 \quad\Rightarrow\quad A = \frac{1}{17}$, $B = \frac{2}{289}$, and the general solution is $y(x) = e^{2x}(c_1\cos 4x + c_2\sin 4x) + e^x\left(\frac{1}{17}x + \frac{2}{289}\right)$.

9. $y_c(x) = c_1e^x + c_2e^{2x}$ so try $y_p(x) = Axe^{2x}$. Then $(4Ax + 4A - 6Ax - 3A)e^{2x} + 2Axe^{2x} = e^{2x} \quad\Rightarrow$ $A = 1$. Thus the general solution is $y(x) = c_1e^x + c_2e^{2x} + xe^{2x}$.

10. Using variation of parameters, $y_c(x) = c_1\cos x + c_2\sin x$, $u_1'(x) = -\csc x\sin x = -1 \quad\Rightarrow\quad u_1(x) = -x$, and $u_2'(x) = \dfrac{\csc x\cos x}{x} = \cot x \quad\Rightarrow\quad u_2(x) = \ln|\sin x| \quad\Rightarrow\quad y_p = -x\cos x + \sin x\ln|\sin x|$. The solution is $y(x) = (c_1 - x)\cos x + (c_2 + \ln|\sin x|)\sin x$.

11. The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2e^{-6x} = k_1 + k_2e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k^2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.

12. The auxiliary equation is $r^2 - 6r + 25 = 0$ and the general solution is $y(x) = e^{3x}(c_1\cos 4x + c_2\sin 4x)$. But $2 = y(0) = c_1$ and $1 = y'(0) = 3c_1 + 4c_2$. Thus the solution is $y(x) = e^{3x}\left(2\cos 4x - \frac{5}{4}\sin 4x\right)$.

13. The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1e^x + c_2e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}\left(e^{4x} - e^x\right)$.

14. $y_c(x) = c_1\cos(x/3) + c_2\sin(x/3)$. For $9y'' + y = 3x$, try $y_{p_1}(x) = Ax + B$. Then $y_{p_1}(x) = 3x$. For $9y'' + y = e^{-x}$, try $y_{p_2}(x) = Ae^{-x}$. Then $9Ae^{-x} + Ae^{-x} = e^{-x}$ or $y_{p_2}(x) = \frac{1}{10}e^{-x}$. Thus the general solution is $y(x) = c_1\cos(x/3) + c_2\sin(x/3) + 3x + \frac{1}{10}e^{-x}$. But $1 = y(0) = c_1 + \frac{1}{10}$ and $2 = y'(0) = \frac{1}{3}c_2 + 3 - \frac{1}{10}$, so $c_1 = \frac{9}{10}$ and $c_2 = -\frac{27}{10}$. Hence the solution is $y(x) = \frac{1}{10}[9\cos(x/3) - 27\sin(x/3)] + 3x + \frac{1}{10}e^{-x}$.

15. Let $y(x) = \sum_{n=0}^{\infty} c_nx^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\dfrac{1}{3}$, $c_5 = \dfrac{(-1)^2}{3\cdot 5}$, $c_7 = \dfrac{(-1)^3}{3\cdot 5\cdot 7} = \dfrac{(-1)^3\,2^3 3!}{7!}$, $\ldots$, $c_{2n+1} = \dfrac{(-1)^n\,2^n\,n!}{(2n+1)!}$ for $n = 0, 1, 2, \ldots$. Thus the solution to the initial-value problem is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_nx^n = \sum_{n=0}^{\infty}\frac{(-1)^n\,2^n\,n!}{(2n+1)!}x^{2n+1}$.

16. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (n+2)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = \dfrac{c_n}{n+1}$ for $n = 0, 1, 2, \ldots$. Given c_0 and c_1, we have $c_2 = \dfrac{c_0}{1}$, $c_4 = \dfrac{c_2}{3} = \dfrac{c_0}{1 \cdot 3}$,
$c_6 = \dfrac{c_4}{5} = \dfrac{c_0}{1 \cdot 3 \cdot 5}, \ldots, c_{2n} = \dfrac{c_0}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = c_0 \dfrac{2^{n-1}(n-1)!}{(2n-1)!}$. Similarly $c_3 = \dfrac{c_1}{2}$,
$c_5 = \dfrac{c_3}{4} = \dfrac{c_1}{2 \cdot 4}$, $c_7 = \dfrac{c_5}{6} = \dfrac{c_1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n+1} = \dfrac{c_1}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} = \dfrac{c_1}{2^n n!}$. Thus the general solution is
$y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = c_0 + c_0 \sum\limits_{n=1}^{\infty} \dfrac{2^{n-1}(n-1)!\,x^{2n}}{(2n-1)!} + c \sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{2^n n!}$. But $\sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{2^n n!} = x \sum\limits_{n=0}^{\infty} \dfrac{\left(\frac{1}{2}x^2\right)^n}{n!} = xe^{x^2/2}$,
so $y(x) = c_1 x e^{x^2/2} + c_0 + c_0 \sum\limits_{n=1}^{\infty} \dfrac{2^{n-1}(n-1)!\,x^{2n}}{(2n-1)!}$.

17. Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0!$. Then $Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A!$. Thus the general solution is $Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}!$. But $0.01 = Q(0) = c_1 + 0.03$ and $0 = Q'(0) = -10c_1 + 10c_2$, so $c_1 = -0.02 = c_2!$. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

18. By Hooke's Law the spring constant is $k = 64$ and the initial-value problem is $2x'' + 16x' + 64x = 0$, $x(0) = 0$, $x'(0) = 2.4$. Thus the general solution is $x(t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t)$. But $0 = x(0) = c_1$ and $2.4 = x'(0) = -4c_1 + 4c_2 \quad \Rightarrow \quad c_1 = 0, c_2 = 0.6$. Thus the position of the mass is given by $x(t) = 0.6e^{-4t} \sin 4t$.

19. (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as follows: $\rho = \dfrac{\text{mass of earth}}{\text{volume of earth}} = \dfrac{M}{\frac{4}{3}\pi R^3}$. If V_r is the volume of the portion of the earth which lies within a distance r of the center, then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \dfrac{Mr^3}{R^3}$. Thus $F_r = -\dfrac{GM_r m}{r^2} = -\dfrac{GMm}{R^3} r$.

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion, $m\dfrac{d^2y}{dt^2} = F_y = -\dfrac{GMm}{R^3} y$, so $y''(t) = -k^2 y(t)$ where $k^2 = \dfrac{GM}{R^3}$. At the surface, $-mg = F_R = -\dfrac{GMm}{R^2}$, so $g = \dfrac{GM}{R^2}$. Therefore $k^2 = \dfrac{g}{R}$.

(c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 18.1 [ET 17.1], not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 18.1 [ET 17.1], the general solution of our differential equation for t is
$y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R \cos kt$ and $y'(t) = -kR \sin kt$. This is simple harmonic motion (see Section 18.3 [ET 17.3]) with amplitude R, frequency k, and phase angle 0. The period is $T = 2\pi/k$.
$R \approx 3960$ mi $= 3960 \cdot 5280$ ft and $g = 32$ ft/s^2, so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3} s^{-1}$ and
$T = 2\pi/k \approx 5079$ s ≈ 85 min.

(d) $y(t) = 0 \quad \Leftrightarrow \quad \cos kt = 0 \quad \Leftrightarrow \quad kt = \frac{\pi}{2} + \pi n$ for some integer $n \quad \Rightarrow$
$y'(t) = -kR \sin\left(\frac{\pi}{2} + \pi n\right) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899$ mi/s $\approx 17{,}600$ mi/h.